IV. Illustrative Examples

Example 1: Let the system

$$\dot{x} = xu + zu, \quad \epsilon\dot{z} = 2xu + zu \qquad (0 < \epsilon \leqslant 0.1).$$

One observes that condition (22) cannot be satisfied for any $P_1 > 0$ and $P_2 > 0$. Hence, Steps 1, 2, and 4 must be applied.

i) Choose $P_1 = 1$ and $P_2 = 1$.

ii) Solve (18a) to obtain $P_3 = -3/(1+\epsilon)$.

Now, using (20)

$$K_1 = 1 - 6/(1+\epsilon) = (\epsilon - 5)/(1+\epsilon)$$
$$K_2 = -3\epsilon/(1+\epsilon) + 1 = (1-2\epsilon)/(1+\epsilon)$$

and so by (19)

$$u^0 = -\left(\frac{\epsilon-5}{1+\epsilon}\right)x^2 - \left(\frac{1-2\epsilon}{1+\epsilon}\right)z^2.$$

Note that one could compute u^0 directly from (5) with $\boldsymbol{P}$ being replaced by $\boldsymbol{P}_\epsilon$ which here is

$$\boldsymbol{P}_\epsilon = \left[\begin{array}{c|c} 1 & -3\epsilon/(1+\epsilon) \\ \hline -3\epsilon/(1+\epsilon) & \epsilon \end{array}\right].$$

Clearly,

$$\det P_\epsilon = \epsilon - 9\epsilon^2/(1+\epsilon)^2 = \epsilon(\epsilon^2 - 7\epsilon + 1)/(1+\epsilon)^2 > 0 \qquad \text{for } 0 < \epsilon \leqslant 0.1.$$

Example 2: Let the system

$$\dot{x} = 3xu + 4zu$$
$$\epsilon\dot{z} = -2xu + zu, \qquad 0 < \epsilon < 0.2.$$

One observes that for $P_1 = 1$ and $P_2 = 2$ the condition (22) is satisfied. Hence, the policy (5) with

$$\boldsymbol{P}_\epsilon = \begin{bmatrix} 1 & 0 \\ 0 & 2\epsilon \end{bmatrix}, \quad \det P_\epsilon = 2\epsilon > 0 \qquad \text{for } \epsilon > 0$$

stabilizes the system, and can be written in the form (19) as

$$u^0 = -(3x^2 + 2z^2).$$

Example 3: Consider the system $\dot{\boldsymbol{X}} = \boldsymbol{N}\boldsymbol{X}u$ with

$$\boldsymbol{N} = \begin{bmatrix} 2 & 0 \\ 1/\epsilon & 3/\epsilon \end{bmatrix}.$$

It is easily seen that condition (22) cannot be satisfied with any

$$\boldsymbol{P}_\epsilon = \begin{bmatrix} \boldsymbol{P}_1 & 0 \\ 0 & \epsilon\boldsymbol{P}_2 \end{bmatrix}$$

positive definite.

However, using the transformation (29a)

$$T_2(\epsilon) = \begin{bmatrix} 2\epsilon-3 & 0 \\ 1 & 1 \end{bmatrix}, \qquad T_2^{-1}(\epsilon) = \frac{1}{(2\epsilon-3)}\begin{bmatrix} 1 & 0 \\ -1 & 2\epsilon-3 \end{bmatrix}$$

we find [see (25)]

$$\hat{N} = T_2^{-1}(\epsilon)NT_2(\epsilon) = \begin{bmatrix} 2 & 0 \\ 0 & 3/\epsilon \end{bmatrix}$$

and the condition $\hat{N}_{12}^T\hat{P}_1 + \hat{P}_2\hat{N}_{21} = 0$ is satisfied for, say, $\hat{P}_1 = \hat{P}_2 = 1$.

