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On a stability criterion for continuous slow adaptation *

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Abstract: A self-contained proof is given of a stability criterion for slow adaptation. The proof is based on a novel open-loop representation of the parameter adjustment feedback.

Keywords: Adaptive control, Parameter adjustment feedback, Slow adaptation, Riedle–Kokotovic stability criterion.

1. Introduction

Many recent studies of adaptive estimation and control focus on conditions for exponential stability of an ideal regime in order to assure its robustness with respect to modeling non-idealities. In the earlier theory an ideal regime was characterized by the SPR (strictly positive real) property of a transfer function and the PE (persistent excitation) property of relevant signals (Morgan and Narendra [11,12], Anderson [1], Yuan and Wonham [15]). However, the SPR property itself is not robust because it can be lost due to the presence of small 'parasitic' time constants which are known to cause several types of instability (Rohrs et al. [14], Ioannou and Kokotovic [7,8]). Among them, the instability of the slow adaptation process is a new phenomenon not observed in feedback systems with fixed parameters. Deeper understanding of this phenomenon is a prerequisite for the robust design of adaptive schemes.

The slow instability, appearing in the form of a drift of adjustable parameters, is possible even with PE signals and at infinitesimally low adaptation gains. As anticipated by Ioannou and Kokotovic [6], and demonstrated by Kokotovic and Riedle [9], this drift is due to an excessive amount of signal energy in a 'bad' part of the spectrum, that is, at frequencies where the real part of the relevant transfer function is negative. An averaging analysis introduced by Astrom [3,4] and Krause et al. [10] was further developed by Riedle and Kokotovic [13]. Using an averaging theorem due to Hale [5], Riedle and Kokotovic [13] replaced the non-robust SPR condition by a signal-dependent positivity criterion, which delineates a sharp stability–instability boundary for slow adaptation. Earlier attempts to relax the SPR requirement, such as those due to Ioannou and Kokotovic [6,7] and Anderson et al. [2], appear in a less specific form and do not deal with the instability aspect of the problem.

This note presents a self-contained proof of the Riedle–Kokotovic stability criterion for slow adaptation. To make the proof more readily accessible, the derivations are given for the case of periodic signals. In this form, the criterion has an extremely simple frequency-domain interpretation. At the expense of additional refinements, the same type of results can be shown to hold for more general classes of signals.

A major simplification achieved in this note is due to a novel representation of the adaptive feedback system as an open-loop connection of a fast subsystem and a slow subsystem whose stability properties can be analyzed by averaging. The lower-triangular transformation (L-transformation) leading to this open-loop representation is defined in Section 2 and a proof of its properties is given in the Appendix. The stability criterion is then derived and interpreted in Section 3.

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2. An open-loop representation of slow adaptation

A first step in establishing the stability properties of most continuous adaptive schemes is an analysis of the linear equations with T-periodic coefficients

$$\begin{bmatrix} \dot{x} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} A & b\phi' \\ -\varepsilon\phi c' & -\varepsilon\phi d\phi' \end{bmatrix} \begin{bmatrix} x \\ \theta \end{bmatrix}, \quad \phi(t) = \phi(t+T), \tag{2.1}$$

where A, b, c', and d represent a scalar transfer function

$$H(s) = d + c'(sI - A)^{-1}b \tag{2.2}$$

for a state $x \in \mathbb{R}^m$, and $\theta \in \mathbb{R}^n$ is the parameter vector being adjusted with rate $\dot{\theta}$ proportional to a scalar gain $\varepsilon \geq 0$. The T-periodic signal vector $\phi(t) \in \mathbb{R}^n$ is assumed to be bounded and independent of x and θ. While the restrictions of $H(s)$ to be scalar and $\phi(t)$ to be periodic are not crucial, the assumption that $\phi(t)$ is bounded and independent of x and θ is essential for the subsequent linear analysis. For those adaptive schemes for which (2.1) is a valid linearization along a 'nominal' solution which corresponds to $x = 0$, $\theta = 0$, and $\phi = \phi(t)$, the results of this analysis establish local stability–instability conditions.

