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Lecture Notes on

An introduction to some Lyapunov designs of global asymptotic stabilizers

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1 Introduction

1.1 What we do

This text is an introduction to some designs of control laws providing global asymptotic stability with, maybe, **disturbance attenuation**[1], for a nonlinear system of the form :

$$\dot{x} = f(x, u, d) \tag{1}$$

with state x in $\mathbb{R}^n$, control u in $\mathbb{R}^m$ and disturbance d in $L^\infty_{loc}([0,\infty), \mathbb{R}^p)$. More specifically, we are interested in those designs which follow directly from the will to make **negative** the time derivative of a **Lyapunov function** also to be designed.

Lyapunov functions are known to be a very efficient tool for stability analysis but of restricted applicability since exhibiting such a function is usually difficult. The point here is different. We are dealing with stabilization and not stability; i.e., we are dealing with synthesis and not analysis. The system we consider is under-determined, the control not being specified a priori. So a possible route is to choose a Lyapunov function first and then to specify the system by designing its control law. This scheme does not work for any Lyapunov function. Those for which it works are called **Control Lyapunov Functions** (**CLF**). This way of designing control laws is called Lyapunov design and has a long history (see for example [4, 19, 29, 40, 41, 49]).

We are dealing with global asymptotic stability. This does not mean that the equilibrium point of interest should have the whole universe as basin of attraction but, more realistically, that we impose a priori an open set, diffeomorphic to $\mathbb{R}^n$, as basin of attraction. Then we are working with the coordinates x given by this diffeomorphism so that, when $|x|$ goes to ∞, the point goes actually to the boundary of the prescribed open set (see Example 217 for an illustration).

Since we do not want here to pay any attention to smoothness, we consider controllers which are only continuous. Systems with continuous only right hand side being less in the common knowledge, we start by recalling some basic facts in section 2.

In section 3, we give a precise definition of Control Lyapunov functions and show how they allow us to design global asymptotic stabilizers. By applying straightforwardly this definition we show how asymptotic stabilization can be obtained for systems whose dynamics have a triangular form, built recursively by **adding differentiators**, and called **feedback form**. The corresponding design is the so called **backstepping** technique.

In section 4, we consider systems which are C^1 dissipative. They can be stabilized by adding damping. By generalizing this property, we can deal with systems whose dynamics have the other triangular form, built recursively by **adding integrators**, and called **feedforward form**. The corresponding design is the so called **forwarding** technique. This section 4 is a reproduction of [56] with some slight modifications.

A short glossary and some generic notations are given at the end of these notes. We do recommend at least a quick look at that section before entering the technical part of this text.

1.2 What we do not do

This text is only an introduction and only to some control designs. The topics we cover are hardly sketched but our references should allow the reader to enter them more deeply. A more

[1] The expressions in bold face are explained in the glossary at the end of these notes.

complete but compact overview can be found in [31].

Also these topics are only in the context of Lyapunov design. There are many other techniques proposed in the literature to address the problem of global asymptotic stabilization or disturbance attenuation. They rely on :

- regular or singular perturbations or averaging (see [30] for instance),
- on the small nonlinear gain theorem (see [28, 72] for instance) or passivity (see [53] for instance),
- on partial or full state feedback linearization (see [24] for instance),
- ...

We deal with the problems of asymptotic stabilization and disturbance attenuation only. The problem of performance is not considered. In other words we tackle with existence problems and not with the choice among existing solutions.

Finally we consider only static state feedback. But many results are also available in output or adaptive feedback (see [17, 35, 37, 47, 50] for instance).

As will be clear after reading the following paragraphs the design of stabilizers requires a preliminary step of finding particular state coordinates allowing us to write the system dynamics in a form with a very specific structure. We do not address this topic at all. It is referable to differential geometry while we are invoking only analysis. See the books [24, 47, 51].

We illustrate our presentation with examples. For the sake of clarity and conciseness, they are academic, although some times inspired from real world applications. But we insist on the fact that all the techniques we are presenting have been at least considered in the design of controllers tested on real world applications. Experiments of backstepping-based controllers are reported for instance for electric motors [10], control of ship position and motion [14] or vehicle control [6]. Forwarding-based controllers are used for instance for controlling downrange distance in guided atmospheric entry [52] or for swinging up 1D or 2D inverted pendulum [71].

2 Basic facts

Let $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ be a continuous function.

Definition 2
Given x in $\mathbb{R}^n$ and d in $L^\infty_{loc}([0,\infty),\mathbb{R}^p)$, $X(x,t;d)$ is called a solution of :

$$\dot{x} \;=\; f(x,d) \quad , \qquad X(x,0;d) \;=\; x \tag{3}$$

if there exists $T > 0$ such that :

- $X(x,\cdot;d) : [0,T) \to \mathbb{R}^n$ *is continuous,*
- $$X(x,t;d) \;=\; x \;+\; \int_0^t f(X(x,;d(s))ds \qquad \forall t \in [0,T) \ . \tag{4}$$

Theorem 5 ([20, Theorem I.5.1] for instance)
For each x in $\mathbb{R}^n$ and each d in $L^\infty_{loc}([0,\infty),\mathbb{R}^p)$, there exists a solution to (3).

Remark 6
The main difference with the more common case where f is Lipschitz continuous lies in the fact that there is no guarantee of uniqueness of solutions •

2.1 Case without disturbance

For the case with no input d, we have :

Definition 7
Let the ***origin*** *be a solution of :*

$$\dot{x} \;=\; f(x) \ . \tag{8}$$

It is said to be :

- *globally stable if there exists a* ***class $\mathcal{K}$*** *function α such that, for each x in $\mathbb{R}^n$, all the solutions $X(x,t)$ are defined on $[0,\infty)$ and satisfy :*

$$|X(x,t)| \;\leq\; \alpha(|x|) \qquad \forall t \geq 0 \ . \tag{9}$$

- *globally asymptotically stable if there exists a* ***class $\mathcal{KL}$*** *function β such that, for each x in $\mathbb{R}^n$, all the solutions $X(x,t)$ are defined on $[0,\infty)$ and satisfy :*

$$|X(x,t)| \;\leq\; \beta(|x|,t) \qquad \forall t \geq 0 \ . \tag{10}$$

Remark 11
Equivalent definitions of global asymptotic stability can be found for instance in [44, Remark 2.4, Proposition 2.5] and [5].