The corresponding stabilizing policy is given by (26b), i.e.,

$$u^0 = -\left\{\hat{x}^T\left(\hat{N}_{11}^T\hat{P}_1\right)\hat{x} + \hat{z}^T\left(\hat{N}_{22}^T\hat{P}_2\right)\hat{z}\right\}$$
$$= -(2\hat{x}^2 + 3\hat{z}^2).$$

To express this policy in terms of the original state variables x and z we first find

$$\boldsymbol{P}_\epsilon = T_2^{-1}(\epsilon)^T\hat{\boldsymbol{P}}T_2^{-1}(\epsilon)$$
$$= \frac{1}{(2\epsilon-3)^2}\begin{bmatrix} 1 & -1 \\ 0 & 2\epsilon-3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \epsilon \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 2\epsilon-3 \end{bmatrix}$$
$$= \frac{1}{(2\epsilon-3)^2}\begin{bmatrix} 1+\epsilon & -\epsilon(2\epsilon-3) \\ -\epsilon(2\epsilon-3) & \epsilon(2\epsilon-3)^2 \end{bmatrix},$$
$$\det \boldsymbol{P} = \epsilon > 0 \qquad \text{for } \epsilon > 0.$$

Then

$$u^0 = -\boldsymbol{X}^T\boldsymbol{N}^T\boldsymbol{P}_\epsilon\boldsymbol{X}$$
$$= -\left[\frac{5x^2}{(2\epsilon-3)^2} + 3z^2 - \frac{6xz}{2\epsilon-3}\right].$$

V. Conclusions

We have studied the problem of stabilizing strictly bilinear singularly perturbed systems, by using state feedback control policies. Our aim was to design reduced-order controllers requiring less computational effort compared to the overall controllers of the form (5). We have seen that, under the conditions of Theorem 1, it suffices to choose an n-dimensional symmetric positive definite matrix $\boldsymbol{P}_1$ in order to stabilize our $(n+m)$-dimensional system (1). Alternatively, choosing $\boldsymbol{P}_1$, $\boldsymbol{P}_2$, and $\boldsymbol{P}_3$ as in Theorem 2, one can construct a stabilizing matrix gain as is shown in (11). If condition (22) is satisfied, then $\boldsymbol{P}_\epsilon$ can be constructed in the simpler form (21). Note that we gain a considerable computational effort in both checking the positive definitness of the gain matrices involved and carrying out the required algebraic matrix computations.

The results of the paper can easily be used when the system at hand involves a linear control term $\boldsymbol{B}u$. The class of fully bilinear systems of the form $\dot{\boldsymbol{X}} = \boldsymbol{A}\boldsymbol{X} + \boldsymbol{N}\boldsymbol{X}u + \boldsymbol{B}u$ is treated in [8], and the class of ϵ-coupled bilinear systems in [7]. We close by remarking that in the strictly bilinear case one can always determine the reduced model upon which the controller is based. In the fully bilinear case this is possible only under certain conditions [2].

References

[1] R. R. Mohler and W. J. Kolodziej, "An overview of bilinear system theory and applications," *IEEE Trans. Syst., Man, Cybern.*, vol. SMC-10, no. 10, pp. 683–688, 1980.

[2] J. M. Guillen and M. A. Armada, "A singular perturbation method for order reduction of large-scale bilinear dynamical system," in *Proc. IFAC Symp. Large-Scale Systems Theory and Applications*, Toulouse, June 1980, pp. 229–236.

[3] M. Susuki and M. Miura, "Stabilizing feedback controllers for singularly perturbed linear constant systems," *IEEE Trans. Automat. Contr.*, pp. 123–124, Feb. 1976.

[4] M. Athans and P. Falb, *Optimal Control.* New York: McGraw-Hill, 1966.

[5] H. Kwakernaak and R. Sivan, *Linear Optimal Control Systems.* New York: Wiley 1972, pp. 46–47.

[6] J. H. Chow and P. V. Kokotovic, "Eigenvalue placement in two-scale systems," in *Proc. IFAC Symp. Large Scale Systems Theory and Applications*, Udine, June 1976, pp. 321–326.