To be able to treat (2.1) as a fundamental equation of *slow adaptation* we need two additional assumptions. First, we let the adaptation gain ε be sufficiently small, that is, the variations of θ are slow compared to those of x. A bound for ε obtained in the Appendix quantifies this assumption. Our second assumption is that A is Hurwitz, that is,

$$\operatorname{Re} \lambda(A) < 0, \tag{2.3}$$

which implies that the *slow adaptation* is meant to estimate *slow* variations of parameters of a stable plant, or to counteract them by a *slow* adjustment of controller parameters rather than to stabilize an unstable plant.

The smallness of $\dot{\theta}$ suggests an analysis in separate time scales: one for the fast signals and transients and the other for the slow parameter adjustment. To separate the fast behavior we introduce a new state variable

$$z = x - L(t, \varepsilon)\theta \tag{2.4}$$

and determine the $m \times n$ matrix $L(t, \varepsilon)$ such that the new equation for z is independent of θ. The substitution of (2.4) into (2.1) shows that this will happen if $L(t, \varepsilon)$ is a bounded solution of

$$\dot{L} = AL + b\phi' + \varepsilon L\phi(d\phi' + c'L). \tag{2.5}$$

Then (2.4) transforms (2.1) into a lower-triangular form

$$\begin{bmatrix} \dot{z} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} A + \varepsilon L\phi c' & 0 \\ -\varepsilon\phi c' & -\varepsilon\phi v' \end{bmatrix} \begin{bmatrix} z \\ \theta \end{bmatrix} \tag{2.6}$$

where the row n-vector v' is defined by

$$v' = v'(t, \varepsilon) = d\phi'(t) + c'L(t, \varepsilon) \tag{2.7}$$

and also appears in the nonlinear term of (2.5). With (2.5), (2.6) we have obtained an equivalent open-loop representation of the feedback system (2.1). The feedback loop in Fig. 1a, which depicts the original system (2.1), is closed by a path from θ to the input $\phi'\theta$ into the block $H(s)$. This path is removed from the transformed system in Fig. 1b, where ϕ' is the input into the block $H(s)$. Its output v', multiplied by ϕ, forms the matrix $\phi v'$ of the θ-subsystems. This multiplicative connection of the L-subsystem with the θ-subsystem, depicted in thick lines in Fig. 1b, determines the stability properties of the θ-subsystem. Since the z-subsystem is independent of θ, the additive input $\phi c'z$ into the θ-subsystem does not close a feedback loop and, hence, does not affect the stability of the θ-subsystem.

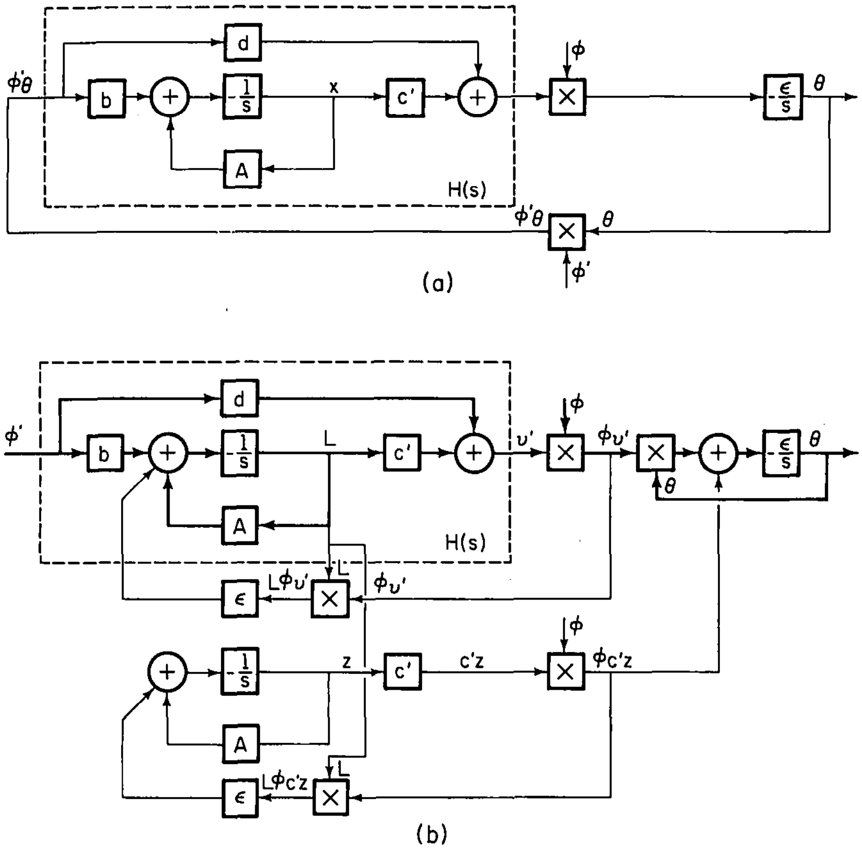

Fig. 1. Block diagrams corresponding to (a) (2.1); and (b) (2.5), (2.6).