The property of global asymptotic stability is by nature robust :

Theorem 12 ([5, 33])
*The origin is a globally asymptotically stable solution of (8) if and only if there exists a continuous **positive definite** function δ such that for any function $f_\delta : \mathbb{R}^n \to \mathbb{R}^n$ which is continuous and such that :*

$$\forall x \in \mathbb{R}^n \ , \ \exists y \in \mathbb{R}^n \quad : \qquad |y - x| \ + \ |f_\delta(x) - f(y)| \ \leq \ \delta(x) \ , \tag{13}$$

the origin is also a globally asymptotically stable solution of :

$$\dot{x} \ = \ f_\delta(x) \ . \tag{14}$$

The interpretation of (13) is that global asymptotic stability is robust to measurement noise and regular perturbation of the differential equation provided they are bounded by a continuous positive definite function of the state which typically goes to 0 as $|x|$ goes to ∞.

With the help of Lyapunov functions, we have sufficient (and even necessary) conditions for the various notions of stability we have mentioned.

Definition 15
A set $\mathcal{S}$ is said to be quasi-invariant if, for all x in $\mathcal{S}$, there exists at least one solution $X(x,t)$ defined on $(-\infty, +\infty)$ taking values in $\mathcal{S}$.

Theorem 16 ([60, **Theorem 2**])
Let :

- *$\mathcal{U}$ be a non empty subset of $\mathbb{R}^n$,*
- *V be a C^1 function satisfying the inequality :*

$$\dot{\overbrace{V(x)}} \ = \ L_f V(x) \ \leq \ 0 \qquad \forall x \in \mathcal{U} \ , \tag{17}$$

- *$X(x,t)$ be a solution, defined on $[0, +\infty)$, bounded and taking values in $\mathcal{U}$.*

Then there exists a real number v such that, when t tends to $+\infty$, $X(x,t)$ tends to the largest quasi-invariant set contained in

$$\{x \in \texttt{closure}(\mathcal{U}) : \ V(x) = v \, , \ L_f V(x) = 0\}$$

*Moreover, if V is positive definite (respectively **proper**), then the origin is (respectively globally) stable.*

Definition 18
The system (8) with continuous output function $h(x)$ is said zero-state detectable if each solution $X(x,t)$, right maximally defined on $[0,T)$, for some $T > 0$, which satisfies :

$$h(X(x,t)) \ = \ 0 \qquad \forall t \in [0,T) \tag{19}$$

is actually defined on $[0, +\infty)$ and converges to the origin as t goes to $+\infty$.

Theorem 20 ([25, p.44])
If the system is zero-state detectable with h as continuous output function and V is a C^1 Lyapunov function satisfying :

$$\overset{\cdot}{\widehat{V(x)}} \;=\; L_f V(x) \;\leq\; -|h(x)| \qquad \forall x \in \mathbb{R}^n \;, \tag{21}$$

then the origin is globally asymptotically stable.

Conversely, we have :

Theorem 22 ([39])
If the origin is globally asymptotically stable, there exists a C^1 Lyapunov function V satisfying :

$$\overset{\cdot}{\widehat{V(x)}} \;\leq\; -V(x) \qquad \forall x \in \mathbb{R}^n \;. \tag{23}$$

2.2 Case with disturbance

When there is an input d, we can quantify its action on solutions as follows :

Definition 24
The system :

$$\dot{x} \;=\; f(x,d) \tag{25}$$

is said to be Input to State Stable (ISS) if there exists a function γ of class $\mathcal{K}$, called the nonlinear L^∞ gain, and a function β of class $\mathcal{KL}$ such that, for each d in $L^\infty_{loc}([0,\infty),\mathbb{R}^p)$ and each x in $\mathbb{R}^n$, all the solutions $X(x,t;d)$ are defined on $[0,\infty)$ and satisfy :

$$|X(x,t;d)| \;\leq\; \max\left\{\beta(|x|,t)\,,\,\gamma\left(\left\| \,d|_{[0,t]}\right\|_\infty\right)\right\} \qquad \forall t\geq 0\;. \tag{26}$$

Example 27 : An ISS system
Consider the system :

$$\left\{\begin{array}{rcl} \dot{x}_1 &=& -x_1^3 \,+\, d^3 \\ \dot{x}_2 &=& -x_2 \,+\, x_3 \,+\, x_1\, d \\ \dot{x}_3 &=& -x_2 \end{array}\right. \tag{28}$$

Let us show that it is ISS.

We look first at the x_1 subsystem. With **Young's inequality** (605), we get :

$$\overset{\cdot}{\widehat{\tfrac{1}{2}x_1^2}} \;=\; -x_1^4 \,+\, x_1\, d^3 \;, \tag{29}$$

$$\leq\; -\tfrac{3}{4}\left[1-\left(\tfrac{1}{2}\right)^4\right] x_1^4 \;-\; \tfrac{3}{4}\left(\tfrac{1}{2}\right)^4\left(x_1^4 - 2^4\, d^4\right)\;. \tag{30}$$

We observe that $\left\|\, d|_{[0,t]}\right\|_\infty$ is a non decreasing function of time and that any solution y of the differential inequality :

$$\overset{\cdot}{\widehat{\tfrac{1}{2}y^2}} \;\leq\; -\left[1-\left(\tfrac{1}{2}\right)^4\right]\left(\tfrac{1}{2}y^2\right)^2 \;, \tag{31}$$

satisfy :

$$|y(t)| \leq \sqrt{\frac{y(0)^2}{2+3\left[1-\left(\frac{1}{2}\right)^4\right] y(0)^2 t}} \ . \qquad (32)$$

It follows that all the solutions of the x_1 subsystem satisfy (see [66]) :

$$|X_1(x_1,t;d)| \leq \max\left\{\sqrt{\frac{x_1^2}{2+3\left[1-\left(\frac{1}{2}\right)^4\right] x_1^2 t}}, 2\left\| d|_{[0,t]}\right\|_\infty\right\} \ . \qquad (33)$$