[7] S. G. Tzafestas and K. Anagnostou, "Stabilization of large-scale bilinear systems via the ϵ-coupling approach," in *Proc. 3rd IFAC/IFORS Symp. on LSSTAI*, Warsaw, July 1983.

[8] ——, Stabilizing controllers for singularly perturbed time-invariant bilinear systems," *Foundations Contr. Eng.*, vol. 8, no. 2, pp. 93–104, 1983.

Towards a Globally Stable Direct Adaptive Control Scheme for Not Necessarily Minimum Phase Systems

LAURENT PRALY

***Abstract*—A direct adaptive control scheme for not necessarily minimum phase systems is presented. The algorithm is based on simultaneous**

Manuscript received March 30, 1983; revised December 5, 1983. This paper is based on a prior submission of June 9, 1982. Paper recommended by G. C. Goodwin, Past Chairman of the Adaptive, Learning Systems, Pattern Recognition Committee.

The author is with CAI—Ecole Nationale Superieure des Mines de Paris, Fontainebleau, France.

0018-9286/84/1000-0946$01.00 ©1984 IEEE

identification of input and output prediction models. This leads to a bilinear parameter estimation problem for which a least-squares criterion minimization is proposed. This framework makes it possible to establish global convergence without any extra condition. We give an example to illustrate the practical features.

I. Introduction

Recently, there has been considerable interest in the problem of direct adaptive control for nonminimum phase systems. A common characteristic of these control schemes is the estimation of more parameters than those effectively needed for control. In [1] or [2], these extra parameters are those of the model. However, this leads to parameter estimation based on a bilinear observation equation. This estimation problem is solved by classical linear model parameter estimation the result of which is used in a second step to linearize the bilinear observation equation. This is, in fact, very close to indirect schemes. These schemes can fail if the estimated model loses stabilizability. In [3] the extra parameters are those of a partial state predictor. The observation of the parameters is linear. But without a persistency of excitation assumption, only local stability may be established.

Here we introduce an input prediction model which together with an output prediction model defines an implicit model which is bilinear in the parameters. To solve this estimation problem a least-squares criterion minimization is proposed.

In Section II, we present the controller structure. In Section III we study the bilinear estimation problem and we mention a global convergence result. In Section IV, a simulation study is presented demonstrating the feasibility of controlling unstable nonminimum phase systems.

II. Direct Adaptive Control Scheme

Assumptions: Consider a system with $y(t), u(t)$ as scalar output and input, respectively. The following assumptions will be used.

A0: There exists (unknown) scalar polynomials in the unit delay operator q^{-1} such that the system can be represented by

$$A(q^{-1})y(t)=B(q^{-1})u(t) \tag{1}$$

with

$$A(q^{-1})=1+a_1q^{-1}+\cdots+a_{n_A}q^{-n_A} \tag{2}$$

$$B(q^{-1})=b_0+b_1q^{-1}+\cdots+b_{n_B}q^{-n_B}. \tag{3}$$

A1: $n\geqslant\max\{n_A,n_B\}$ is known.

A2: An upper bound (which does not need to be small) of the coefficients of $A(q^{-1}), B(q^{-1})$ is known.

A3: Given an (known) exponentially stable polynomial $R(q^{-1})$

$$R(q^{-1})=1+r_1q^{-1}+\cdots+r_{n_R}q^{-n_R},\qquad n_r\leqslant n. \tag{4}$$

There exists (unknown) polynomials $C(q^{-1}), D(q^{-1})$ of degree $n,(n-1)$, respectively, with $C(0)=1$, such that:

$$A(q^{-1})C(q^{-1})+q^{-1}B(q^{-1})D(q^{-1})=R(q^{-1}). \tag{5}$$

Moreover, an upper bound of the coefficients of $C(q^{-1}), D(q^{-1})$ is known.