Prior to our stability analysis of (2.6), we must guarantee the existence, continuity, and boundedness properties of $L(t, \varepsilon)$. They follow from the assumption (2.3) and boundedness of $\phi(t)$ because (2.5) is an ε-perturbation of the linear equation

$$\dot{L}_0(t) = AL_0(t) + b\phi'(t). \tag{2.8}$$

Hence $L(t, \varepsilon)$ can be written as

$$L(t, \varepsilon) = L_0(t) + \varepsilon L_1(t, \varepsilon) \quad \forall \varepsilon \in [-\varepsilon^*, \varepsilon^*], \tag{2.9}$$

where $L_1(t, \varepsilon)$ satisfies the equation (A.1) in the Appendix. Clearly, to each bounded T-periodic $\phi(t)$ there corresponds a unique T-periodic $L_0(t)$ satisfying (2.8). Its frequency-domain expression is

$$L_0(j\omega) = (j\omega I - A)^{-1} b\phi'(j\omega). \tag{2.10}$$

A classical perturbation theorem due to Poincaré can be used to establish the existence and uniqueness of a T-periodic solution $L(t, \varepsilon)$ of (2.5) for sufficiently small ε. To obtain a bound on the perturbation term $L_1(t, \varepsilon)$ and an estimate of the segment $[-\varepsilon^*, \varepsilon^*]$ in which the transformation (2.4) is defined, the following lemma is proved in the Appendix.

Lemma 2.1. *Under the assumption* Re $\lambda(A) < 0$ *and for each* ε *in the segment* $[-\varepsilon^*, \varepsilon^*]$, *where* ε^* *is given by* (A.5), *there exists a unique T-periodic solution* $L_1(t, \varepsilon)$ *of* (A.1), *bounded by* ρ_4 *given in* (A.5). □

Thus, choosing $L(t, \varepsilon)$ as the T-periodic solution of (2.5) uniquely defines the transformation of (2.4) of (2.1) into (2.6). We note that the initial value of z at $t = 0$ is uniquely defined in (2.4) by the values of x

and θ at $t=0$, and $L(0, \varepsilon)$, which is the value of the T-periodic solution $L(t, \varepsilon)$ at $t=0$. Another useful observation from (2.9) is that the derivative of $L(t, \varepsilon)$ with respect to ε at $\varepsilon=0$ is $L_1(t, 0)$.

3. Proof and interpretation of the stability criterion

With $L(t, \varepsilon)$ uniquely defined for each bounded T-periodic $\phi(t)$, the stability properties of the original system (2.1) are identical to those of the lower-triangular system (2.6), and can be established by separately analyzing the z-subsystem and the θ-subsystem of (2.6). For the z-subsystem

$$\dot{z} = (A + \varepsilon L(t, \varepsilon)\phi(t)c')z \tag{3.1}$$

a proof of the following lemma is given in the Appendix.

Lemma 3.1. *Under the assumption* Re $\lambda(A) < 0$ *and for all* $\varepsilon \in [-\varepsilon^*, \varepsilon^*]$, *where* ε^* *is given by* (A.5), *the system* (3.1) *is exponentially stable.* □