So this subsystem is ISS with a nonlinear L^∞ gain :

$$\gamma_1(s) \leq 2\,s \ . \qquad (34)$$

We consider now the (x_2,x_3) subsystem with input (x_1,d). We let :

$$V(x_2,x_3) = x_2^2 - x_2 x_3 + x_3^2 \ . \qquad (35)$$

By using the inequality :

$$x_2^2 + x_3^2 \leq 2V \ , \qquad (36)$$

and **completing the squares**, we get the derivative :

$$\begin{aligned}
\overbrace{V(x_2,x_3)}^{\cdot} &= -V(x_2,x_3) + (2x_2 - x_3)\, x_1\, d \ , && (37)\\
&\leq -V(x_2,x_3) + \tfrac{1}{4}\left(x_2^2 + x_3^2\right) + 5\, x_1^2\, d^2 \ , && (38)\\
&\leq -\tfrac{1}{2}\, V(x_2,x_3) + 5\, x_1^2\, d^2 \ , && (39)\\
&\leq -\tfrac{1}{2}\, V(x_2,x_3) + \tfrac{5}{6}\, x_1^4 + \tfrac{15}{2}\, d^4 \ . && (40)
\end{aligned}$$

With the notations :

$$V(t) = V(X_2(x_2,x_3,t;x_1,d), X_3(x_2,x_3,t;x_1,d)) \quad , \qquad x_1(t) = X_1(x_1,t;d) \ , \qquad (41)$$

it follows :

$$\begin{aligned}
V(t) &\leq \exp(-t/2)\, V(0) + \int_0^t \exp(-(t-s)/2)\left[\tfrac{5}{6}\, x_1(s)^4 + \tfrac{15}{2}\, d^4(s)\right] ds \ , && (42)\\
&\leq \exp(-t/2)\, V(0) && (43)\\
&\qquad + \int_0^t \exp(-(t-s)/2)\left[\tfrac{5}{6}\max\left\{\beta_1(|x_1|,t)^4, 16\left(\left\| d|_{[0,t]}\right\|_\infty\right)^4\right\} + \tfrac{15}{2}\, d^4(s)\right] ds \ , \\
&\leq \left(\exp(-t/2)\, V(0) + \tfrac{5}{6}\int_0^t \exp(-(t-s)/2)\beta_1(|x_1|,t)^4\right) + 47\left(\left\| d|_{[0,t]}\right\|_\infty\right)^4 \ . && (44)
\end{aligned}$$

With (36), we conclude that there exists a function β_{23} of class $\mathcal{KL}$ such that, for all the solutions $X_{23}(x_2,x_3,t;x_1,d)$, we have :

$$|X_{23}(x_2,x_3,t;x_1,d)| \leq \beta_{23}((|x_1|+|x_2|+|x_3|),t) + 9\left(\left\| d|_{[0,t]}\right\|_\infty\right)^2 \ . \qquad (45)$$

Collecting (33) and (45), we have (26); i.e., the system (28) is ISS.

To summarize, the key point of this example is the use of Lyapunov functions to exhibit the ISS property (see Theorem 75) •

Remark 46

1. Other ways of quantifying the action of disturbances on solutions have been proposed and studied (see for instance [68] for the deterministic case and [35] for the stochastic case).

2. In the following we are dealing with systems in the form :

$$\dot{x} \;=\; f(x,u,d) \tag{47}$$

and we are looking for continuous functions $\phi : \mathbb{R}^n \to \mathbb{R}^m$ such that the the closed loop system

$$\dot{x} \;=\; f(x,\phi(x),d) \tag{48}$$

is ISS with a nonlinear L^∞ gain γ as small as possible. This is the problem of disturbance attenuation. We shall look at it only in the L^∞ case (i.e. the ISS framework). But it has received attention in others cases like the L^2 case (see for instance [24, Section 9.5] or [17, 35, 54, 61, 75] and the references therein) •

With the ability of designing controllers attenuating the effect of a disturbance, we can tackle the more complex global asymptotic stabilization problem. In particular we can "hide" in d things which are too intricate or poorly known, namely we work with a model of reduced complexity (see Example 342). To check if such an approach is successful we can apply the following small gain Theorem (see also [25, 23]) :

Theorem 49 ([28])
Consider the following two systems :

$$\begin{aligned} \dot{x} &= f(x,d)\ , && (50)\\ \dot{y} &= g(y,e)\ . && (51) \end{aligned}$$

Assume they are ISS. In particular, given two continuous functions h and k which are zero at the origin, let β_e and β_d, of class $\mathcal{KL}$, and γ_e and γ_d, of class $\mathcal{K}$, be such that, for each d in $L^\infty_{loc}([0,\infty),\mathbb{R}^{p_d})$, each e in $L^\infty_{loc}([0,\infty),\mathbb{R}^{p_e})$, each x in $\mathbb{R}^n$ and each y in $\mathbb{R}^m$, all the solutions $X(x,t;d)$ and $Y(y,t;e)$ are defined on $[0,\infty)$ and satisfy, for almost all $t \geq 0$:

$$\begin{aligned} |h(Y(y,t;e),e(t))| &\leq \max\left\{\beta_d(|y|,t)\,,\,\gamma_d\left(\left\|\,e|_{[0,t]}\right\|_\infty\right)\right\}\ , && (52)\\ |k(X(x,t;d))| &\leq \max\left\{\beta_e(|x|,t)\,,\,\gamma_e\left(\left\|\,d|_{[0,t]}\right\|_\infty\right)\right\}\ . && (53) \end{aligned}$$

Under these conditions, if :

$$\gamma_e(\gamma_d(s)) \;<\; s \qquad (\textit{resp. } \gamma_d(\gamma_e(s)) \;<\; s) \qquad \forall s > 0\ , \tag{54}$$

then the origin is a globally asymptotically stable solution of the interconnection :

$$\left\{\begin{array}{rcl} \dot{x} &=& f(x,d)\ ,\\ e &=& k(x)\ , \end{array}\right. \quad , \quad \left\{\begin{array}{rcl} \dot{y} &=& g(y,e)\ ,\\ d &=& h(y,e)\ . \end{array}\right. \tag{55}$$

Remark 56
Local versions of this result are available. See [74, 72] for instance.