System Reparametrization: Using Assumption A3, let

$$\begin{aligned}A(q^{-1})&=1+q^{-1}\overline{A}(q^{-1})\\C(q^{-1})&=1+q^{-1}\overline{C}(q^{-1})\\R(q^{-1})&=1+q^{-1}\overline{R}(q^{-1}).\end{aligned} \tag{6}$$

From (1), (5) we get the following:

$$\overline{R}(q^{-1})u(t)=\begin{cases}+\overline{C}(q^{-1})u(t)+D(q^{-1})y(t)+\overline{A}(q^{-1})u(t)\\+\overline{A}(q^{-1})\left[\overline{C}(q^{-1})u(t-1)+D(q^{-1})y(t-1)\right]\end{cases} \tag{7.a}$$

$$\overline{R}(q^{-1})y(t)=B(q^{-1})\left[u(t)+\overline{C}(q^{-1})u(t-1)+D(q^{-1})y(t-1)\right]. \tag{7.b}$$

Given $A(q^{-1}), B(q^{-1}), C(q^{-1}), D(q^{-1}), R(q^{-1})$, this is an implicit input–output prediction model. Given $R(q^{-1}),\{u(t)\},\{y(t)\}$, this is a bilinear observation equation in the coefficients of $A(q^{-1})$, $B(q^{-1})$, $C(q^{-1})$, $D(q^{-1})$.

Feedback Control Law: If the input $u(t)$ is generated by the causal feedback control law

$$u(t+1)=-\overline{C}(q^{-1})u(t)-D(q^{-1})y(t)+ER(q^{-1})y^M(t) \tag{8}$$

where $\{y^M(t)\}$ is an arbitrary bounded set point sequence and E is a scalar, then the resulting closed loop is

$$R(q^{-1})\left[u(t)-A(q^{-1})Ey^M(t-1)\right]=0 \tag{9}$$

$$R(q^{-1})\left[y(t)-B(q^{-1})Ey^M(t-1)\right]=0. \tag{10}$$

It is exponentially stable since $R(q^{-1})$ is exponentially stable and achieves zero tracking error if

$$1=B(1)E. \tag{11}$$

Adaptive Control Scheme: When $A(q^{-1}), B(q^{-1})$ are unknown, at each time t, we proceed in the following two steps.

1) Identification of both system and controller polynomials using prediction model (7). This gives time varying polynomials $\overline{A}(t,q^{-1}), B(t,q^{-1}), \overline{C}(t,q^{-1}), D(t,q^{-1})$.

2) Computation of the control as

$$u(t+1)=\overline{C}(t,q^{-1})u(t)-D(t,q^{-1})y(t)+E(t)R(q^{-1})y^M(t) \tag{12}$$

with

$$E(t)=\begin{cases}\dfrac{1}{B(t,1)} & \text{if } |B(t,1)|>\epsilon\\ 1 & \text{if not.}\end{cases} \tag{13}$$

III. Bilinear Estimation and Stability

Let ψ^*,θ^* be the true system and controller parameter vector, respectively,

$$\theta^*=(d_0\cdots d_{n-1}c_1\cdots c_n)' \tag{14}$$

$$\psi^*=(a_1\cdots a_nb_0\cdots b_n)'. \tag{15}$$

For some other value of these vectors say ψ,θ we may define from (7) the following prediction error vector

$$\epsilon(t,\theta,\psi)=z(t)-H(t)'F(\theta,\psi) \tag{16}$$

where

$$z(t)=\left(\overline{R}(q^{-1})u(t),R(q^{-1})y(t)\right)' \tag{17}$$

$$H(t)=\begin{pmatrix}0 & y(t)\\ & \cdots\\ & y(t-2n+1)\\ \cdots & u(t)\\ & \cdots\\ 0 & u(t-2n+1)\\ y(t-1) & 0\\ \cdots & \\ y(t-2n) & \\ & \cdots\\ u(t) & \\ \cdots & \\ u(t-2n) & 0\end{pmatrix}. \tag{18}$$

