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Direct adaptive tuning of robust controllers with guaranteed stability properties

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Abstract: We address the following problem: given a fixed regulator that insures closed-loop stability for a partially known plant, design a mechanism to adaptively tune the controller parameters in order to improve performance. Our main concern is to insure that global $\mathscr{L}_\infty$-stability of the overall system is preserved. The proposed parameter update law includes a signal normalization [1] and a σ-modification [2]. The condition for global $\mathscr{L}_\infty$-stability relates the margin of stability of the closed-loop linear system, the speed of adaptation and the size of the residual set for the tracking error.

Keywords: Robust adaptive control, Adaptive tuning, Stability of adaptive controllers.

1. Introduction and problem formulation

In this paper we are interested in the following problem: given a linear time invariant plant and a known fixed stabilizing regulator, desing a mechanism to adaptively tune the controller parameters in order to improve performance retaining global stability of the overall system. The motivation to address this problem stems from the fact that stringent closed-loop performance demands require very accurate models. Since the actual dynamics are likely to differ from the available model, it is of practical importance to be able to tune the controller on-line. Furthermore a fairly complete theory is available for the design of stabilizing controllers for partially known systems, see e.g. [6]. The adaptive tuning procedure will therefore search a controller parametrization with improved performance. The main contribution of the paper is the establishment of a condition under which global $\mathscr{L}_\infty$-stability is preserved.

Although we use the term tuning to refer to the proposed control design, we are of course really discussing an adaptive controller. In this case, we consider continuous controller adjustments, however the theory developed here applies also mutatis mutandi to the practically important case of infrequent parameter update [7].

The system to be adaptively tuned is the single-input–single-output LTI plant described by

$$y(t) = G(p)u(t) + v(t) \tag{1}$$

where $p = \mathrm{d}/\mathrm{d}t$, $G(p)$ is the process transfer function, $u(t)$ and $y(t)$ are the process input and output respectively and $v(t)$ represents the effect of bounded disturbances as seen at the output. The input to the plant is taken as

$$u(t) = \hat{\theta}^{\mathrm{T}}(t)\phi(t) \tag{2}$$

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where $\phi(t) \in \mathbb{R}^n$ is an auxiliary vector, possibly containing filtered input, output and reference, $\hat{\theta}(t) \in \mathbb{R}^n$ is the control parameter vector to be adjusted on-line. The controller structure may, for instance, be taken such that a model reference objective is attained. In that case n is determined by the apriori estimate of the process order. Details on this structure can be found elsewhere, e.g. [3].

In this paper we will pursue a model following objective, that is we will require the process output to track the output of a reference model as close as possible. The design involves two steps: the first is the determination of a vector θ_0 such that for $\hat{\theta}(t) = \theta_0$ the closed-loop system is stable. This step evidently requires some prior knowledge of the plant, as described for instance in [6]. Notice that we do not require this controller to attain a good tracking error, but just to stabilize the plant. The second step is to design an estimator which will search in a neighborhood of θ_0 in the parameter space for the best available regulator. Our objective in this paper is to propose an estimator which guarantees that the on-line tuning does not upset the overall stability and eventually will lead to performance improvement.

To this end, we propose the following update law for the parameters:

$$\dot{\hat{\theta}}(t) = -\sigma\hat{\theta}(t) - \gamma\frac{\phi(t)e(t)}{\rho(t)^2} + \sigma\theta_0, \quad \hat{\theta}(0) = \theta_0, \quad \sigma, \gamma > 0, \tag{3}$$

where $e(t)$ is the tracking error and $\rho(t)$ is a normalizing factor satisfying

$$\rho(t) > \varepsilon \quad \text{for some } \varepsilon > 0, \tag{3a}$$

$$\dot{\rho}(t) \geq -\mu\rho(t), \quad \mu > 0, \tag{3b}$$

$$\frac{1}{\rho(t)} \max_i |\phi_i(t)| \leq m_\rho. \tag{3c}$$

We need the following assumption:

A.1. The plant (1) with the controller

$$u(t) = \theta_0^{\mathrm{T}}\phi(t)$$

has closed-loop poles λ_i such that

$$\mathrm{Re}\{\lambda_i\} < -\mu.$$

Remark 1. Notice that if θ_0 is a 'good' estimate for the controller, i.e. $e(t)$ is small, then $\hat{\theta}(t)$ will remain close to θ_0. Otherwise it will depart from θ_0 at a speed essentially determined by γ.

