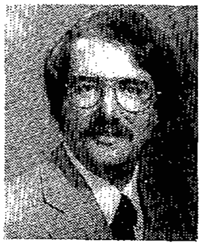

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# Robustness of Discrete-Time Direct Adaptive Controllers

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*Abstract*—**The problem of preserving stability of discrete-time adaptive controllers in spite of reduced-order modeling and output disturbances is addressed in this paper. Conditions for global stability (convergence of the tracking error with bounded signals) are derived for a discrete-time pole-zero placement adaptive controller where the parameter estimator is modified in terms of normalized signals. Following an input–output perpective, the overall system is decomposed into two subsystems reflecting the parameter estimation and modeling errors, respectively, and its stability is studied using the sector stability and passivity theorems. First the analysis is carried for the class of disturbances and reference inputs that are either decaying or can be exactly nulled by a linear controller of the chosen structure. In this $\mathcal{L}_2$-framework, it is shown that the only substantive assumption to assure stability is the existence of a linear controller such that the closed-loop transfer function verifies certain conicity conditions. The convergence speed and alertness properties of various parameter adaptation algorithms regarding this condition are discussed. The results are further extended to a broader class of $\mathcal{L}_\infty$ disturbances and reference inputs.**

## I. Introduction

THE fundamental practical issue which motivates the entire body of feedback design is how to achieve desired levels of performance in the face of plant uncertainties. Two aspects of the problem must be distinguished: choosing a mathematically convenient representation of the modeling error [generically referred to as model-process mismatch (MPM)] and capturing both the uncertainty and performance aspects in a single problem statement. These constitute the essential difficulty of a successful design technique.

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In a very general way, we can distinguish three specific classes of MPM leading to different mathematical problems. Optimal control of *stochastic models* when disturbances arise from small independent linearly combined fluctuations. Adaptive control, where MPM is represented in terms of a set membership statement *for the parameters* of a suitably choosen structure, e.g., an otherwise known linear time-invariant (LTI) system. Robust control theory which characterizes uncertainty by a set membership statement *for the input–output (I/O) operator*, e.g., the process transfer function.

Intense research activity has been devoted to the control of stochastic models with parametric uncertainty. Single-stage optimization schemes for scalar LTI invertible systems have been shown to be globally stable under fairly reasonable assumptions provided the system noise dynamics verifies a positivity condition and the underlying model structure has been suitably chosen. Equivalence of single-stage optimal stochastic and pole-zero placement deterministic adaptive controllers is now well established; see, e.g., [11]. It has been shown in [25] that bounded output disturbances (BOD), and more recently in [4], [21], that reduced-order modeling (ROM) could make the closed-loop adaptive system unstable. Since such violations are the rule and not the exception in practice, these results raised the interest of studying the controllers ability to retain adequate performance when faced with *other classes of MPM besides parametric uncertainty*. We will refer to this case as the mismatched case in contrast to the *matched case* where no disturbances are present and an upper bound on the process order is known.

Since in the mismatched case it is no longer possible to ensure convergence to zero of the tracking error for all BOD and reference sequences, a revised notion of *acceptable performance* is required. Three fundamental, if modest, requirements are the following. 1) Assure tracking error cancelation with bounded signals for all BOD and reference sequences for which a linear robust servobehavior is possible, i.e., the tracking error can be exactly nulled by a linear controller of the same structure. 2) When perfect tracking error cancelation is not possible, preserve its boundedness for "sufficiently small" BOD. 3) Since the key property of an adaptive regulator is to track variations in process

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dynamics, gain decreasing estimation schemes should be discarded. Convergence to a constant value of the estimated parameters and the capability of reflecting the MPM level in the stability conditions are further desirable properties.

### *A. Background*

Robustness results of adaptive controllers were first available for the output disturbance problem [25], [18], [19]. Fairly complete results in a state-space setting were obtained in [21] for the case when the reduced-order model residuals are a parasitic system. Ad hoc modifications to the adaptation laws were presented in [18], [19], [21]. Although in the latter the MPM is characterized by a well-defined scalar parameter (the ratio of the dominant versus parasitic frequencies) in none of the aforementioned schemes is it straightforward to establish the validity of the prior information required nor to incorporate *a priori* knowledge about the process. Any attempt to treat "less structured" uncertainties from a *state-space approach* seems doomed from the outset not to yield useful results.

In contrast to adaptive control theory, research in robust control [1], [2], [17] has preceded from an *operator model formulation.* This allows natural accommodation of uncertain model order and provides an adequate framework to incorporate *a priori* knowledge to quantify the MPM. Conic bounded transfer functions to deal with coarsely defined systems are used to characterize uncertainty. In this approach the input–output map is assumed to be in a ball in the frequency domain, whose center is the plant parametric model and the radius defines, by a known frequency function, the error induced by the unstructured uncertainty.

The key to the successful application of the powerful I/O stability theorems [9] in an adaptive context is to find, as was done for the nominal stability analysis of model-reference adaptive controllers [6], a suitable operator-theoretic description of the systems isolating the parametric error. To treat robustness problems, the effects of the *modeling and parameter estimation error must be effectively isolated.* This was first clearly stated in [10] for a class of continuous-time adaptive controllers leading to stability conditions given in terms of passivity requirements of an MPM-related operator. Stabilizability of the process by a fixed gain regulator (with the same structure as the adaptive one), which is an obvious requirement, is used in [10] to ensure boundedness of the regressor vector. The first discrete-time robustness results using an I/O approach were reported in [8]. There, a small gain formulation is proposed to study the robustness of the self-tuning controller. Unfortunately, the results are incomplete, since besides the small gain requirement an intricately signal-dependent assumption has to be made, specifically, it was assumed that the regressor signals are *a priori* known to be bounded. The same flaw is present in [5], [15] where sectoricity theory was proposed for robustness analysis. The $\mathcal{L}_2$ results of [8] have been translated to an $\mathcal{L}_\infty$ framework in [23]; however, the signal-dependent assumption remained unsolved.

Departing from the operator-theoretic approach, a signal-to-noise ratio formulation of the robustness problem was introduced in [28]. It allows one to derive results for both ROM and BOD [24] using a modified version of the adaptation law introduced in [25]. The results obtained are however more of a qualitative rather than quantitative nature.

Some local stability conditions have been reported in [22]. This type of approach, which may lead to more practical results, complements the global one where the goal is to define the limits of the adaptive schemes in its widest possible formulation.

### *B. Contributions of the Paper*

The purpose of our robustness studies is to determine a class of modeling errors (besides parameter uncertainty) for which the adaptive scheme retains acceptable performance (as defined above).

The *framework* proposed in this paper, largely inspired by [10], is of the system theoretic type and is based on conic sectors. Our main technical device is the sector stability theorem [2], [17] which states that the feedback interconnection of two conic bounded operators is globally stable if one is strictly inside a cone and the inverse of the other one outside it. This theorem is applied to the error model derived in [5] which is similar to the ones in [8], [10]. The operator representing the parameter adaptation algorithm (PAA) is in feedback interconnection with an LTI operator. The latter operator is the transfer function from the delayed reference sequence to the system output.