$F(\theta,\psi)$ is a vector whose entries are the coefficients of

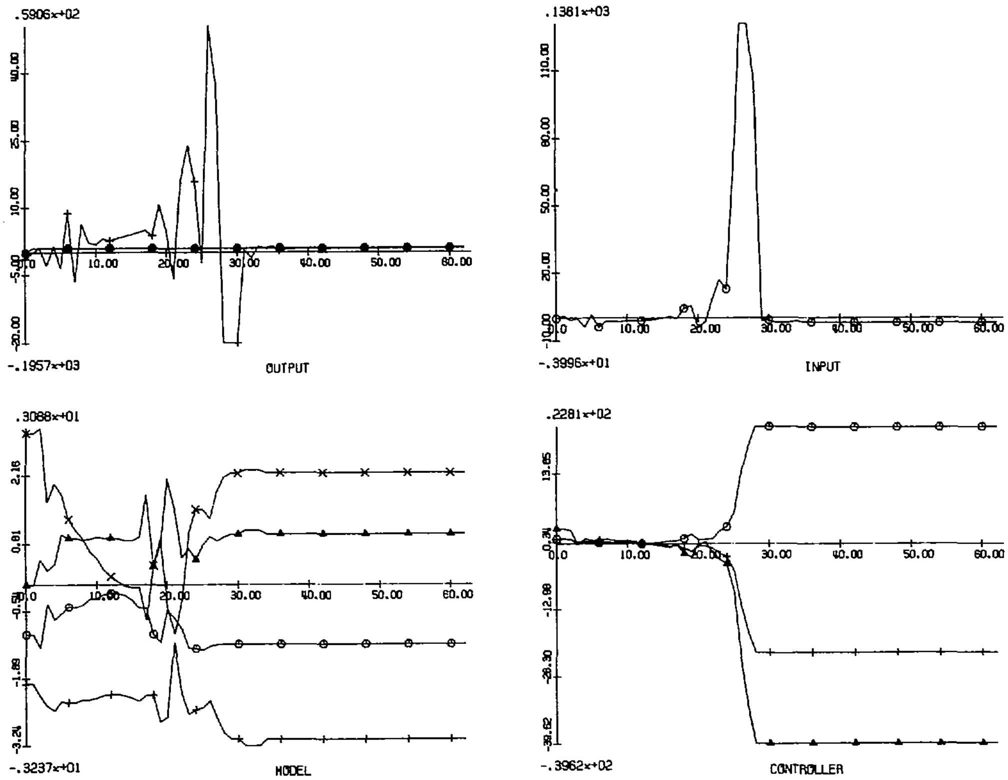

Fig. 1.

$$A(q^{-1})D(q^{-1}), A(q^{-1})C(q^{-1}), B(q^{-1})D(q^{-1}), B(q^{-1})C(q^{-1}).$$

Let $J(t,\theta,\psi)$ be the least-squares criterion with forgetting factor defined recursively as follows:

$$J(t,\theta,\psi)=\mu J(t-1,\theta,\psi)+\epsilon(t,\theta,\psi)'Q\epsilon(t,\theta,\psi), 0<\mu<1 \tag{19}$$

$$J(0,\theta,\psi)=[F(\theta,\psi)-F(\theta(0),\psi(0))]'P_o[F(\theta,\psi)-F(\theta(0),\psi(0))] \tag{20}$$

where Q, P_o are positive definite matrices, $(\theta(0),\psi(0))$ is *a priori* estimates.

Let J_o be a positive scalar given by boundedness assumption in A2, A3, such that

$$J(0,\theta^*,\psi^*)<J_o. \tag{21}$$

Let M_t be the set defined as follows:

$$M_t=\{(\theta,\psi)|J(t,\theta,\psi)<\mu^t J_o\}. \tag{22}$$

Then at time t we take the estimates $(\theta(t),\psi(t))$ as the element of M_t that minimizes $\|\theta-\theta(t-1)\|^2$, i.e., $(\theta(t),\psi(t))$ is given by solving:

$$(\theta(t),\psi(t))\in \arg\min_{(\theta,\psi)\in M_t}\|\theta-\theta(t-1)\|^2. \tag{23}$$

With this estimation procedure, the following global convergence may be proved (see [4]).