Example 57 : Application of the small gain Theorem
Consider the system :

$$\begin{cases} \dot{x}_1 &= -x_1 + x_1^3 x_2 \,, \\ \dot{x}_2 &= -x_2 - x_1^6 x_2 + y^2 \,, \\ \dot{y} &= -y^3 + \frac{1}{2} |x_1|^{\frac{3}{2}} \,. \end{cases} \tag{58}$$

Let us show that the origin is a globally asymptotically stable solution by applying the small gain Theorem[2].

To do this we look at the system (58) as being made of the interconnection of the x and y subsystems.

To establish the ISS property for the x subsystem, we let :

$$V_x(x_1, x_2) = \frac{1}{2} \left(x_1^2 + x_2^2 \right) \,. \tag{59}$$

We get, by completing the squares,

$$\begin{aligned} \dot{\overbrace{V_x(x_1, x_2)}} &= -\left(x_1^2 + x_2^2 \right) + x_1 \left(x_1^3 x_2 \right) - x_1^6 x_2^2 + x_2 \, y^2 \,, &(60)\\ &\leq -\frac{1}{2} x_1^2 - \frac{1}{2} x_2^2 - \frac{1}{2} x_1^6 x_2^2 + \frac{1}{2} y^4 \,, &(61)\\ &\leq -V_x + \frac{1}{2} y^4 \,. &(62) \end{aligned}$$

This implies, with the same notation as in (41),

$$V_x(t) \leq \exp(-t) \, V_x(0) + \frac{1}{2} \int_0^t \exp(-(t-s)) \, y(s)^4 ds \,. \tag{63}$$

This establishes that the x subsystem is ISS and proves the existence of a function β_e of class $\mathcal{KL}$ such that, for all the solutions and all positive times, we have :

$$|X_1((x_1, x_2); y, t)| \leq \max \left\{ \beta_e(|x_1| + |x_2|, t) \,,\, \gamma_e \left(\left\| \, y|_{[0,t]} \right\|_\infty \right) \right\} \,, \tag{64}$$

where :

$$\gamma_e(s) = s^2 \,. \tag{65}$$

Similarly, for the y subsystem, we let :

$$V_y(y) = \frac{1}{2} y^2 \,. \tag{66}$$

Young's inequality (605) says :

$$y \, |x_1|^{\frac{3}{2}} \leq \frac{1}{4} y^4 + \frac{3}{4} x_1^2 \,. \tag{67}$$

[2] Global asymptotic stability can also be established with the Lyapunov function $x_1^2 + x_2^2 + y^2$. But this is not the point here.

This implies, for any ε in $(0,1)$,

$$\begin{aligned}
\overbrace{V_y(y)}^{\cdot} &= -y^4 + \frac{1}{2}\, y\, |x_1|^{\frac{3}{2}} , && (68)\\
&\leq -\frac{7}{8} y^4 + \frac{3}{8} x_1^2 , && (69)\\
&\leq -\frac{7\varepsilon}{2} V_y(y)^2 - \frac{7(1-\varepsilon)}{2} \left(V_y(y)^2 - \frac{3}{28(1-\varepsilon)} x_1^2 \right) . && (70)
\end{aligned}$$

This establishes that the y subsystem is ISS. In particular this implies the existence of a function β_d of class $\mathcal{KL}$ such that, for all the solutions and all positive times, we have :

$$|Y(y,t;x_1)| \leq \max\left\{ \beta_d(|y|,t) \,,\, \gamma_d\left(\left\| \, x_1|_{[0,t]} \right\|_\infty \right) \right\} . \tag{71}$$

where :

$$\gamma_d(s) = \left(\frac{3}{7(1-\varepsilon)} \right)^{\frac{1}{4}} s^{\frac{1}{2}} . \tag{72}$$

Having established the inequalities (53) and (52), it remains to check the small gain condition (54). We have, for s strictly positive,

$$\begin{aligned}
\gamma_e(\gamma_d(s)) &= \left(\left(\frac{3}{7(1-\varepsilon)} \right)^{\frac{1}{4}} s^{\frac{1}{2}} \right)^2 , && (73)\\
&= \left(\frac{3}{7(1-\varepsilon)} \right)^{\frac{1}{2}} s < s . && (74)
\end{aligned}$$

With Theorem 49, this implies that the origin is globally asymptotically stable for the system (58) •

As for asymptotic stability, Lyapunov functions are an efficient tool to deal with problems on ISS. For example, we have :

Theorem 75 ([69])
The system (25) is ISS if and only if there exist a C^1 Lyapunov function V, a class $\mathcal{K}^\infty$ function[3] a and a class $\mathcal{K}$ function b such that :

$$\overbrace{V(x)}^{\cdot} = \frac{\partial V}{\partial x}(x)\, f(x,d) \leq -a(V(x)) + b(|d|) \qquad \forall (x,d) \in \mathbb{R}^n \times \mathbb{R}^p . \tag{76}$$

More specifically, the above inequality implies that the nonlinear L^∞ gain is smaller than the function $\alpha^{-1} \circ a^{-1} \circ b$, where α is a class $\mathcal{K}^\infty$ function satisfying :

$$\alpha(|x|) \leq V_x(x) . \tag{77}$$

We have already used this result in Example 27 to show that the system (28) is ISS. In this context of the ISS property written in terms of Lyapunov functions, we have another version of the small gain Theorem which is helpful for design (see Example 154).

[3] Without loss of generality, we can take $a(s) = s$ (see [59]).