In order to apply the conic sector theory, conic sector conditions must be established for the PAA. In [7], [14] these tools were applied to analyze the stability of the self-tuning controller. The conic sectors derived in those papers are critically dependent on the $\mathcal{L}_\infty$ norm of the regressor vector. The assumption of a bounded regressor vector leaves the results incomplete. To remove this defect we use, as in [25], normalized signals in the PAA and following the approach of [24], we modify the least squares algorithm by regularizing the covariance matrix. In this way, signal-independent conic sectors are established for constant gain (CG) and regularized least squares (RLS) estimation schemes. It is worth mentioning that the regularization in the least squares algorithm is required only for the $\mathcal{L}_\infty$-stability analysis. For the $\mathcal{L}_2$-stability analysis of the weighted least squares PAA, see [31].

$\mathcal{L}_2$-stability, that is tracking error cancellation, may be ensured for reference inputs and disturbances that are either $\mathcal{L}_2$ signals or such that linear robust servobehavior is possible. To treat the more realistic situation of arbitrary reference inputs and BOD, an $\mathcal{L}_\infty$ formulation is required. Analogously to [23], we use exponentially weighted techniques [9] to extend the $\mathcal{L}_2$ result to a $\mathcal{L}_\infty$ framework. In both cases a tradeoff between altertness of the PAA and robustness arises.

Direct application of the sector stability theorem to the normalized error model allows us to derive conditions for the stability of the normalized signals. To be able to conclude stability of the adaptive scheme from stability of the normalized error model, two additional results are needed. First, the conditions ensuring stability of the normalized scheme, which are given in terms of normalized operators, must be translated to the original operators. Second, conditions under which stability of the normalized scheme implies stability of the original one must be established. This is done by referring to multiplier theory [9, p. 202]. The problem basically reduces to proving that the regressor vector is bounded, which ensures that the normalization factor qualifies as a multiplier. Arguments similar to the ones in [24] are used for this part of the proof.

The *main contributions of the paper* are the following. 1) An extension of the I/O approach pioneered in [7], [8], [10] for analyzing the effects of ROM and BOD in discrete-time adaptive controllers. 2) Establishment of a well-defined class of ROM errors and BOD for which robust stability is ensured. 3) Use of a normalized approach to parameter estimation for improved robustness. The latter completes the results of [5], [8], [23].

The paper is organized as follows. The type of MPM and the regulator structure studied are presented in Section II together with the error equations. The implications of the presence of MPM in the PAA selection and the I/O properties of a class of PAA's are discussed in Section III. In Section IV the need to normalize the PAA signals is motivated. The main stability theorems are given in Section V. Some concluding remarks are presented in Section VI.

## II. Problem Formulation

In order to carry out the objective presented in Section I-B we must isolate the effects of the modeling and parameter estimation errors. This is done by reconfiguring the adaptive system into two subsystems: the parameter adaptation algorithm (PAA) and an LTI subsystem independent of the parametric error.

In this section we will first define the MPM representation considered in the paper. A standard pole-zero placement adaptive controller is introduced later. Before proceeding to describe the PAA, which is left to Section III, error equations suitable to the robust stability analysis are then established. Assuming linear stabilizability of the process, the stability problem of the adaptive case is reduced to the analysis of a feedback arrangement around the PAA; this arrangement is suitable to the application of I/O stability theorems [2], [9], [17].

### A. The Plant

It is assumed that the plant to be controlled is described by

$$A(q^{-1})Y_t = q^{-d}B(q^{-1})U_t + \xi_t \tag{2.1}$$

where $A$, $B$ are polynomials in $q^{-1}$. $A$ is monic, $U_t$, $Y_t$, $\xi_t$ are the input, output, and disturbance sequences, and $d$ is known. The order of each polynomial and its coefficients are unknown and $\xi_t$ is bounded, i.e., $\xi_t \in \mathcal{L}_\infty$.

### B. The Controller Structure

We will pursue a pole-placement all-zero canceling objective with the desired closed-loop poles being the roots of a polynomial $C_R$. Defining a filtered tracking error

$$e_t \triangleq C_R Y_t - \omega_t \tag{2.2}$$

our objective is to ensure that $e_t$ tends to 0 as $t$ tends to infinity.

Choosing two integers $n_S$ and $n_R$ we use the regulator structure

$$\hat{S}_t U_t = \omega_{t+d} - \hat{R}_t Y_t$$

where $\hat{S}_t$ and $\hat{R}_t$ are polynomial functions in $q^{-1}$ of degrees $n_S$ and $n_R$, respectively, with time-varying coefficients and $\omega_t$ is the reference signal assumed known $d$ steps ahead. In compact notation the control law may be written as

$$\omega_{t+d} = \hat{\theta}_t^T \phi_t \tag{2.3}$$

with

$$\phi_t \triangleq [U_t,\ U_{t-1},\ \cdots,\ U_{t-n_S};\ Y_t,\ Y_{t-1},\ \cdots,\ Y_{t-n_R}]^T. \tag{2.4}$$

Before proceeding with the process reparameterization, let us introduce the following stabilizability assumption that will justify the choice of the regulator given above.

*Assumption A.1:* Let $S_*$, $R_*$ be polynomials of given orders $n_S$, $n_R$. Let $\mu \in (0, 1)$ be a scalar. Define the polynomial coefficients vector

$$\theta_* \triangleq [S_0^*,\ S_1^*,\ \cdots,\ S_{n_S}^*,\ r_0^*,\ r_1^*,\ \cdots,\ r_{n_R}^*]^T$$

and the polynomial

$$C \triangleq S_* A + q^{-d} R_* B. \tag{2.5}$$

With these notations, we assume that there exists a *nonempty* set $\theta_{LS}$ defined as

$$\Theta_{LS} \triangleq \{\Theta_* \in \mathbb{R}^n : C(q) \neq 0,\ \forall\ q \in C,\ |q| > \mu^{1/2}\} \neq \emptyset$$

where $n \triangleq n_S + n_R + 2$. □

*Remark 2.1:* The set $\theta_{LS}$ defines the fixed gain regulators which ensure that the systems closed-loop poles are within a disk of radius $\mu^{1/2}$, where $\mu$ is a designer chosen parameter to be defined later. The elements of this set, which we will call the linear stabilizing set, the corresponding polynomials and associated signals will be denoted with an asterisk. Notice that for $\mu = 1$ this assumption simply states that the system may be stabilized by a linear regulator of the chosen structure. If $\Theta_{LS}$ is empty the plant cannot be stabilized even when it is perfectly known.

### C. Error Equations

Combining (2.5) with (2.1) and using (2.4)

$$CY_t = B\theta_*^T \phi_{t-d} + S_* \xi_t \tag{2.6a}$$

$$CU_t = A\theta_*^T \phi_t - R_* \xi_t. \tag{2.6b}$$

Define

$$\psi_t \triangleq (\hat{\theta}_{t-d} - \theta_*)^T \phi_{t-d} \triangleq \tilde{\theta}_{t-d}^T \phi_{t-d} \tag{2.7}$$

where $\tilde{\theta}_t$ is the difference between the actual parameters [see (2.3)] and a vector of stabilizing parameters. From (2.2), (2.3), (2.6), and (2.7) we see that the error model may be expressed as

$$e_t = -H_2 \psi_t + e_t^* \tag{2.8}$$

where

$$e_t^* \triangleq (H_2 - 1)\omega_t + C_R C^{-1} S_* \xi_t \tag{2.9a}$$

$$H_2 \triangleq C_R C^{-1} B. \tag{2.9b}$$

The regressor vector can analogously be written as

$$\phi_{t-d} = -W_1 \psi_t + \phi_{t-d}^* \tag{2.10}$$

where

$$\phi_{t-d}^* \triangleq W_1 \omega_t + W_2 \xi_t \tag{2.11a}$$

$$W_1 \triangleq C^{-1}[A,\ q^{-1}A,\ \cdots,\ q^{-n_R}A;\ q^{-d}B,\ q^{-d-1}B,\ \cdots,\ a^{-d-n_S}B]^T \tag{2.11b}$$

$$W_2 \triangleq C^{-1}[-q^{-d}R_*,\ -q^{-d-1}R_*,\ \cdots,\ -q^{-d-n_R}R_*;\ q^{-d}S_*,\ q^{-d-1}S_*,\ \cdots,\ q^{-d-n_S}S_*]. \tag{2.11c}$$

*Remark 2.2:* Notice that in the matched case there exists $S_*$ and $R_*$ such that $C_* \equiv C_R B$, see (2.5), so that $H_2 = 1$. Furthermore, since $\xi_t = 0$, then $e_t^* = 0$. It is reasonable to expect that the stability conditions in the mismatched case will require "$H_2$ close to 1" and "small" $e_t^*$. Our problem is to formalize these notions and to provide conditions to ensure its verification.