Remark 2. A normalizing factor that satisfies (3a)–(3c) is given in [4] for a model reference controller. Similarly to [5], to insure property (3c) with $m_\rho = 1$ the frequency content of the reference and process output must be restricted by means of low pass filters. The procedure is shown in the example of Section 4. We will assume in the sequel that $m_\rho = 1$. This without loss of generality, since as will be seen in the proof the result applies without modification to the standard case without filters, i.e. $m_\rho \neq 1$.

We will find it convenient to write the estimator in the equivalent operator form:

$$\tilde{\theta}(t) = H_1(p)\frac{\phi(t)e(t)}{\rho(t)^2} \tag{4a}$$

where

$$H_1(p) \triangleq \frac{-\gamma}{p+\sigma}, \qquad \tilde{\theta}(t) \triangleq \hat{\theta}(t) - \theta_0.$$

Note that $\tilde{\theta}(t)$ denotes the deviation of the actual controller parameter with respect to the stabilizing parameters.

Writing (2) in terms of $\tilde{\theta}(t)$, replacing in (1) and arranging terms we obtain the standard structure for the error model [3],

$$e(t) = H_2(p)\tilde{\theta}^{\mathrm{T}}(t)\phi(t) + e_0(t), \tag{4b}$$

$$\phi(t) = H_3(p)\tilde{\theta}(t)^{\mathrm{T}}\phi(t) + \phi_0(t), \tag{4c}$$

where $H_2(p) \in \mathbb{R}(p)$, $H_3(p) \in \mathbb{R}^n(p)$ are the transfer functions, $u(t) \to y(t)$ and $u(t) \to \phi(t)$ respectively, when $\hat{\theta}(t)$ is held fixed at θ_0. As seen from (4b), (4c), $e_0(t)$ and $\phi_0(t)$ are the resulting tracking error and regressor for the fixed controller. In view of assumption A.1, $H_2(p)$ is stable and $e_0(t) \in \mathscr{L}_\infty$.

The conditions to insure stability of $H_3(p)$ and $\phi_0(t)$ depend on the particular structure of the controller (2).

For model reference controllers the transfer function vector $H_3(p)$ is of the form (see e.g. [3,4])

$$H_3(p) = \left[0, \frac{1}{\Lambda(p)}G(p)^{-1}, \ldots, \frac{p^{n/2-2}}{\Lambda(p)}G(p)^{-1}, \frac{a}{p+a}, \frac{1}{\Lambda(p)}, \ldots, \frac{p^{n/2-2}}{\Lambda(p)}\right]^{\mathrm{T}} H_2(p)$$

where $G(p)$, $\Lambda(p)$ and $a/(p+a)$ are the transfer functions of the process, filter signal generator and low-pass output filter respectively. It is clear then that the condition of stability of $H_3(p)$ imposes a restriction of process stable invertibility.

Notice that since $H_1(p)$, $H_2(p)$ and $H_3(p)$ are stable operators, the effect of its initial conditions will decay exponentially and will not upset the stability.

In Section 2 we present a preliminary lemma, then in Section 3 our main result, namely a condition for global $\mathscr{L}_\infty$-stability of (4) is derived. In Section 4 the theorem's applicability is illustrated with an example of a model reference adaptive controller.

2. A preliminary lemma

The following preliminary lemma will be useful in the derivation of the main result.

Lemma. *Assume $H_2(p-\mu)$ is stable and let $h_2^\mu(t)$ denote its impulse response, that is*

$$\mathscr{L}\{h_2^\mu(t)\} = H_2(s-\mu)$$

and

$$\exists m, \lambda > 0: \quad |h_2^\mu(t)| \le m\, \mathrm{e}^{-\lambda t}. \tag{5}$$