Thus, if (2.1) can be unstable for $0 < \varepsilon \le \varepsilon^*$, the instability can occur only in the θ-subsystem of (2.6), that is, in the system

$$\dot{\theta} = -\varepsilon\phi(t)v'(t, \varepsilon)\theta - \varepsilon\phi(t)c'z, \tag{3.2}$$

whose matrix $\phi(t)v'(t, \varepsilon)$ is a bounded T-periodic function of t for each $\varepsilon \in [-\varepsilon^*, \varepsilon^*]$. It is well known that, if the state transition matrix $\mathscr{F}(t, t_0)$ of this system,

$$\dot{\mathscr{F}} = -\varepsilon\phi(t)v'(t, \varepsilon)\mathscr{F}, \quad \mathscr{F}(t_0, t_0) = I, \tag{3.3}$$

evaluated over a period T, say $\mathscr{F}(T, 0) = \mathscr{F}_T$, has all its eigenvalues inside the circle, that is, if

$$|\lambda_i(\mathscr{F}_T)| < 1, \quad i = 1, \dots, n, \tag{3.4}$$

then the system (3.2) is *exponentially stable*. If any one of these eigenvalues is outside the unit circle,

$$|\lambda_j(\mathscr{F}_T)| > 1, \quad \text{for some } j \in \{1, \dots, n\}, \tag{3.5}$$

then the system (3.2) is *unstable*. The usefulness of (3.4), (3.5) as a stability criterion depends on our ability to express $\mathscr{F}_T$ in terms of some practically meaningful quantities. One such quantity is the signal $\phi(t)$. As a second meaningful quantity, we choose the signal

$$v_0'(t) = d\phi'(t) + c'L_0(t), \tag{3.6}$$

which, by (2.7)–(2.10), is the output of the transfer function $H(s)$ for input ϕ'.

Lemma 3.2. *The matrix* $\mathscr{F}_T = (T, 0)$ *can be expressed as*

$$\mathscr{F}_T = I - \varepsilon\left(\int_0^T \phi(t)v_0'(t)\,\mathrm{d}t + \varepsilon M\right) \tag{3.7}$$

where the matrix M *is a bounded function of* ε.

Proof. Solving (3.3) by successive approximations

$$[\mathscr{F}(t, 0)]_{k+1} = I - \varepsilon\int_0^t \phi(\tau)v'(\tau, \varepsilon)[\mathscr{F}(\tau, 0)]_k\,\mathrm{d}\tau; \quad k = 0, 1, 2, \dots, \tag{3.8}$$

with $[\mathscr{F}(t, 0)]_0 = I$, we obtain as $k \to \infty$,

$$\mathscr{F}_T = I - \varepsilon\int_0^T \phi(t)v'(t, \varepsilon)\,\mathrm{d}t + \varepsilon^2 M_1, \tag{3.9}$$

where a bound for

$$M_1=\sum_{k=2}^{\infty}(-\varepsilon)^{k-2}\int_0^T\phi(\tau_1)v'(\tau_1,\varepsilon)\int_0^{\tau_1}\phi(\tau_2)v'(\tau_2,\varepsilon)\cdots\int_0^{\tau_{k-1}}\phi(\tau_k)v'(\tau_k,\varepsilon)\,\mathrm{d}\tau_k\cdots\mathrm{d}\tau_1 \tag{3.10}$$

is readily available from the bounds for $\phi(t)$ and $v'(t,\varepsilon)$. The substitution of

$$v'(t,\varepsilon)=v_0'(t)+\varepsilon c'L_1(t,\varepsilon), \tag{3.11}$$

which follows from (2.7), (2.9), and (3.6), into (3.9), and the notation

$$M=M_1-\int_0^T\phi(t)c'L_1(t,\varepsilon)\,\mathrm{d}t \tag{3.12}$$

prove (3.7). □

A practically meaningful stability criterion for slow adaptation can now be formulated as follows.

Theorem 3.1 (Stability criterion). *For any bounded T-periodic PE signal $\phi(t)$, and under the assumption* Re $\lambda(A)<0$, *there exists ε^{**} such that for all $\varepsilon\in(0,\varepsilon^{**}]$ the system* (2.1) *is **exponentially stable** if*

$$\operatorname{Re}\lambda_i\left(\int_0^T\phi(t)v_0'(t)\,\mathrm{d}t\right)>0,\quad i=1,\dots,n, \tag{3.13}$$

*and is **unstable** if*

$$\operatorname{Re}\lambda_j\left(\int_0^T\phi(t)v_0'(t)\,\mathrm{d}t\right)<0,\quad \textit{for some } j\in\{1,\dots,n\}. \tag{3.14}$$