In Fig. 1 the complete error model is depicted. $H_1$ denotes a relation defined by the PAA. One important difference arises with respect to the continuous-time error model developed in [10], namely that defining $\psi_t$ in terms of the delayed signals [see (2.7)], allows us to obtain a transfer function $H_2$ of relative degree zero, i.e., proper. This will prove to be of fundamental importance in the analysis of the stability conditions implications.

*Remark 2.3:* It is easy to show that $H_2 = C_R Y_t^* / \omega_{t+d}$; that is, $H_2$ represents the transfer function of the process in closed-loop with a stabilizing regulator. $e_t^*$ and $\phi_t^*$ are the corresponding tracking error and regressor signals for that linear scheme. Notice that they can be interpreted as inputs to the error model [10] which are bounded in view of Assumption A.1. Henceforth, the establishment of tracking error convergence conditions for the overall system reduces to ensuring stability for the feedback interconnection of the blocks $H_1$, $H_2$. Boundedness of $\phi_t$ will follow if the former conditions are $\phi_t$-independent.

## III. The Parameter Adaptation Algorithms

We intend to obtain stability conditions in terms of conic bounds in the presence of MPM. In addition, we will attempt to satisfy performance requirements. Our key technical device to

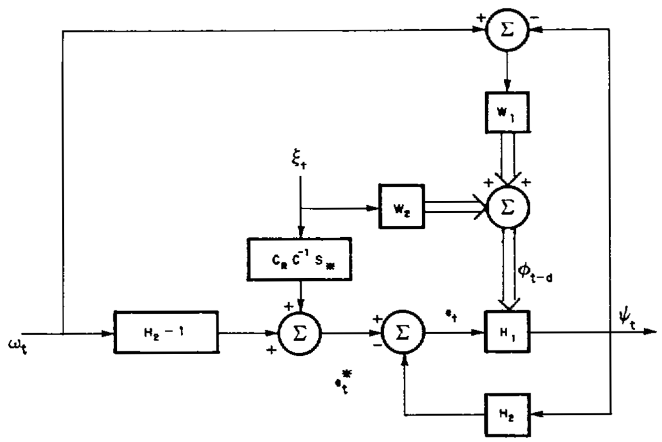


Fig. 1.

study the feedback interconnection is the conic sector stability theorem [17] (see also [2]). It is required then to choose a PAA such that sector conditions may be established for the relation $H_1: e_t \rightarrow \psi_t$.

It will be shown below that to obtain $\phi_t$-independent properties for the PAA (see Remark 2.3) normalization of $e_t$ and $\phi_t$ are compulsory. In the following $(\bar{\cdot})$ *will be used to denote normalized* variables and corresponding operators and are defined as:

$$\bar{\phi}_{t-d} \triangleq \rho_t^{-1/2}\phi_{t-d},\ \bar{e}_t \triangleq \rho_t^{-1/2}e_t;\ \bar{\psi}_t \triangleq \rho_t^{-1/2}\psi_t \quad (3.0a)$$

$$\bar{H}_i \triangleq \rho_t^{-1/2}H_i[\rho_t^{1/2}\ \cdot];\qquad i=1,\ 2. \quad (3.0b)$$

The normalization factor $\rho_t$ is introduced in Section V.

To gain some insight into the problem of the selection of the PAA we will consider first the approaches and motivations of the matched case, that is when no ROM or BOD are present. A class of PAA for which suitable I/O properties have been established is later presented and its properties stated and proved.

### A. The Matched Case

Most adaptive schemes reported in the literature use an integral PAA of the form

$$\hat{\theta}_t = \hat{\theta}_{t-1} + F_t\phi_{t-d}e_t^p \quad (3.1)$$

where $F_t$ is a time-varying matrix (the matrix gain) and $e_t^p$ is an estimate of the prediction error. The increasing complexity of the treated cases required increasing information fed through $e_t^p$ into the PAA. Therefore, the choice of $e_t^p$ may be thought of as reflecting the evolution of the adaptive control theory. It was initially taken equal to the tracking error to solve the unitary delay case. Later it was shown that using this same error, a physically realizable globally stable solution was still possible for $d = 2$, by proper replacement of $\hat{\theta}_t$ by the multiplier operator $P_L(\hat{\theta}_t)$.[1]. This last modification was required to ensure the positive real condition of the error model. The ingenious inclusion of the augmented error model allowed proof of convergence of the tracking error by taking

$$e_t^p = (C_R Y_t - \hat{\theta}_{t-1}^T\phi_{t-d})/(1+\phi_{t-d}^T F_t\phi_{t-d}). \quad (3.2)$$

However, this new form of $e_t^p$ posed the new stability problem of *ensuring boundedness* of the auxiliary signal, which was later proved for invertible systems [13] by showing that $\Delta\hat{\theta}_t \in \mathcal{L}_2$. Similar results were obtained in [25], [26].

The introduction of the *a posteriori* error representation [6], [11] allows a clear-cut interpretation of the stability proofs, either Lyapunov or Popov based, available in the literature. Due to the structure of the integral PAA it is easy to show that in the matched case $e_t^p$ as given in (3.2) is equal to $-\tilde{\theta}_t^T\phi_{t-d}$, the *a posteriori* error. Since the operator $H_1: e_t^p \rightarrow \tilde{\theta}_t^T\phi_{t-d}$ is passive (for a constant gain matrix), even for unbounded $\phi_t$, direct application of the passivity theorem leads to the stability of $\tilde{\theta}^T\phi_{t-d}$. The proof is completed by showing that $\tilde{\theta}_t^T\phi_{t-d} \rightarrow 0$ implies $e_t \rightarrow 0$ with bounded $\phi_t$. A similar procedure will be required below when we will seek to prove stability of the adaptive scheme from the stability of the normalized signals.

*Remark 3.1:* It can also be shown that when $d > 1$ an interlaced version of (3.1) avoids the necessity of using the augmented error in (3.2) since for that scheme

$$\frac{e_t}{1+\phi_{t-d}^T F_t\phi_{t-d}} = -\tilde{\theta}_t^T\phi_{t-d}.$$

### B. PAA Sector Conditions

Given our objective of uniform asymptotic stability we disregard proportional components in the PAA. In addition, gain decreasing PAA are discarded to *preserve the alertness* of the adaptive scheme. Extrapolating from current usage we consider integral interlaced PAA of the form

$$\tilde{\theta}_t = \tilde{\theta}_{t-d} + \mathfrak{F}\bar{\phi}_{t-d}\bar{e}_t \quad (3.3)$$

where $\mathfrak{F}$ takes one of the following forms.