Theorem 78 ([58])
Assume the y system (51) is ISS. In particular, given a continuous function h which are zero at the origin, let β_d, of class $\mathcal{KL}$ and γ_d, of class $\mathcal{K}$ be such that, for each e in $L^\infty_{loc}([0,\infty),\mathbb{R}^p)$ and each y in $\mathbb{R}^n$, all the solutions $Y(y,t;e)$ are defined on $[0,\infty)$ and satisfy, for almost all $t \geq 0$:

$$|h(Y(y,t;e),e(t))| \;\leq\; \max\left\{\beta_d(|y|,t)\,,\,\gamma_d\left(\left\|\, e\big|_{[0,t]}\right\|_\infty\right)\right\}\;. \tag{79}$$

For the x system (50), assume there exist V_x, a C^1 Lyapunov function, γ_x, a class $\mathcal{K}$ function, and λ, a strictly positive real number, satisfying :

$$\overbrace{V_x(x)}^{\cdot} \;=\; \frac{\partial V_x}{\partial x}(x)\,f(x,d) \;\leq\; -\lambda\,V_x(x) \;+\; \gamma_x(|d|) \qquad \forall (x,d)\in\mathbb{R}^n\times\mathbb{R}^p\;. \tag{80}$$

Under these conditions, if, for some ε in $(0,1)$, we have :

$$\gamma_x(\gamma_d(|k(x)|)) \;\leq\; (1-\varepsilon)\,\lambda\,V_x(x) \qquad \forall x\in\mathbb{R}^n\;, \tag{81}$$

then the origin is a globally asymptotically stable solution of the interconnection (55).

3 Control Lyapunov functions and application to systems in feedback form

3.1 Control Lyapunov functions (CLF)

Consider a continuous control system which is affine in the control u :

$$\dot{x} \;=\; f(x) \;+\; g(x)\,u\;. \tag{82}$$

Let V be a C^1 Lyapunov function. Its derivative along the solutions of (82) is :

$$\dot{\overbrace{V(x)}} \;=\; L_fV(x) \;+\; L_gV(x)\,u\;. \tag{83}$$

For each point x where $|L_gV(x)|$ is not zero, this derivative can be made strictly negative for instance by taking the control as :

$$u \;=\; -\frac{L_fV(x)+|x|}{|L_gV|^2}\,L_gV(x)^T\;. \tag{84}$$

But for all x where $|L_gV(x)]$ is zero, the control has no action on this derivative and we get :

$$\dot{\overbrace{V(x)}} \;=\; L_fV(x)\;. \tag{85}$$

It follows that, for the given Lyapunov function V to be eligible to get a global asymptotic stabilizer, we must have at least the implication :

$$|L_gV(x)| \;=\; 0 \qquad\Longrightarrow\qquad L_fV(x) \;\le\; 0\;. \tag{86}$$

In other words, the Lyapunov function V must be such that the restriction of the function L_fV to the set $\{x:\ |L_gV(x)|=0\}$ has non positive values.

Definition 87
A C^1 Lyapunov function V is called a Control Lyapunov Function (CLF) for the system (82) if we have :

$$\{x\;\neq\;0\;,\;|L_gV(x)|\;=\;0\}\qquad\Longrightarrow\qquad L_fV(x)\;<\;0\;. \tag{88}$$

Definition 89
A C^1 Lyapunov function V is said to satisfy the Small Control Property (SCP) for the system (82) if we have[4] :

$$\limsup_{|x|\to 0}\frac{L_fV(x)}{|L_gV(x)|}\;\le\;0\;. \tag{90}$$

Example 91 : Verification of the CLF property and the SCP
Consider the system :

$$\left\{\begin{array}{rcl}\dot{x}_1 &=& x_1\,x_2\;,\\ \dot{x}_2 &=& -x_2\;+\;u\;,\end{array}\right. \tag{92}$$

[4] In (90), the limit may very well be $-\infty$.

and the Lyapunov function :

$$V(x_1,x_2) \;=\; \tfrac{1}{2}\left(x_1^2+(x_2+x_1^2)^2\right) \ . \tag{93}$$

Let us check that V is a CLF. We have :

$$\left\{\begin{array}{lcl} L_fV(x_1,x_2) &=& x_1^2x_2 \;+\; (x_2+x_1^2)(-x_2+2x_1^2x_2)\ , \\ L_gV(x_1,x_2) &=& x_2+x_1^2\ . \end{array}\right. \tag{94}$$

It follows that $L_gV(x_1,x_2)$ is zero if and only if :

$$x_2 \;=\; -x_1^2\ . \tag{95}$$

In this case, we have :

$$L_fV(x_1,x_2) \;=\; -x_2^2 \tag{96}$$

which, with (95), is strictly negative when (x_1,x_2) is not at the origin.

Also we have :

$$L_fV(x_1,x_2) \;=\; -x_2^2 \;-\; 2x_1^4\,L_gV(x_1,x_2) \;+\; 2x_1^2\,L_gV(x_1,x_2)^2\ . \tag{97}$$

It follows that the SCP holds •

Theorem 98 ([65, 17, 21])
Let V be a CLF for the system (82).

- *If V satisfies the SCP, the functions ϕ_S and ϕ_F below are continuous and give global asymptotic stabilizers :*

$$\left\{\begin{array}{lcll} \phi_S(x) &=& \phi_F(x) \;=\; 0 & \text{if}\;\; |B| \;=\; 0\ , \\ \phi_S(x) &=& -\dfrac{A+\sqrt{A^2+|B|^4}}{|B|^2}\,B^T & \text{if}\;\; |B| \;\neq\; 0\ , \\ \phi_F(x) &=& -\dfrac{\max\{A+|B|^2\,,\,0\}}{|B|^2}\,B^T & \text{if}\;\; |B| \;\neq\; 0\ , \end{array}\right. \tag{99}$$

with the notation :

$$A \;=\; L_fV(x) \quad , \qquad B \;=\; L_gV(x)\ . \tag{100}$$

- *If there exists $\alpha>1$ such that :*

$$\limsup_{|x|\to 0} \frac{L_fV(x)}{|L_gV(x)|^\alpha} \;<\; +\infty\ , \tag{101}$$

then there exists a C^1 positive definite and proper function ℓ whose derivative is positive and such that :

$$\phi_L(x) \;=\; -|L_g\ell(V)(x)|^{\alpha-2}\,L_g\ell(V)(x)^T \tag{102}$$

is a continuous global asymptotic stabilizer.