Then, for all functions $\rho(t)$ satisfying (3a), (3b), *the $\mathscr{L}_\infty$-gain of the operator $\rho^{-1}(t)H_2(p)\rho(t)$ satisfies*

$$\gamma_\infty\{\rho^{-1}(t)H_2(p)\rho(t)\} \le m/\lambda. \tag{6}$$

Proof. Let $Y(t) = H_2(p)u(t)$. Then from (5),

$$\mathrm{e}^{\mu t}|y(t)| \le \int_0^t |h_2^\mu(t-\zeta)|\, \mathrm{e}^{\mu\zeta}|u(\zeta)|\, \mathrm{d}\zeta \le m\int_0^t \mathrm{e}^{-\lambda(t-\zeta)}\, \mathrm{e}^{\mu\zeta}|u(\zeta)|\, \mathrm{d}\zeta$$

which implies that

$$|\rho^{-1}(t)y(t)| \leqslant m\int_0^t \mathrm{e}^{-\lambda(t-\zeta)}\, \mathrm{e}^{\mu(\zeta-t)}\rho(t)^{-1}\rho(\zeta)\,|\rho(\zeta)^{-1}u(\zeta)|\, \mathrm{d}\zeta. \tag{7}$$

From (3b) we have

$$\mathrm{e}^{\mu\zeta}\rho(\zeta) \le \mathrm{e}^{\mu t}\rho(t), \quad \forall \zeta \le t,$$

which together with (7) gives

$$|\rho^{-1}(t)y(t)| \leqslant m\int_0^t e^{-\lambda(t-\zeta)}|\rho(\zeta)^{-1}u(\zeta)|\,d\zeta.$$

This completes the proof. □

3. Main result

We are in position to present our main result.

Theorem. *Consider the adaptive system* (1), (2), (3). *Assume* A.1 *holds*, $H_3(p)$ *is stable and* $\phi_0(t)$ *is bounded. Then there always exists* $\gamma/\sigma > 0$ *sufficiently small such that the overall adaptive system is globally* $\mathscr{L}_\infty$*-stable, that is*

$$e(t),\ \phi(t),\ \tilde{\theta}(t) \in \mathscr{L}_\infty.$$

Proof. The proof proceeds as follows. First we find, using (4a), (4b) and the lemma above, an upperbound for $\|\tilde{\theta}(t)\|_\infty$. Then replacing in (4c) we use a small-gain argument to show that $\phi(t)$ is bounded. Finally $e(t)$ is shown also to be bounded appealing to stability of $H_2(p)$ and (4b).

From (4a),

$$\|\tilde{\theta}(t)\|_\infty \leq \gamma_\infty\{H_1(p)\}\left\|\frac{\phi(t)e(t)}{\rho(t)^2}\right\|_\infty = \frac{\gamma}{\sigma}\left\|\frac{\phi(t)e(t)}{\rho(t)^2}\right\|_\infty \tag{8}$$

Using (4b) we can write

$$\frac{\phi(t)e(t)}{\rho(t)^2} = \frac{\phi(t)}{\rho(t)}\left\{\left[\frac{1}{\rho(t)}H_2(p)\rho(t)\right]\left[\tilde{\theta}(t)^{\mathrm{T}}\frac{\phi(t)}{\rho(t)}\right] + \frac{e_0(t)}{\rho(t)}\right\}.$$

Taking the $\mathscr{L}_\infty$-norm,

$$\left\|\frac{\phi(t)e(t)}{\rho(t)^2}\right\|_\infty \leq \frac{m}{\lambda}\|\tilde{\theta}(t)\|_\infty + \left\|\frac{e_0(t)}{\rho(t)}\right\|_\infty, \tag{9}$$

where we have used the previous lemma and (3c), i.e. $\|\phi(t)/\rho(t)\|_\infty \leq 1$.

Substituting (9) in (8) gives

$$\|\tilde{\theta}(t)\|_\infty \leq \frac{\gamma}{\sigma}\left\{\frac{m}{\lambda}\|\tilde{\theta}(t)\|_\infty + \left\|\frac{e_0(t)}{\rho(t)}\right\|_\infty\right\}.$$

Choosing γ/σ such that $(\gamma/\sigma)(m/\lambda) < 1$ we can write

$$\|\tilde{\theta}(t)\|_\infty \leq \frac{\gamma/\sigma}{1-\frac{\gamma}{\sigma}\frac{m}{\lambda}}\left\|\frac{e_0(t)}{\rho(t)}\right\|_\infty,$$

which proves that $\tilde{\theta}(t)$ is bounded.