1) Constant gain (CG) PAA: $\mathfrak{F}$ is a scalar

$$\mathfrak{F} \triangleq f > 0. \quad (3.4a)$$

2) Regularized least squares (RLS) PAA: $\mathfrak{F}$ is a time-varying matrix

$$\mathfrak{F} \triangleq F_t \quad (3.4b)$$

where (see [24] for further details)

$$F_t = \left(1-\frac{\lambda_0}{\lambda_1}\right)\left[F_{t-d} - \frac{F_{t-d}\bar{\phi}_{t-d}\bar{\phi}_{t-d}^T F_{t-d}}{\lambda + \bar{\phi}_{t-d}^T F_{t-d}\bar{\phi}_{t-d}}\right] + \lambda_0 I \quad (3.4c)$$

and $\lambda_0 < \lambda_1$, $\lambda$ are strictly positive scalars.

The eigenvalues of $F_t$ are all contained in the chosen interval $[\lambda_0, \lambda_1]$.

Equations (3.3) and (3.4) define an operator $\bar{H}_1: \bar{e}_t \rightarrow \bar{\psi}_t$ (see Fig. 2). Besides this operator we will consider for the RLS/PAA, its exponentially weighted counterpart $\bar{H}_1^\alpha: \bar{e}_t^\alpha \rightarrow \bar{\psi}_t^\alpha$ where the superscript $\alpha$ denotes

$$X_t^\alpha \triangleq \alpha^t X_t:\qquad \alpha > 0.$$

The I/O properties of the two operators are summarized in the following lemma. Similar results were obtained earlier in [7], [14], [15], [24]. Notice that $\bar{H}_1^\alpha = \bar{H}_1$ when $\alpha = 1$.

*Lemma 3.1 (I/O Properties of the PAA):*

1) *CG/PAA:* If $\mathfrak{F}$ is given by (3.4a), then

$$\bar{H}_1 + \frac{1}{2}\bar{\sigma}_{CG}\ \text{is passive}$$

for all $\bar{\sigma}_{CG}$ such that

$$\bar{\sigma}_{CG} \geq f\|\bar{\phi}_t^T\bar{\phi}_t\|_\infty. \quad (3.5)$$

2) *RLS/PAA:* If $\mathfrak{F}$ is given by (3.4b), (3.4c), then

$$\bar{H}_1^\alpha\ \text{is outside CONE}\ (-1,\ \sqrt{1-\bar{\sigma}_{RLS}})$$

[1] This section's discussion, although restricted to discrete-time systems, is further simplified by choosing the following structure for the operator: $P_L: P_L(\hat{\theta}_t) \triangleq q^d\hat{\theta}_t q^{-d}$ (see [13]) so that the operator retains the basic concepts of continuous and hybrid schemes.

for $\alpha$ verifying

$$\lambda \max \left[ F_t^{-1}\left( F_{t-d} - \frac{F_{t-d}\bar{\phi}_{t-d}\bar{\phi}_{t-d}^T F_{t-d}}{\lambda + \bar{\phi}_{t-d}^T F_{t-d}\bar{\phi}_{t-d}} \right) \right] \cdot \alpha^{2d} \leq 1 \tag{3.6}$$

and all $\bar{\sigma}_{\mathrm{RLS}}$ satisfying

$$\bar{\sigma}_{\mathrm{RLS}} \geq \frac{\lambda_1 \bar{\phi}_t^T \bar{\phi}_t}{\lambda + \lambda_1 \bar{\phi}_t^T \bar{\phi}_t}. \tag{3.7}$$

*Proof:* The proof is given in two parts. The passivity property for the CG/PAA is first established. The conic sector for the RLS/PAA is later derived.

1) Consider the quadratic function

$$V_t \triangleq \frac{1}{2}\tilde{\theta}_t^T f^{-1}\tilde{\theta}_t$$

direct manipulation of (3.3) and (3.4a) gives

$$V_t - V_{t-d} = \bar{\psi}_t \bar{e}_t + \frac{1}{2}\bar{\phi}_{t-d}^T f \bar{\phi}_{t-d}(\bar{e}_t)^2.$$

It can be readily seen that

$$\left\langle \frac{1}{2}\bar{\sigma}_{\mathrm{CG}}\bar{e}_t + \bar{\psi}_t | \bar{e}_t \right\rangle_N \geq -V_{-d} - \cdots - V_{-1}$$

which completes the first part of the proof.

2) Let the matrix $F_t'$ and the scalars $V_t$, $V_t'$ be defined as

$$F_t' \triangleq F_{t-d} - \frac{F_{t-d}\bar{\phi}_{t-d}\bar{\phi}_{t-d}^T F_{t-d}}{\lambda + \bar{\phi}_{t-d}^T F_{t-d}\bar{\phi}_{t-d}} \tag{3.8}$$

$$V_t \triangleq \frac{\tilde{\theta}_t^T F_t^{-1}\tilde{\theta}_t}{\lambda}, \quad V_t' \triangleq \frac{\tilde{\theta}_t^T F_t'^{-1}\tilde{\theta}_t}{\lambda}.$$

We have (see the Appendix)

$$V_t \leq \lambda \max (F_t^{-1}F_t') \cdot V_t'$$

and after some algebra (see [30] for example).

$$V_t' - V_{t-d} = (\bar{\psi}_t + \bar{e}_t)^2 - \frac{\lambda}{\lambda + \bar{\phi}_{t-d}^T F_{t-d}\bar{\phi}_{t-d}}\bar{e}_t^2.$$

Now from (3.4c), (3.6) it follows that:

$$\alpha^{2d}\lambda \max (F_t^{-1}F_t') \leq 1, \quad \bar{\phi}_{t-d}^T F_{t-d}\bar{\phi}_{t-d} \leq \lambda_1 \bar{\phi}_{t-d}^T \bar{\phi}_{t-d}.$$

Hence,

$$\alpha^{2d}V_t \leq \alpha^{2(t-d)}V_{t-d} + \alpha^{-2d}\left[ (\bar{\psi}_t^\alpha + \bar{e}_t^\alpha)^2 - \frac{\lambda}{\lambda + \bar{\phi}_{t-d}^T F_{t-d}\bar{\phi}_{t-d}}\bar{e}_t^{\alpha^2} \right].$$

Summing from 0 to $N$ leads to the result

$$\sum_{t=0}^{N} (\bar{\psi}_t^\alpha + \bar{e}_t^\alpha)^2 \geq \sum_{t=0}^{N} \frac{\lambda}{\lambda + \bar{\phi}_{t-d}^T F_{t-d}\bar{\phi}_{t-d}}\bar{e}_t^{\alpha^2} - \sum_{t=-1}^{-d} \alpha^{2d}V_t. \quad \square$$

*Remark 3.2:* From (3.5), (3.7) we see that the PAA's properties are critically dependent on the boundedness of $\bar{\phi}_t$. This indicates that the normalization factor $\rho_t$ in (3.0) should ensure a finite $\mathcal{L}_\infty$-norm for $\bar{\phi}_t$. We will assume from now on that $\rho_t$ is such that

$$\|\bar{\phi}_t\|_\infty \leq 1. \tag{3.9}$$

A sequence $\rho_t$ giving this property will be presented in Section V. With (3.9), the radius of the cone for the RLS/PAA does not vanish. It is exactly at this point that our result differs from [5], [8], [15], [23].