Remark 103

1. Theorem 98 has been extended to the case where the control is subject to some magnitude limitations (see [45] and the references therein).

2. In [12] it is shown that the condition (88) can be relaxed. Specifically, the result in Theorem 98 is still right if the Lyapunov function V satisfies only :

$$\limsup_{|L_gV(x)|\to 0} \frac{L_fV(x)}{|L_gV(x)|} \;\leq\; 0 \; . \tag{104}$$

But then we must add that, for each non negative real number v, the largest quasi invariant set of :

$$\dot{x} \;=\; f(x) \tag{105}$$

contained in the set :

$$\{x\in\mathbb{R}^n \;:\; V(x)\leq v\,,\; L_fV(x)=|L_gV(x)|=0\}$$

is reduced to the origin.

3. The controller (102) is often called an **L_gV-controller** or a **gradient controller** (see [34, 64]). When $\alpha=2$, its interest lies in the fact that the system :

$$\left\{\begin{array}{rcl} \dot{x} &=& f(x) \;+\; g(x)\,[\phi_L(x)+v] \;, \\ y &=& \phi_L(x) \;, \end{array}\right. \tag{106}$$

with input v and output y is strictly **passive**. This property guarantees for instance robustness of the global asymptotic stabilization to some neglected actuator dynamics. However, with the presence of ℓ' which has to be large enough, an L_gV controller requires a higher effort (see [64, Example 3.36]). For more details, see [64] (see also [21]).

4. If SCP or (101) does not hold, we may loose continuity at the origin and even local boundedness of the controllers. But these controllers may still be used to make a neighborhood of the origin globally asymptotically stable •

Example 107 : Lyapunov design from a CLF
For the system (92), we have seen in Example 91 that :

$$V(x_1,x_2) \;=\; \tfrac{1}{2}\left(x_1^2+(x_2+x_1^2)^2\right) \tag{108}$$

is a CLF which meets the SCP. We conclude from Theorem 98 that the functions :

$$\left\{\begin{array}{rcll} \phi_S(x) &=& 0 & \text{if} \quad |x_2+x_1^2| \;=\; 0\;, \\ \phi_S(x) &=& -\frac{x_1^2x_2+(x_2+x_1^2)(-x_2+2x_1^2x_2)+\sqrt{[x_1^2x_2+(x_2+x_1^2)(-x_2+2x_1^2x_2)]^2+|x_2+x_1^2|^4}}{x_2+x_1^2} & \text{if} \quad |x_2+x_1^2| \;\neq\; 0\;, \end{array}\right. \tag{109}$$

or :

$$\left\{\begin{array}{rcll} \phi_F(x) &=& 0 & \text{if} \quad |x_2+x_1^2| \;=\; 0\;, \\ \phi_F(x) &=& -\frac{\max\{x_1^2x_2+(x_2+x_1^2)(-x_2+2x_1^2x_2)+(x_2+x_1^2)^2\,,\,0\}}{x_2+x_1^2} & \text{if} \quad |x_2+x_1^2| \;\neq\; 0\;, \end{array}\right. \tag{110}$$

are continuous and give global asymptotic stabilizers. We can also take advantage of the particular decomposition (97) of $L_f V$ to design a controller from the CLF. Indeed, we get :

$$\overbrace{V(x_1,x_2)}^{\cdot} \;=\; -x_2^2 \;+\; L_gV(x_1,x_2)\left(-2x_1^4 + 2x_1^2 L_gV(x_1,x_2) + u\right) \;. \tag{111}$$

So :

$$\phi(x_1,x_2) \;=\; -\left(-2x_1^4 + 2x_1^2 L_gV(x_1,x_2)\right) \;-\; L_gV(x_1,x_2) \tag{112}$$

is a continuous global asymptotic stabilizer.

The steps above are typical in Lyapunov design. Namely, they consist in writing an upperbound for the derivative $\dot{V}$ as the sum :

$$\dot{V} \;\leq\; T_- \;+\; L_gV \;\times\; (u \;+\; T\,) \;, \tag{113}$$

where T_- is a non positive term and T is an arbitrary term. Such a decomposition is not unique. For instance, (111) can also be written as :

$$\overbrace{V(x_1,x_2)}^{\cdot} \;=\; -x_1^4 \;+\; L_gV(x_1,x_2)\left(x_1^2 - x_2 + 2x_1^2 x_2 + u\right) \;. \tag{114}$$

From such decompositions, we can proceed with **cancellation** as above. Namely the control cancels all the terms in factor of L_gV and add an extra negative term multiplied by L_gV :

$$u \;=\; -\,T \;-\; Q_- \;\times\; L_gV \;. \tag{115}$$

where Q_- is a non positive term. We can also take advantage of inequalities like (605) and proceed with **domination**. For instance :

- in the decomposition (111), we see that there is no need to cancel the product

  $$L_gV(x_1,x_2)\left(-2x_1^4 + 2x_1^2 L_gV(x_1,x_2)\right)$$

  when it is non positive. So a domination design gives (compare with (110)) :

  $$\phi(x_1,x_2) = -\frac{\max\{L_gV(x_1,x_2)\left(-2x_1^4 + 2x_1^2 L_gV(x_1,x_2)\right) + L_gV(x_1,x_2)^2\,,\,0\}}{L_gV(x_1,x_2)} \;. \tag{116}$$

- in the decomposition (114), by completing the squares, the product $L_gV x_1^2$ can be upperbounded as :

  $$L_gV(x_1,x_2)\,x_1^2 \;\leq\; \tfrac{1}{2}\,x_1^4 \;+\; \tfrac{1}{2}\,(L_gV(x_1,x_2))^2 \;. \tag{117}$$

  So we get the inequality :

  $$\overbrace{V(x_1,x_2)}^{\cdot} \;\leq\; -\tfrac{1}{2}x_1^4 \;+\; L_gV(x_1,x_2)\left(\tfrac{1}{2}L_gV(x_1,x_2) - x_2 + 2x_1^2 x_2 + u\right) \;. \tag{118}$$