Proof. Let $\alpha+\mathrm{j}\beta$ be an eigenvalue of the matrix $TR+\varepsilon M$, where

$$R=\frac{1}{T}\int_0^T\phi(t)v_0'(t)\,\mathrm{d}t. \tag{3.15}$$

From (3.7) the modulus of the corresponding eigenvalue of $\mathscr{F}_T$ is

$$|\lambda(\mathscr{F}_T)|=\left[1-2\varepsilon\alpha+(\alpha^2+\beta^2)\varepsilon^2\right]^{1/2}=1-\varepsilon\alpha+\mathrm{O}(\varepsilon^2) \tag{3.16}$$

and, because $TR+\varepsilon M$ is an ε-perturbation of TR, we have

$$\alpha=\operatorname{Re}\lambda(TR)+\mathrm{O}(\varepsilon^\nu),\quad \nu\geq 1/n. \tag{3.17}$$

If the eigenvalues of TR are distinct, then $\nu=1$. The proof of (3.13) and (3.14) follows from the substitution of (3.16), (3.17) into (3.4) and (3.5). □

The correlation-like positivity condition (3.13) is much less restrictive than the requirement that $H(s)$ be SPR. Given a general non-SPR transfer function $H(s)$, there exists a class of signals $\phi(t)$ satisfying the condition (3.13) and, hence, leading to exponentially stable slow adaptation. For the same transfer function another class of signals $\phi(t)$ will satisfy the condition (3.14) and make the slow adaptation unstable. A sharp boundary between 'good' and 'bad' signals $\phi(t)$ willl become more explicit if we express $\phi(t)$ as

$$\phi(t)=\sum_{k=-\infty}^{\infty}\varphi_k\,\mathrm{e}^{\mathrm{j}\omega_k t},\quad \omega_k=\frac{2\pi k}{T},\quad \varphi_{-k}=\bar{\varphi}_k, \tag{3.18}$$

where $\bar{\varphi}_k$ is the conjugate of the complex vector φ_k, and note that

$$\sum_{k=-\infty}^{\infty}\varphi_k\bar{\varphi}_k'>0 \tag{3.19}$$

because $\phi(t)$ is PE. If $\phi(t)$ is the input into $H(j\omega)$, its output is

$$v_0(t) = \sum_{k=-\infty}^{\infty} \varphi_k H(j\omega_k)\, e^{j\omega_k t} \tag{3.20}$$

and, hence, the matrix in the stability criterion is

$$R = \frac{1}{T}\int_0^T \phi(t)\, v_0'(t)\, dt = \sum_{k=-\infty}^{\infty} \bar{\varphi}_k H(j\omega_k)\varphi_k'. \tag{3.21}$$

A sufficient condition for this nonsymmetric matrix R to satisfy the stability condition (3.13) is that its symmetric part $\frac{1}{2}(R + R')$ be positive definite, that is,

$$\sum_{k=-\infty}^{\infty} \operatorname{Re} H(j\omega_k) \operatorname{Re} \varphi_k\bar{\varphi}_k' > 0. \tag{3.22}$$

In view of (3.19) this means that the sum of terms with Re $H(j\omega_k) > 0$ dominates the sum of terms with Re $H(j\omega_k) < 0$. Hence, the 'good' signals $\phi(t)$ are those whose H-weighted 'energy' in the frequency range where Re $H(j\omega) > 0$ is larger than that in the range where Re $H(j\omega) < 0$. Hence the condition (3.22) gives a precise meaning to the notion of 'dominantly rich inputs' introduced by Ioannou and Kokotovic [6].

The form of the condition (3.22) also suggests that the stability properties of the slow parameter adjustment can be enhanced by appropriate filtering. Figure 2 shows two modifications for which the matrix R in the stability criterion becomes

$$R = \sum_{k=-\infty}^{\infty} \bar{\varphi}_k \overline{F(j\omega_k)} H(j\omega_k) F(j\omega_k)\varphi_k' \tag{3.23}$$

and the condition (3.22) becomes

$$\sum_{k=-\infty}^{\infty} |F(j\omega_k)|^2 \operatorname{Re} H(j\omega_k) \operatorname{Re} \varphi_k\bar{\varphi}_k' > 0. \tag{3.24}$$

A prior knowledge of the frequency range where Re $H(j\omega_k) < 0$ can be used to design $F(s)$ so that the magnitude of the 'bad' terms in the sum in (3.24) is reduced and hence the stability properties of the *slow* adaptation are improved.