*Remark 3.3:* Another interesting property for our study would be to have $\alpha > 1$ in (3.6). Clearly from (3.4c) we have

$$F_t \geq F_t'.$$

Therefore, in any case

$$\alpha \geq 1. \tag{3.10}$$

In some circumstances, the stronger property "$\alpha > 1$" is also satisfied. In the Appendix we show that, in the case $d = 1$, this is achieved at least for $\bar{\phi}_t$ persistently spanning in the following sense: there exist $0 < \beta < 1$, $\epsilon > 0$, $N_o$ such that:

$$\sum_{t=0}^{N} \beta^{N-t}\bar{\phi}_t\bar{\phi}_t^T \geq \epsilon I \qquad \forall N \geq N_0. \tag{3.11}$$

Unfortunately this is a signal-dependent condition. However, it is usually satisfied for $\lambda$ large enough (slow adaptation) and for all period of time such that $\hat{\theta}_t \in \Theta_{LS}$ provided the reference input is persistently exciting.

## IV. Stability of the Normalized Error Model

$\mathcal{L}_2$ and $\mathcal{L}_\infty$-stability results for the normalized system are given below. Discussion on the stability conditions is deferred to the following section, where stability of the adaptively controlled system is derived from the stability of the normalized error model.

### A. $\mathcal{L}_2$-Stability

Combining Lemma 3.1 and the sector stability theorem we get the following $\mathcal{L}_2$ result for the normalized system.

*Lemma 4.1:* Consider the feedback interconnection

$$\bar{\psi}_t = \bar{H}_1\bar{e}_t \tag{4.1a}$$

$$\bar{e}_t = -\bar{H}_2\bar{\psi}_t + \bar{e}_t^*. \tag{4.1b}$$

If $\bar{H}_2$ is strictly inside $\mathcal{C} \triangleq \mathrm{CONE}\ (C_A, R_A)$, where

$$(C_A, R_A) = \begin{cases} (1/\bar{\sigma}_{\mathrm{CG}}, 1/\bar{\sigma}_{\mathrm{CG}}) & \text{for the CG/PAA} \quad (4.2a) \\ (1/\bar{\sigma}_{\mathrm{RLS}}, \sqrt{1-\bar{\sigma}_{\mathrm{RLS}}}/\bar{\sigma}_{\mathrm{RLS}}) & \text{for the RLS/PAA} \quad (4.2b) \end{cases}$$

for any

$$\bar{\sigma}_{\mathrm{CG}} \geq f \text{ and } \bar{\sigma}_{\mathrm{RLS}} \geq \frac{\lambda_1}{\lambda + \lambda_1} \tag{4.3}$$

then

$$\bar{e}_t, \bar{\psi}_t \in \mathcal{L}_2 \qquad \text{for all } \bar{e}_t^* \in \mathcal{L}_2.$$

*Proof:* This is a straightforward application of [17, Theorem 2a p. 234]. $\square$

### B. $\mathcal{L}_\infty$-Stability

The $\mathcal{L}_\infty$ extension of the previous result using the RLS/PAA follows below.

*Lemma 4.2:* Consider the feedback system (4.1) for the RLS/PAA. Assume $\rho_t$ is bounded away from zero. Under these conditions, if

$\bar{H}_2^\alpha \triangleq \alpha^t\bar{H}_2[\alpha^{-t}] \cdot$ is strictly inside $\mathcal{C}$ [with $\mathcal{C}$ as in (4.2b)]

with $\alpha > 1$ satisfying (3.b), then there exists a scalar $K_2$ such that

$$\bar{\psi}_N^2 \leq \frac{K_2}{\min_t \rho_t}\frac{\|e_t^{*^2}\|_\infty}{(1-\bar{\alpha}^2)}.$$

*Proof:* $\mathcal{L}_2$-stability of the map $(\bar{e}_t^*)^\alpha \rightarrow \bar{\psi}_t^\alpha$ (see Fig. 2) is ensured from Lemma 3.1 and the sector stability theorem. That is, $\exists\, K_2 < \infty$ such that

$$\|\bar{\psi}_t^\alpha\|_N^2 \leq K_2 \|(\bar{e}_t^*)^\alpha\|_N^2, \qquad \forall\, N \geq 0. \tag{4.4}$$

Notice that

$$\|\bar{\psi}_t^\alpha\|_N^2 \geq (\alpha^N \bar{\psi}_N)^2 \tag{4.5}$$

and

$$\|(\bar{e}_t^*)^\alpha\|_N^2 \leq \|\bar{e}_t^{*2}\|_\infty \sum_{t=0}^{N} \alpha^{2t} \leq \|\bar{e}_t^{*2}\|_\infty \frac{\alpha^{2N}}{1-\alpha^{-2}} \tag{4.6}$$

since $\alpha > 1$. Combining (4.4)–(4.6) we can conclude that uniformly in $N$

$$\bar{\psi}_N^2 \leq \frac{K_2}{\boldsymbol{\rho}(1-\alpha^{-2})} \|e_t^{*2}\|_\infty \tag{4.7}$$

where

$$\boldsymbol{\rho} \triangleq \min_t \rho_t > 0. \qquad \square \tag{4.8}$$

*Remark 4.1:* The same types of arguments were used in [23] to prove the boundedness of $e_t$ assuming *a priori* constraints in the regressor vector.

## VI. Main Results

In this section we will determine the conditions under which stability is preserved for the plant (2.1) in closed loop with the time-varying regulator (2.3) and adaptive law (3.3), (3.4). For this purpose we will introduce the following normalization factor:

$$\rho_t = \mu\rho_{t-1} + \max\,(|\phi_{t-d}|^2,\ \boldsymbol{\rho}), \qquad \boldsymbol{\rho} > 0,\ \mu \in (0,\ 1) \tag{5.1}$$

which together with (3.0) completes the description of the PAA.

*Remark 5.1:* This type of multiplier was introduced in [25], and its importance for robustness established in [24], [30]. $\boldsymbol{\rho}$ is a small positive constant that defines a lower bound to $\rho_t$. The choice of the time constant $\mu$ will prove to be a compromise between PAA alertness and robustness.

The problem is solved by analyzing the error models depicted in Figs. 1 and 2. It should be recalled (see Remark 2.3) that under the stabilizability Assumption A.1 the key point is proving stability of $\psi_t$ [see (2.8), (2.10)]. The proof proceeds as follows. First we prove using the Bellman–Gronwall lemma that $\mathcal{L}_2$-stability of $\bar{\psi}_t$ (given by Lemma 4.1) implies $\psi_t \in \mathcal{L}_\infty$. This in its turn assures that the regressor vector is bounded. As a consequence, the normalizing factor $\rho_t$ is bounded and proceeding from the multiplier theory $\mathcal{L}_2$-stability of the normalized error model implies $\mathcal{L}_2$-stability of the adaptive system. For the $\mathcal{L}_\infty$-stability proof, boundedness of $\bar{\psi}_t$, as shown in Lemma 4.2, is used to establish boundedness of $\psi_t$.

The stability conditions derived in Lemma 4.1 and 4.2 are translated in terms of the designer chosen parameters ($n_S$, $n_R$, $C_R$, $\mu$) and the MPM ($H_2$, $\xi_t$).