Global Stability Theorem: Subject to Assumptions A0, A1, A2, A3, let $\theta(t),\psi(t)$ be given by (22), (23), and let $u(t+1)$ be given by (12).

Then we have the following:

i) $u(t), y(t)$ remain bounded
ii) $\lim_{t\to\infty} R(q^{-1})y(t)-B(t,q^{-1})y^*(t-1)=0$ (24)
with

$$y^*(t)=E(t)R(q^{-1})y^M(t). \tag{25}$$

IV. Towards an Implementable Algorithm

The algorithm presented in the previous section is conceptual. At each time t, it requires the minimization of a quadratic criterion over an implicitly defined nonconvex set. However, it should be noted that, given θ (respectively, ψ), $\min_\psi J(t,\theta,\psi)$ [respectively, $\min_\theta J(t,\theta,\psi)$] is a classical quadratic minimization problem. Consequently, an alternate minimization in θ,ψ is a candidate for an efficient descent algorithm. This leads to the following estimation algorithm.

Algorithm: At each time t:

$$i=0, \theta_o=\theta(t-1), \psi_o=\psi(t-1)$$

1.1—$\theta_{i+1}=\arg\min_\theta J(t,\theta,\psi_i)$

1.2—$\psi_{i+1}=\arg\min_\psi J(t,\theta_{i+1},\psi)$

2—if $i>i_{\max}$ then $\theta(t)=\theta_{i+1}, \psi(t)=\psi_{i+1}$ end

—if not $i=i+1$ return to 1.

The computational complexity of this algorithm consists in 1.1 and 1.2. A positive symmetric linear system has to be computed (order of n^2 operations) and solved.

This algorithm does not solve (22), (23). However, our simulation experience indicates that it gives good performance. In particular, let us consider a very difficult example presented in [5].

The following system is considered:

$$A(q^{-1})=1.0-1.2q^{-1}$$
$$B(q^{-1})=1.0-3.1q^{-1}+2.2q^{-2}.$$

The objectives are

$$R(q^{-1})=1.0$$
$$y^M(t)=1.0 \qquad 1<t<60.$$

It follows that the controller is defined by

$$C(q^{-1}) = 1.0 + 22.8q^{-1} - 39.6q^{-2}$$
$$D(q^{-1}) = -21.6.$$

Note the large coefficients. This is due to the proximity of roots of $A(q^{-1})$, $B(q^{-1})$. We use the following conditions:

$$\mu = 0.85, i_{\max} = 14, \text{initial signals} \equiv 0$$

Initial parameters:

$$A(q^{-1}) = 1.0 - 1.0q^{-1}$$
$$B(q^{-1}) = -2.0q^{-1} + 3.0q^{-2}$$
$$C(q^{-1}) = 1.0 + 1.0q^{-1} + 3.0q^{-2}$$
$$D(q^{-1}) = 1.0.$$

P_o is obtained by simulating the closed-loop plant given by initial parameters with $y^M(t)$ a white noise and by taking

$$P_o = \sum_{t=0}^{T} \mu^{T-t} H(t) \quad H(t)'.$$

The optimization is worked out only when the *a priori* output error prediction is greater than 0.01.

Fig. 1 shows the output and input signals and the estimated parameters. We can see that the output and the parameters converge to their true values.

V. Conclusion

An adaptive direct control scheme is obtained with a pole placement as underlying design method. The characteristics of this technique are:

i) estimation of both model and controller parameters,

ii) an estimation procedure which is bilinear in these parameters and which is obtained from both an input and output prediction error model.

A conceptual least-squares criterion minimization allows one to establish global convergence with very weak assumptions: stabilizability of the system, knowledge of an upper bound of the parameters and of the system order. This is only a theoretical existence result. Nevertheless, a more implementable algorithm leads to encouraging simulation results.

References

[1] K. J. Astrom, "Direct methods for non minimum phase systems," in *Proc. IEEE 19th Conf. on Decision Contr.*, Dec. 1980.

[2] L. Ljung and E. Trulsson, "Direct minimization methods for adaptive control of non-minimum phase systems," in *Proc. 6th IFAC Symp. on Identification Syst. Parameter Estimation*, June 1982.