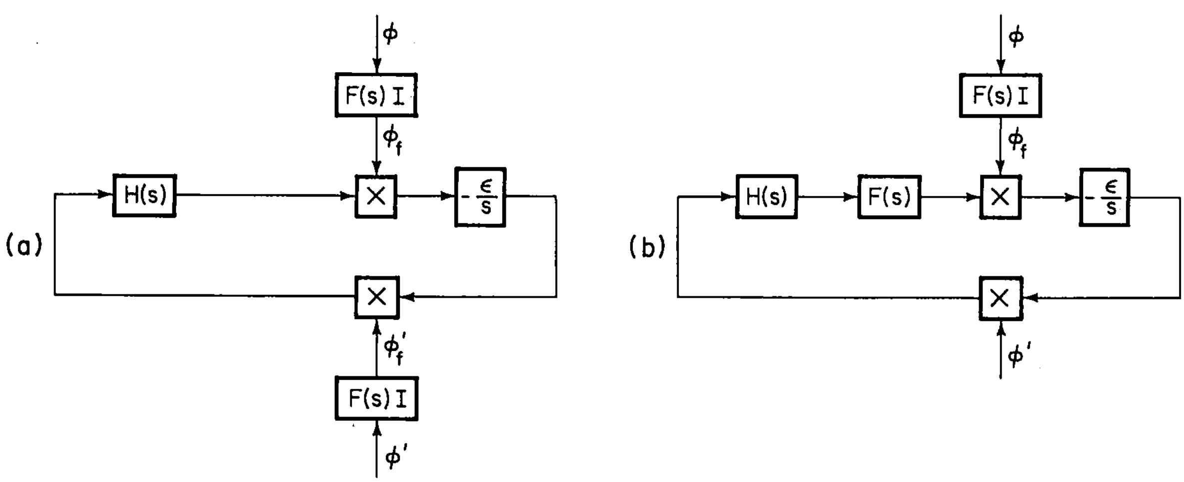

Fig. 2. Modifications which change (3.21) to (3.23).

Concluding remarks

A self-contained proof of the Riedle–Kokotovic stability criterion for slow adaptation has been given for the case of bounded T-periodic $\phi(t)$. For cases when ϕ belongs to a more general class of signals, we can derive a stability criterion following essentially the same steps as in this note. Applying the L-transformation with L in the same class of signals as ϕ, we get the same open-loop system as in Fig. 1b. The stability of (3.1) again depends on the stability of (3.3). For almost periodic signals, an averaging result applies and the stability criterion depends on the eigenvalues of the average of $\phi(t)v_0'(t)$. The interpretation remains the same with R in (3.21) defined for the limit as $T \to \infty$. For more general classes of signals, more work must be done to get a meaningful criterion.

Finally, we point out that in adaptive control the assumption that ϕ is bounded and independent of x and θ is crucial. It restricts the validity of the stability results to a neighborhood of a solution for which this assumption is valid.

Appendix

Proof of Lemma 2.1. Substituting (2.9) into (2.5) yields

$$\dot{L}_1 = AL_1 + L_0(t)\phi(t)v_0'(t) + f(t, L_1, \varepsilon) \tag{A.1}$$

where $v_0'(t)$ is defined by (3.6) and

$$f(t, L_1, \varepsilon) = \varepsilon\left(L_1\phi(t)v_0'(t) + L_0(t)\phi(t)c'L_1\right) + \varepsilon^2 L_1\phi(t)c'L_1. \tag{A.2}$$

Let $\mathscr{L}$ be the Banach space of $m \times n$ matrices with continuous T-periodic entries defined on $t \in (-\infty, \infty)$, and note that $\phi(t)$ and $L_0(t)$ are T-periodic and can be defined for all $t \in (-\infty, \infty)$. Equipping $\mathscr{L}$ with the norm

$$\|Y\|_{\mathscr{L}} \triangleq \sup_{t \in (-\infty, \infty)} \|Y(t)\| \tag{A.3}$$

where $\|\cdot\|$ is the induced Euclidean norm, we consider the mapping $\mathscr{M}: \mathscr{L} \to \mathscr{L}$ pointwise in t defined by (A.1) for any element Y of $\mathscr{L}$, namely,

$$(\mathscr{M}Y)(t) = \int_{-\infty}^{t} e^{A(t-\tau)}\left[L_0(\tau)\phi(\tau)v_0'(\tau) + f(\tau, Y(\tau), \varepsilon)\right] d\tau. \tag{A.4}$$