### A. $\mathcal{L}_2$-Stability

*Theorem 5.1:* Consider $\bar{\psi}_t$ given in (3.0), (5.1) and $\phi_t$ as in (2.10), (2.11). Under these conditions if Assumption A.1 of Section II-B is verified, then

$$\bar{\psi}_t \in \mathcal{L}_2 \Rightarrow \psi_t \in \mathcal{L}_\infty.$$

*Proof:* Define the exponentially weighted signals [9, p. 251]

$$X_t^\mu \triangleq \mu^{-t} X_t. \tag{5.2}$$

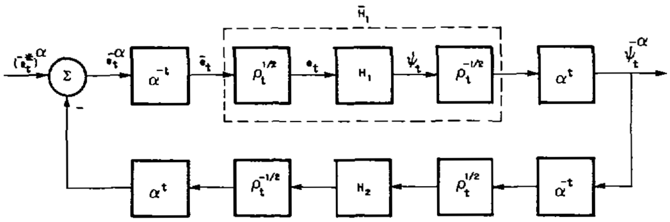


Fig. 2.

From (5.1)

$$\mu^{-N}\rho_N \leq \rho_0 + \|\phi_{t-d}^\mu\|_N^2 + \mu^{-N}\boldsymbol{\rho}/(1-\mu). \tag{5.3}$$

Applying the truncated $\mathcal{L}_2$ norm to the exponentially weighted version of (2.10) and taking into account A.1

$$\|\phi_{t-d}^\mu\|_N \leq \gamma_2' [\|\omega_t^\mu\|_N + \|\psi_t^\mu\|_N] + \gamma_2'' \|\xi_t^\mu\|_N \tag{5.4a}$$

where $\gamma_2'$, $\gamma_2''$ are $\mathcal{L}_2$-gains defined as

$$\gamma_2' \triangleq \gamma_2\{W_1[(\mu^{1/2}q)^{-1}]\}, \quad \gamma_2'' \triangleq \gamma_2\{W_2[(\mu^{1/2}q)^{-1}]\}. \tag{5.4b}$$

From the definition of $\bar{\psi}_t$, (3.0a) and (5.3), (5.4) we get

$$|\psi_N|^2 \geq \frac{\mu^{-N}\bar{\psi}_N^2}{\rho_0 + \mu^{-N}\boldsymbol{\rho} + 2\{[\gamma_2']^2(\|\omega_t^\mu\|_N^2 + \|\psi_t^\mu\|_N^2) + [\gamma_2'']^2\|\xi_t^\mu\|_N^2}.$$

Since $\bar{\psi}_t \in \mathcal{L}_2$ by assumption, $\bar{\psi}_t \rightarrow 0$, so that for all $\delta > 0$, $\exists\, N_0$ such that for all $N \geq N_0$,

$$\bar{\psi}_N^2 \leq \delta. \tag{5.5}$$

Therefore,

$$\mu^{-N}\bar{\psi}_N^2 \leq \delta\{\rho_0 + \mu^{-N}\boldsymbol{\rho} + [\gamma_2']^2\|\omega_t^\mu\|_N^2 + 2[\gamma_2'']^2\|\xi_t^\mu\|_N^2 + 2[\gamma_2']^2(\mu^{-N}\bar{\psi}_N^2 + \|\psi_t^\mu\|_{N-1}^2)\}.$$

If we choose $\delta$ such that $1 - 2\delta[\gamma_2']^2 > \mu$, we get

$$\mu^{-N}\psi_N^2 \leq \delta^2 K_1 \mu^{-N} + \frac{2\delta[\gamma_2']^2}{1-2\delta[\gamma_2']^2} \sum_{t=N_0}^{N-1} \mu^{-t}\psi_t^2 \tag{5.6}$$

where we have used the fact that $\rho_0$, $\boldsymbol{\rho}$, $\omega_t$ $\{\psi_t\}_0^{N0}$, $\xi_t \in \mathcal{L}_\infty$ to bound them by $\delta K_1 \mu^{-N}$.

Applying the Bellman–Gronwall lemma to (5.6)

$$\mu^{-N}\psi_N^2 \leq \delta^2 K_1 \mu^{-N} + \frac{2\delta[\gamma_2']^2}{1-2\delta[\gamma_2']^2} \sum_{t=N_0}^{N-1} \cdot \left[\frac{1}{1-2\delta[\gamma_2']^2}\right]^{N-t-1} \delta^2 K_1 \mu^{-t}$$

which may also be written as

$$\psi_N^2 \leq \delta^2 K_1 \left\{1 + 2\delta(\gamma_2')^2 \sum_{t=0}^{N-1} \left[\frac{\mu}{1-2\delta(\gamma_2')^2}\right]^{N-t}\right\}. \tag{5.7}$$

The term inside the brackets is smaller than 1 and the series is convergent, therefore, we can conclude that $\psi_t \in \mathcal{L}_\infty$. $\square$

*Corollary 5.1:* If $\bar{\psi}_t \in \mathcal{L}_2$, $\omega_t$, $\xi_t \in \mathcal{L}_\infty$ and A.1 holds, then $\phi_t \in \mathcal{L}_\infty$ and consequently $\rho_t \in \mathcal{L}_\infty$.

*Proof:* Follows immediately from Theorem 5.1, (2.10), and (5.1). $\square$

The following lemma will help us to find the *conicity conditions over* $H_2$ ensuring the ones required in Lemma 4.1.

*Lemma 5.1:* Let us consider the operator $H$: $\gamma_t \rightarrow \eta_t$. If $H[(\mu^{1/2}q)^{-1}]$ is inside the CONE $(C, R)$, then $H \triangleq \rho_t^{-1/2} H \rho_t^{1/2}$

(i.e., $H:\bar{\gamma}_t \to \bar{\eta}_t$) with $\rho_t$ as in (5.1) is inside the same CONE ($C$, $R$).

*Proof:* See also [14]. Define

$$Z_t \triangleq (\gamma_t - C\eta_t)^2 - (R\eta_t)^2$$
$$\bar{Z}_t \triangleq (\bar{\gamma}_t - C\bar{\eta}_t)^2 - (R\bar{\eta}_t)^2 = \rho_t^{-1} Z_t.$$

Taking the sum

$$\sum_{t=0}^{N} \bar{Z}_t = \sum_{t=0}^{N} \mu^t \rho_t^{-1} \mu^{-t} Z_t = \rho^{N+1} \rho_{N+1}^{-1} \sum_{t=0}^{N} \mu^{-t} Z_t$$
$$+ \sum_{j=0}^{N} \left[ \left( \sum_{t=0}^{j} \mu^{-t} Z_t \right) (\mu^j \rho_j^{-1} - \mu^{j+1} \rho_{j+1}^{-1}) \right].$$

The proof is completed noting that $\mu^t \rho_t^{-1}$ is decreasing since

$$\mu^{-(t+1)} \rho_{t+1} = \mu^{-t} \rho_t + \mu^{-(t+1)} \max \; [\boldsymbol{\rho}, |\phi_{t-d+1}|^2]$$

and the implications

$$H[(\mu^{1/2} q)^{-1}] \in \text{CONE } (C, R) \Rightarrow \sum_{t=0}^{N} [(\mu^{-t/2} \eta_t - C\mu^{-t/2} \gamma_t)^2$$
$$- (R\mu^{-t/2} \gamma_t)^2] < 0 \Rightarrow \sum_{t=0}^{N} \mu^{-t} Z_t < 0.$$

We establish that $\Sigma_{t=0}^{N} \bar{Z}_t < 0$, and consequently $\bar{H} \in$ CONE ($C$, $R$). □

We are now in position to present our main $\mathcal{L}_2$-result.