On the other hand, from (3b),

$$\|\phi(t)\|_\infty \leq \gamma_\infty\{H_3(p)\}\,\|\tilde{\theta}(t)\|_\infty\,\|\phi(t)\|_\infty + \|\phi_0(t)\|_\infty$$

and consequently $\phi(t)$ is bounded if

$$\gamma_\infty\{H_3(p)\}\frac{\gamma}{\sigma}\left\|\frac{e_0(t)}{\rho(t)}\right\|_\infty \Bigg/ \left(1-\frac{\gamma}{\sigma}\frac{m}{\lambda}\right)<1,$$

which is verified for sufficiently small γ/σ.

The proof is completed with (4b), stability of $H_2(p)$ and boundedness of $\tilde{\theta}(t)$, $\phi(t)$ and $e_0(t)$. □

Remark 3. The stability analysis does not rely on any of the following assumptions on the process: knowledge of the high-frequency gain and relative degree, finite dimensionality or strict separation of time scales. Neither we assume almost periodic or persistently exciting inputs nor rely on the inclusion of probing signals, parameter projections or dead zones.

Remark 4. The stability theorem relates the following properties of the estimator: adaptation speed (γ), size of the residual set for the error (σ) and 'memory' of the normalizing factor (μ) with the relative stability of the linear system (m/λ) and the size of the linear system error (e_0).

4. An example: A model reference adaptive controler

An example is provided in this section to illustrate the application of the stability theorem.

Consider the process described by

$$y(t)=\frac{K_p}{(p+p_1)(p+p_2)}u(t)$$

with K_p, p_1 and p_2 unknown constants. The reference model is

$$y_m(t)=\frac{K_m}{p+p_m}r_f(t),\quad y_m(0)=r_f(0)=r(0),$$

$$r_f(t)=\frac{a}{p+a}r(t),\quad a\ll p_m.$$

The control is of the form

$$u(t)=\hat{\theta}_1(t)r_f(t)+\hat{\theta}_2(t)y_f(t)=\hat{\theta}(t)^T\phi(t),\qquad y_f(t)=\frac{a}{p+a}y(t),\quad y_f(0)=y(0).$$

Let the parameter update law be

$$\dot{\hat{\theta}}(t)=-\sigma\hat{\theta}(t)+\gamma\frac{\phi(t)e(t)}{\rho(t)^2}+\sigma\theta_0,\quad \hat{\theta}(0)=\theta_0,$$

$$\dot{\rho}(t)=-\mu\rho(t)+a[|y(t)|+|r(t)|]+1,\quad \rho(0)=|r(0)|+|y(0)|,\quad 0<\mu<a,$$

$$e(t)=y(t)-y_m(t),$$

where θ_0 is such that A.1 holds.

Notice that without the low-pass filters we can not ensure (3c) with $m_\rho=1$ for all values of $y(t)$ and $r(t)$. Take for instance a step change in both signals.

It can be easily shown that for this example

$$H_2(p)=\frac{K_p(p+a)}{(p+a)(p+p_1)(p+p_2)-\theta_{20}ak_p},\qquad e_0(t)=\left[\theta_{10}H_2(p)-\frac{K_m}{p+p_m}\right]r_f(t),$$

$$H_3(p)=[0,\ H_2(p)]^T,\qquad \phi_0(t)=[1,\ \theta_{10}H_2(p)]^T r_f(t).$$

The conditions for $\mathscr{L}_\infty$-stability may be therefore easily derived from an analysis of the transfer function $H_2(p)$.

5. Concluding remarks

Conditions have been presented for global $\mathscr{L}_\infty$-stability of an adaptive tuning mechanism intended to improve performance of a plant stabilizable with a fixed known controller.

The estimator (4a) contains a leakage and signal normalization. The input–output formalism is used to derive the results. The allows us to treat the problem with wide generality and still provide interpretative stability conditions. It is the authors' belief that the results may be extended to treat discrete-time, multivariable or time-varying systems. Current research is under way on that direction.

The modifications to the estimator are essential for the analysis. Namely, the normalization is used to insure boundedness of $\phi(t)/\rho(t)$ and $\gamma_\infty\{\rho^{-1}(t)H_2(p)\rho(t)\}$. Also the leakage is required to have a finite $\mathscr{L}_\infty$-gain of $H_1(p)$.

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