Using the constants $\rho_1, \ldots, \rho_4$ and ε^*,

$$\|\phi(t)\| \le \rho_1, \quad \|L_0(t)\| \le \rho_2, \quad \|v_0(t)\| \le \rho_3 \qquad \forall t,$$
$$\rho_4 = 2\frac{K}{\alpha}\rho_1\rho_2\rho_3, \qquad \varepsilon^* = \frac{\sqrt{3}-1}{2}\,\frac{\alpha}{K\rho_1(\rho_3 + \rho_2\|c\|)} \tag{A.5}$$

where K and α are the bounds for the exponentially stable linear part

$$\|e^{A(t-s)}\| \le K\,e^{-\alpha(t-s)} \quad \forall t \ge s, \tag{A.6}$$

we see that, if $Y_1 \in \mathscr{L}$, $Y_2 \in \mathscr{L}$ satisfy $\|Y_1\|_{\mathscr{L}} \le \rho_4$, $\|Y_2\|_{\mathscr{L}} \le \rho_4$, then for each $\varepsilon \in [-\varepsilon^*, \varepsilon^*]$,

$$\|f(\cdot, Y_1, \varepsilon) - f(\cdot, Y_2, \varepsilon)\|_{\mathscr{L}} \le \left[\varepsilon^*\rho_1(\rho_3 + \rho_2\|c\|) + 2(\varepsilon^*)^2\rho_1\rho_4\|c\|\right]\|Y_1 - Y_2\|_{\mathscr{L}} \le \frac{\alpha}{2K}\|Y_1 - Y_2\|_{\mathscr{L}}. \tag{A.7}$$

Hence we have

$$\|\mathscr{M}Y_1\|_{\mathscr{L}} \le \sup_{t \in (-\infty, \infty)} \int_{-\infty}^{t} K\,e^{-\alpha(t-\tau)}\left(\rho_1\rho_2\rho_3 + \frac{\alpha}{2K}\rho_4\right) d\tau \le \frac{K}{\alpha}\left(\rho_1\rho_2\rho_3 + \frac{\alpha}{2K}\rho_4\right) = \rho_4, \tag{A.8}$$

$$\|\mathscr{M}Y_1 - Y_2\|_{\mathscr{L}} \le \frac{K}{\alpha}\|f(\cdot, Y_1, \varepsilon) - f(\cdot, Y_2, \varepsilon)\|_{\mathscr{L}} \le \tfrac{1}{2}\|Y_1 - Y_2\|_{\mathscr{L}}, \tag{A.9}$$

which proves that $\mathscr{M}$ is a contraction mapping of the compact set $\Omega = \{Y \in \mathscr{L}, \|Y\|_{\mathscr{L}} \leq \rho_4\}$ into itself. Therefore, for each $\varepsilon \in [-\varepsilon^*, \varepsilon^*]$ there exists unique $L_1(\cdot, \varepsilon) \in \Omega$ such that $L_1(t, \varepsilon) = (\mathscr{M} L_1)(t, \varepsilon)$, which satisfies (A.1).

Proof of Lemma 3.1. For each $\varepsilon \in [-\varepsilon^*, \varepsilon^*]$ the perturbation term in (3.1) is bounded by

$$\|\varepsilon L(t, \varepsilon)\phi(t)c'\| \leq \varepsilon^*(\rho_2 + \varepsilon^*\rho_4)\rho_1\|c\| \leq \frac{\sqrt{3}-1}{2}\frac{\alpha}{K} \tag{A.10}$$

which, by a standard perturbation theorem (such as Theorem 2.3 in Chapter III of Hale [5]), shows that

$$\|z(t)\| \leq K\|z(0)\| \exp\left[-\left(\frac{3-\sqrt{3}}{2}\right)\alpha t\right], \tag{A.11}$$

that is, (3.1) is exponentially stable.

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