[3] H. Elliott, "Direct adaptive pole placement with application to nonminimum phase systems," *IEEE Trans. Automat. Contr.*, vol. AC-27, June 1982.

[4] L. Praly, "Towards a direct adaptive control scheme for not necessarily minimum phase systems," in *Proc. IEEE 22nd Conf. on Decision Contr.*, Dec. 1983.

[5] G. C. Goodwin and K. S. Sin, "Adaptive control of non minimum phase system," *IEEE Trans. Automat. Contr.*, vol. AC-26, Apr. 1981.

Computation of Additive Decompositions of ARMA Spectra

BOAZ PORAT

Abstract —**The spectra of vector autoregressive moving-average processes are shown to yield certain additive decompositions. An algorithm for computing such decompositions is given. Contrary to some other methods, the proposed algorithm is based on purely rational operations. In particular, knowledge of the poles of the spectrum is not required.**

Manuscript received December 20, 1984; revised March 9, 1984 and April 16, 1984.

The author is with the Department of Electrical Engineering, Technion—Israel Institute of Technology, Haifa, Israel.

I. Introduction

Stationary random processes whose power spectra are rational play an important role in many engineering and statistics applications. For discrete-time processes, the most general rational model is the so-called autoregressive moving average (ARMA) model. A vector random process $\{y_t\}$ of dimension m is said to be a (p, q) ARMA process if it satisfies the difference equation

$$y_t + \sum_{k=1}^{p} A_k y_{t-k} = \sum_{k=0}^{q} B_k u_{t-k} \tag{1}$$

where $\{u_t\}$ is an m-dimensional zero-mean unit-variance white noise, and $\{A_1, A_2, \cdots, A_p\}$, $\{B_0, B_1, \cdots, B_q\}$ are constant $m \times m$ matrices. All the roots of the polynomial

$$\det\left(z^p + \sum_{k=1}^{p} z^{p-k} A_k\right)$$

are required to be strictly inside the unit circle. The lth lag covariance of $\{y_t\}$ is defined by

$$R_l = R_{-l}^T = E y_t y_{t-l}^T; \qquad -\infty < l < \infty. \tag{2}$$

The spectrum of the ARMA process given in (1) is defined as

$$S(z) = \sum_{l=-\infty}^{\infty} R_l z^{-l}. \tag{3}$$

For vector processes, additive decompositions of the corresponding spectra can be defined in various ways, corresponding to different ways of splitting the matrix R_0. For our purposes it will be convenient to introduce the following definition. Let

$$R_{0+} = \begin{cases} (R_0)_{ij}; & i < j \\ \frac{1}{2}(R_0)_{ij}; & i = j \\ 0; & i > j. \end{cases} \tag{4}$$

Then define

$$S_+(z) = R_{0+} + \sum_{l=1}^{\infty} R_l z^{-l}. \tag{5}$$

$S_+(z)$ will be called the causal part of the spectrum. Clearly, $S(z)$ can be written as

$$S(z) = S_+(z) + \left[S_+(z^{-1})\right]^T. \tag{6}$$

This is known as the additive decomposition of the spectrum. Additive decompositions are useful, e.g., in Wiener filtering theory [1]. Another potential use is in spectral estimation of ARMA processes. A standard technique for obtaining additive decompositions of rational spectra is by partial fraction expansions. This requires knowledge of the poles of the spectrum and can be quite tedious to carry out in practice.

In this note we propose an efficient computational algorithm for obtaining the additive decomposition of the spectrum of a given ARMA process. The algorithm is based on a recently proposed algorithm for a related problem, namely, the computation of the covariances of a vector autoregressive (AR) process [2]. In contrast to partial fraction expansion, the new algorithm is purely rational, using only standard linear algebra operations. In Section II we state and prove the basic theorem, on which the algorithm is based. Section III briefly describes the algorithm, omit-

0018-9286/84/1000-0949$01.00 ©1984 IEEE