  Hence another global asymptotic stabilizer is :

  $$\phi(x_1,x_2) \;=\; -\left(L_gV(x_1,x_2) - x_2 + 2x_1^2 x_2\right) \;. \tag{119}$$

Now to know if there is an L_gV or gradient controller, we check whether or not condition (101) holds. We observe from the decomposition (97) that we have :

$$L_fV(x_1,x_2) \;=\; -\left(x_2^2+2x_1^4x_2+2x_1^6\right) \;+\; 2x_1^2\left(L_gV(x_1,x_2)\right)^2 \,. \tag{120}$$

The first term in parentheses in the right hand side is a positive definite quadratic form of x_2 for all $|x_1|<\sqrt{2}$. This implies :

$$\limsup_{|x_1|+|x_2|\to 0} \frac{L_fV(x_1,x_2)}{|L_gV(x_1,x_2)|^2} \;\leq\; \limsup_{|x_1|+|x_2|\to 0} 2x_1^2 \;=\; 0 \,. \tag{121}$$

So, from (102), we get the following L_gV or gradient controller :

$$\phi_L(x_1,x_2) \;=\; -(x_2+x_1^2)\ell'(V(x_1,x_2)) \,, \tag{122}$$

where the function ℓ' is to be found. By using the inequality (605), it can be shown that, with this controller, the derivative $\dot{\overbrace{V(x_1,x_2)}}$ in (114), is negative definite if $\ell'(V)$ satisfies the following constraint :

$$(1+x_2)^2 \;<\; 1 \;+\; \ell'(V(x_1,x_2)) \,. \tag{123}$$

On the other hand, we get from (108) :

$$(1+x_2)^2 \;\leq\; 2 \;+\; 4\,V(x_1,x_2) \;+\; V(x_1,x_2)^2 \,. \tag{124}$$

So a possible expression for ℓ' is :

$$\ell'(v) \;=\; v^2 \;+\; 4\,v \;+\; 3 \,. \tag{125}$$

To summarize, the key points of this example are :

- the decomposition of the derivative $\dot{V}$ as in (113).
- The design of the asymptotic stabilizer from (113) via a cancellation design as in (115) or via a domination design obtained by further upperbounding the right hand side of (113) with in particular the possibility of getting an L_gV or gradient controller •

For systems without disturbance, we have seen that a CLF allows us to design global asymptotic stabilizers. For the case with disturbance, the same procedure can be used (see also [37, Lemma 5.2]) :

Theorem 126 ([70])
Let V be a CLF for the system (82) which satisfies also the SCP. There exists a continuous function ϕ which makes the following closed loop system ISS :

$$\dot{x} \;=\; f(x) \;+\; g(x)\,\phi(x) \;+\; p(x)\,d \,, \tag{127}$$

if and only if we have :

$$\lim_{\substack{|x|\to\infty \\ L_gV(x)=0}} \; -\frac{L_fV(x)}{|L_pV(x)|} \;=\; +\infty \,. \tag{128}$$

Remark 129

1. The proof of this Theorem is constructive in that the condition (128) guarantees the existence of a class $\mathcal{K}^\infty$ function α such that :

$$\{x \neq 0\,,\ |L_gV(x)| = 0\} \qquad \Longrightarrow \qquad L_fV(x) + |L_pV(x)|\,\alpha(|x|) < 0\,. \tag{130}$$

From there, the controller is given for instance by ϕ_S or ϕ_F in Theorem 98 with A given now by :

$$A = L_fV(x) + |L_pV(x)|\,\alpha(|x|)\,. \tag{131}$$

2. For the general case where the system is not affine in d as in (127), the condition (128) is more involved. Those Lyapunov functions which still give rise to a controller design are called robust control Lyapunov functions (RCLF). They have been introduced and studied in [17] •

Example 132 : Making a system ISS by feedback
For the system :

$$\left\{\begin{array}{rcl} \dot{x}_1 &=& x_1\,x_2 + d\,, \\ \dot{x}_2 &=& -x_2 + u\,, \end{array}\right. \tag{133}$$

we look for a continuous control law making the closed loop system ISS.

From Example 91, we know that the Lyapunov function :

$$V(x_1,x_2) = \tfrac{1}{2}\left(x_1^2 + (x_2+x_1^2)^2\right) \tag{134}$$

is a CLF satisfying SCP (for the undisturbed system). So let us see if condition (128) holds.

We have :

$$\left\{\begin{array}{rcl} L_fV(x_1,x_2) &=& x_1^2x_2 + (x_2+x_1^2)(-x_2+2x_1^2x_2)\,, \\ L_gV(x_1,x_2) &=& x_2+x_1^2\,, \\ L_pV(x_1,x_2) &=& x_1 + 2\,x_1\,(x_2+x_1^2)\,. \end{array}\right. \tag{135}$$

Hence, for each point (x_1,x_2) so that $L_gV(x_1,x_2)$ is zero, we have :

$$L_fV(x_1,x_2) = -x_1^4 \quad , \qquad L_pV(x_1,x_2) = x_1 \tag{136}$$

This implies :

$$\lim_{\substack{|x|\to\infty \\ L_gV(x)=0}} -\frac{L_fV(x)}{|L_pV(x)|} = \lim_{|x_1|\to\infty} |x_1|^3 = +\infty\,. \tag{137}$$

So we know the existence of an appropriate feedback. To get an expression, we write (see (114)), with Young's inequality (605),

$$\begin{array}{rcl} \overset{\cdot}{\overbrace{V(x_1,x_2)}} &=& -x_1^4 + x_1\,d + L_gV(x_1,x_2)\left(x_1^2 - x_2 + 2x_1^2x_2 + u + 2x_1d\right)\,, \qquad (138) \\ &\leq& -\varepsilon\,x_1^4 + L_gV(x_1,x_2)\left(x_1^2 - x_2 + 2x_1^2x_2 + u + \dfrac{27}{256\varepsilon^3}x_1^4L_gV(x_1,x_2)^3\right) + a\,|d|^{\frac{4}{3}}\,, \end{array}$$

with ε any real number in $(0,1)$ and

$$a \;=\; \left(\frac{3}{4[4(1-\varepsilon)]^{\frac{1}{4}}} + \varepsilon\right) \; . \tag{139}$$