*Theorem 5.2:* Consider the process (2.1) in closed loop with the adaptive regulator (2.3), (2.4), whose parameters are updated according to (2.2), (3.3), (3.4) with the normalization (3.0), (5.1). If for given $n_S$, $n_R$ and $\mu$, Assumption A.1 holds and

i) $H_2[(\mu^{1/2} q)^{-1}]$ is strictly inside A (as defined in Lemma 4.1)
ii) $\omega_t$, $\xi_t \in \mathcal{L}_\infty$ are such that $e_t^* \in \mathcal{L}_2$ then

$$\psi_t, \; e_t \in \mathcal{L}_2 \text{ and } \phi_t \in \mathcal{L}_\infty.$$

*Proof:* Condition i) and Lemma 5.1 ensure the stability of the normalized error model (Lemma 4.1). Stability of the adaptive system (Fig. 1) may be concluded using multiplier theory [9] if $\rho_t$ qualifies as a multiplier, e.g., $\rho_t \in L_\infty$ (Fig. 2 with $\alpha = 1$). This is ensured by condition ii) and Corollary 5.1 since $e_t^* \in L_2 \Rightarrow \bar{e}_t^* \in L_2$, and consequently $\bar{\psi}_t \in L_2$. □

*Discussion:*

1) Theorem 5.2 may be stated in the following way. Given an LTI process of known delay, choosen $n_S$, $n_R$, $\mu$ and desired closed-loop poles, the adaptive system will exactly cancel the tracking error if there exists a value for the regulator parameters (an element of $\Theta_{LS}$) such that for this linear scheme. a) The Nyquist locus of the closed-loop transfer function ($Y_t^*/\omega_{t+d}$) is "sufficiently close" to the desired one ($1/C_R$). b) Robust servobehavior is possible. The notion of "sufficiently close" is precisely defined in terms of disks in the complex plane for the locus of the transfer function evaluated at $|q| = \mu^{1/2}$.

2) *The key modification to the PAA* used in this paper is the normalization. One of the main stumbling blocks to establish robust stability results for the RLS/PAA was the impossibility of proving that $\sigma_{\text{RLS}}$, in Lemma 3.1, is strictly smaller than 1 (see, e.g., [25], [14], [8], [23], [15]). This is necessary to disallow a vanishing radius for the cone. Normalization removes this defect, but then the error model is only in terms of normalized signals.

3) Notice that the cone $\mathfrak{C}$ depends only on designer chosen parameters [$\sigma_{\text{CG}}$ and $\sigma_{\text{RLS}}$ in (4.2)]. In the limit the *conicity condition i) coincides with a positivity condition.* Thus robustness enhancement occurs at the expense of reducing the speed of convergence of the PAA

4) The coefficient $\mu$ establishes an *alertness-robustness tradeoff*. Its robustness effects appear in the conicity conditions. PAA alertness is directly affected since $\mu$ is the normalization filter time constant (5.1). See [24] for further discussion.

5) The restriction on the tuned tracking error: $e_t^* \in \mathcal{L}_2$ imposes requirements on $H_2 - 1$, $\omega_t$, and $\xi_t$. If the nature of the reference and disturbance signals is known, incorporating an internal model in the design [16] allows one to ensure that this condition is met. In particular, it is verified for constant reference input and BOD if the open-loop system is type-1. In the following section we carry the analysis for the more interesting and practical case of $e_t^* \in \mathcal{L}_\infty$.

*B. $\mathcal{L}_\infty$-Stability*

The $\mathcal{L}_\infty$ result is given for the RLS/PAA (3.4b), (3.4c).

*Theorem 5.3:* Consider the adaptive system analyzed in Theorem 5.2 with a RLS/PAA.

If for $n_S$, $n_R$, $\lambda$, $\lambda_0$, $\lambda_1$, and $\mu$.

i) Condition i) of Theorem 5.2 holds
ii) $(\lambda_{\max} F_t^{-1} F_t')^2 \leq \mu^d$

then there always exists a $\boldsymbol{\rho}$ (5.1) such that

$$\psi_t, \; e_t, \; \phi_t \in \mathcal{L}_\infty \quad \text{for all } \omega_t, \; \xi_t \in \mathcal{L}_\infty.$$

*Proof:* Consider the normalized exponentially weighted feedback interconnection of Fig. 2. Notice that for $\alpha^2 = \mu^{-1}$ i) and ii) above imply the conditions of Lemma 4.2. Hence,

$$\bar{\psi}_N^2 \leq \frac{K_2}{\boldsymbol{\rho}(1-\mu)} \|e_t^{*2}\|_\infty. \tag{5.8}$$

The Bellman–Gronwall lemma may be now applied as in Theorem 5.1 proceeding from (5.5) with $\delta$ substituted by the right-hand side of (5.8). It becomes clear that the condition ensuring the boundedness of $\psi_t$ becomes

$$1 - 2\frac{K_2}{\boldsymbol{\rho}(1-\mu)} (\gamma_2')^2 \|e_t^{*2}\|_\infty \geq \mu$$

which may be rewritten as

$$(1-\mu)^2 > 2K_2 \frac{1}{\boldsymbol{\rho}} \|e_t^{*2}\|_\infty (\gamma_2')^2. \tag{5.9}$$

Since all the terms in the numerator of the right-hand side are bounded and $\mu$ ranges in (0, 1), there exists a $\boldsymbol{\rho}$ which will make (5.9) true. This completes the proof. □

*Discussion:*

1) Condition ii) has been discussed in Remark 3.3. We know that it is met if a persistence of excitation condition is satisfied.

2) Inequality (5.9) defines the class of (non-$\mathcal{L}_2$) disturbances under which $\mathcal{L}_\infty$-stability is preserved. Notice that $K_2$ quantifies the stability margin of the $H_1$, $H_2$ feedback interconnection (4.7). $\gamma_2$ is the gain of the map $\xi_t \to \phi_{t-d}^*$ (2.11), (5.4b); that is, it measures the effect of the BOD on the regressor in the linear scheme. The conicity condition and (5.9) impose contradictory requirements in the choice of $\mu$. The scalar $\boldsymbol{\rho}$ defines a lower bound for the normalization factor, hence directly affects the gain of the PAA. From (5.9) it appears to be interesting to have slow adaptation. A contradictory requirement would be given in case of a time-varying plant.

3) In a recent paper [29] $\mathcal{L}_\infty$-stability of the error model has been established incorporating into the PAA a parameter projection operation analogous to the one in [25]. This requires additional prior knowledge but allows one to extend the stability analysis without condition ii) and without the restriction (5.9) on the $\mathcal{L}_\infty$-norm of $e_t^*$.

## VI. Concluding Discussion and Further Research

To conclude let us summarize the results reported in the paper. A proof of robust stability for a discrete-time adaptive controller with a normalized estimator has been presented. Systems with arbitrary relative degree may be considered (in contrast to the continuous-time robustness studies [10], [21]) however we require the latter to be known. The stability conditions reduce to the existence of a linear regulator (of the chosen structure) such that: 1) the closed-loop tracking transfer function "approaches" the desired closed-loop behavior; 2) "good" disturbance rejection properties are attainable. Increasing the speed of adaptation renders these requirements more stringent.

Although the two previous conditions preserve the essence of the usual performance (in the sense of pole-placement) and disturbance rejection design objectives, they unfortunately do not offer any engineering design guidelines. The primary culprit here is the notion of transfer function vicinity (as stated in 1) above) which requires that the phase-shift between the attainable and the desired transfer functions should not exceed 90°, at all frequencies. This has been referred to in the literature as the positive real condition (of $H_2$).

One fundamental difference arises at this point between continuous and discrete-time robustness results. In the latter the assumption of known delay permits us to obtain a parametrization where $H_2$ has the relative degree zero. In terms of the Nyquist locus this implies that for all stably invertible processes the overall phase shift contribution is zero, i.e., the locus starts and ends in the same side of the complex plane. Therefore, since phase modification (usually phase lead) is only required over a limited frequency range, it will always be possible by proper filtering to satisfy the positivity condition. Two important questions remain however to be solved. How should we incorporate the available prior knowledge to convert the conicity conditions into tests for robustness? The second question is more disturbing. How should we deal with nonstably invertible process, very likely to appear in a discrete-time context?

## APPENDIX

From (3.4c), (3.8), (3.9), $d = 1$, we have the following property.

*Lemma:* If there exist $\epsilon > 0$ and $N_0$ such that

$$\sum_{i=0}^{N} \beta^{N-t}\bar{\phi}_t\bar{\phi}_t^T \geq \epsilon I \qquad \forall\, N \geq N_0$$

with

$$\beta = \frac{\lambda(\lambda_1-\lambda_0)}{\lambda(\lambda_1-\lambda_0)+\lambda_1(\lambda+\lambda_0)}$$

then we have

$$\max_x \frac{x^T F_t^{-1} x}{x^T F_t'^{-1} x} \leq 1 - \frac{\epsilon\lambda_0\lambda_1}{\lambda(\lambda_1-\lambda_0)+\lambda_1(\lambda+\lambda_0)}.$$

*Proof:* Let us remark some facts.

i) $F_t'$, $F_t$ are invertible for any finite $t$ and

$$F_{t+1}'^{-1} = F_t^{-1} + \frac{\bar{\phi}_t\bar{\phi}_t^T}{\lambda}, \qquad \lambda > 0 \tag{A.1}$$

$$F_t = \left(1-\frac{\lambda_0}{\lambda_1}\right)F_t' + \lambda_0 I, \qquad 0<\lambda_0<\lambda_1 \tag{A.2}$$

$$\|\bar{\phi}_t\| \leq 1. \tag{A.3}$$

Hence, by induction, if we choose $F_0$ such that

$$\lambda \max F_0 < \lambda_1$$

then we have for any finite $t$

$$\lambda \max F_t < \lambda_1.$$

Therefore $\lambda_1 F_t^{-1} - I$ is positive definite for any finite $t$.

ii) $F_t'$ has a symmetric positive definite square root $F_t'^{1/2}$ and we have

$$F_t^{-1}F_t' = F_t'^{1/2}F_t^{-1}F_t'^{1/2} = F_t'F_t^{-1}.$$

Hence, if we let:

$$y = F_t'^{1/2}x$$

we have

$$\frac{x^TF_t^{-1}x}{x^TF_t'^{-1}x} = \frac{y^TF_t'^{1/2}F_t^{-1}F_t'^{1/2}y}{y^Ty}.$$

This proves that:

$$\max_x \frac{x^TF_t^{-1}x}{x^TF_t'^{-1}x} = \lambda \max F_t^{-1}F_t'.$$

iii) If $A$ is a symmetric positive definite matrix, then

$$x^TAx \leq (1+\lambda \max A)x^TA(I+A)^{-1}x, \qquad \forall\, x.$$

This is proved by noticing that we can choose a symmetric positive definite square root $A^{1/2}$ which commutes with $(I + A)^{-1}$. Then with

$$y = A^{1/2}x.$$

The inequality becomes simply

$$y^Ty \leq y^T(I+A^{-1})y.\quad (1+\lambda \max A), \qquad \forall\, y.$$

Let us now study the matrix $G_t$ defined as

$$G_t = I - F_t^{-1}F_t'.$$

From fact ii), $G_t$ is symmetric, with eigenvalues smaller than 1 [see (A.2)]. With (A.2), we have

$$F_t^{-1} = \frac{1}{\lambda_1}I + \frac{1}{\lambda_0}\left(1-\frac{\lambda_0}{\lambda_1}\right)G_t.$$

Hence, from fact i), $G_t$ is positive definite. We have also

$$F_{t+1}'^{-1} = \frac{1}{\lambda_1}I + \frac{1}{\lambda_0}\left(1-\frac{\lambda_0}{\lambda_1}\right)G_t + \frac{\bar{\phi}_t\bar{\phi}_t^T}{d}.$$

Therefore, since

$$F_{t+1}'^{-1}F_{t+1} = \left(1-\frac{\lambda_0}{\lambda_1}\right)I + \lambda_0F_{t+1}'^{-1}$$

we have

$$F_{t+1}'^{-1}F_{t+1} = I + \left(1-\frac{\lambda_0}{\lambda_1}\right)G_t + \frac{\lambda_0}{\lambda_1}\bar{\phi}_t\bar{\phi}_t^T - (I-G_{t+1}).$$

This proves that

$$G_{t+1} = \left[\left(1-\frac{\lambda_0}{\lambda_1}\right)G_t + \frac{\lambda_0}{\lambda}\bar{\phi}_t\bar{\phi}_t^T\right]\left[I + \left(1-\frac{\lambda_0}{\lambda_1}\right)G_t + \frac{\lambda_0}{\lambda}\bar{\phi}_t\bar{\phi}_t^T\right]^{-1}.$$

We remark that, with the properties of $G_t$ and (A.3), we have

$$\lambda \max \left[\left(1-\frac{\lambda_0}{\lambda_1}\right)G_t + \frac{\lambda_0}{\lambda}\bar{\phi}_t\bar{\phi}_t^T\right] \leq 1 - \frac{\lambda_0}{\lambda_1} + \frac{\lambda_0}{\lambda}.$$

Then with fact iii), it follows that (in the sense of quadratic form)

$$G_{t+1} \geq \frac{\lambda_1 \lambda}{\lambda_1(\lambda+\lambda_0)+\lambda(\lambda_1-\lambda_0)} \left[ \left(1-\frac{\lambda_0}{\lambda_1}\right) G_t + \frac{\lambda_0}{\lambda} \bar{\phi}_t \bar{\phi}_t^T \right].$$

This implies

$$G_{t+1} \geq \sum_{i=0}^{t} \left( \frac{\lambda(\lambda_1-\lambda_0)}{\lambda_1(\lambda+\lambda_0)+\lambda(\lambda_1-\lambda_0)} \right)^{t-i} \cdot \frac{\lambda_0 \lambda_1}{\lambda_1(\lambda+\lambda_0)+\lambda(\lambda_1-\lambda_0)} \bar{\phi}_i \bar{\phi}_i^T.$$

The conclusion follows from the assumption and the properties of $G_t$. □

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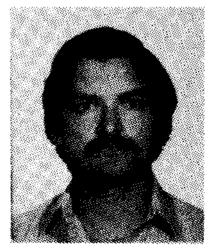

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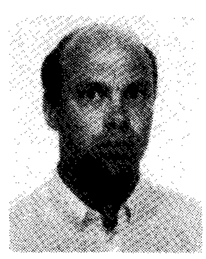

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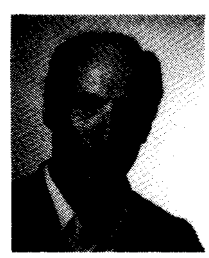

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