It follows that, by picking the control as :

$$u \;=\; \phi(x_1,x_2) \;=\; -\left(x_1^2 - x_2 + 2x_1^2 x_2 + \frac{27}{256\varepsilon^3} x_1^4 L_g V(x_1,x_2)^3\right) \;-\; \varepsilon\, L_g V(x_1,x_2)^3 \; , \tag{140}$$

we get :

$$\begin{aligned} \dot{\overbrace{V(x_1,x_2)}} \;&\leq\; -\varepsilon\left(x_1^4 + L_g V(x_1,x_2)^4\right) \;+\; a\,|d|^{\frac{4}{3}} \; , && (141) \\ &\leq\; -2\,\varepsilon\, V(x_1,x_2)^2 \;+\; a\,|d|^{\frac{4}{3}} \; . && (142) \end{aligned}$$

Then, since we have :

$$|x_1| \;+\; |x_2| \;\leq\; 2\,V(x_1,x_2) \;+\; 2\sqrt{2\,V(x_1,x_2)} \; , \tag{143}$$

Theorem 75 gives that the nonlinear L^∞ gain γ of the closed loop system satisfies :

$$\gamma(s) \;\leq\; \sqrt{2a}\, s^{\frac{2}{3}} \;+\; 2\,(2a)^{\frac{1}{4}}\, s^{\frac{1}{3}} \; . \tag{144}$$

Note that this gain cannot be made arbitrarily small since a is lower bounded by $\frac{3}{4\sqrt{2}}$ •

What is again apparent with the condition (128), is that what can be achieved with a Lyapunov function is completely dictated by the restriction to the set $\{x : \; L_g V(x) = 0\}$. This has been made precise in [75]. In particular, if $L_p V(x)$ is zero when $L_g V(x)$ is zero, then there is no limitation in the disturbance attenuation. Precisely, we have :

Theorem 145 **([59])**
Let V be a CLF for the system (82) which satisfies also the SCP. If there exists a continuous function $\rho : \mathbb{R}^n \to [0,+\infty)$ such that :

$$|L_p V(x)| \;\leq\; |L_g V(x)|\, \rho(x) \; , \tag{146}$$

then, for any class $\mathcal{K}^\infty$ functions γ_u and γ_x, there exist a continuous function ϕ and class $\mathcal{KL}$ functions β_u and β_x such that, for each d in $L^\infty_{loc}([0,\infty),\mathbb{R}^p)$ and each x in $\mathbb{R}^n$, all the solutions $X(x,t;d)$ of the closed loop system :

$$\dot{x} \;=\; f(x) \;+\; g(x)\,\phi(x) \;+\; p(x)\,d \tag{147}$$

are defined on $[0,\infty)$ and satisfy, for all $t \geq 0$,

$$|X(x,t;d)| \;\leq\; \max\left\{\beta_x(|x|,t)\, ,\, \gamma_x\left(\left\| \, d|_{[0,t]} \right\|_\infty\right)\right\} \; , \tag{148}$$

and, when $\rho(s) \leq s$,

$$|\phi(X(x,t;d))| \;\leq\; \max\left\{\beta_u(|x|,t)\, ,\, (\mathtt{Id} + \gamma_u)\left(\left\| \, d|_{[0,t]} \right\|_\infty\right)\right\} \; . \tag{149}$$

Remark 150 .

1. Again the proof of this Theorem is constructive (see Example 154).

2. The nonlinear L^∞ gain γ_x being arbitrary, (148) shows that the action of d on the state can be arbitrarily attenuated. And, when ρ is not larger than the identity function, this can be done with a control whose norm is arbitrarily close to the L^∞ norm of d.

3. The condition (146) is one of these conditions called matching condition. Indeed, if d were known, we could completely counteract (match) its contribution to the positive terms in the derivative of V by taking the control :

$$\phi(x) \;=\; -|d|\,\rho(x)\,\frac{L_gV(x)^T}{|L_gV(x)|}\ . \tag{151}$$

4. As noticed in [66], when the CLF V and an associated global asymptotic stabilizer ϕ are such that the function :

$$W(x) \;=\; -\left(L_fV(x)+L_gV(x)\phi(x)\right) \tag{152}$$

is proper[5], then the simple controller :

$$\phi_d(x) \;=\; \phi(x) \;-\; \rho(x)^2\,L_gV(x)^T \tag{153}$$

makes the system ISS but with no control on the nonlinear L^∞ gain γ_x •

Example 154 : Disturbance attenuation to cope with partial state feedback
Consider the system :

$$\left\{\begin{array}{rcl}\dot{x}_1 &=& -x_1^3 \,+\, x_4^3\ ,\\ \dot{x}_2 &=& -x_2 \,+\, x_3 \,+\, x_1\,x_4\ ,\\ \dot{x}_3 &=& -x_2\ ,\\ \dot{x}_4 &=& u \,+\, x_3 \,+\, x_1\,x_4\ .\end{array}\right. \tag{155}$$

We look for a continuous global asymptotic stabilizer which depends only on x_4.

Our approach to solve this problem follows from the two observations :

1. The (x_1, x_2, x_3) subsystem with x_4 as input is nothing but the system (28) which we have shown to be ISS in Example 27.

2. In the x_4 subsystem, the "disturbance" (x_3, x_1) is matched, (i.e. cancelable by u if it were known). From Theorem 145, we know that we can design a stabilizer ϕ, depending only on x_4, attenuating arbitrarily the action of this disturbance on this coordinate.

Consequently, we should be able to match a small gain condition like (54) or (81). More specifically, we know, from Theorem 78, that our problem will be solved if, for the system (see the x_4 subsystem) :

$$\dot{x} \;=\; u \,+\, d_2 \,+\, d_1\,x\ , \tag{156}$$

[5] which can always be achieved (see [66].

we can find a CLF V and a continuous function ϕ such that the following inequality holds for the derivative :

$$\overbrace{V(x)}^{\cdot} \;\leq\; -\lambda\, V(x) \;+\; \gamma_1(|d_1|) \;+\; \gamma_2(|d_2|) \;, \tag{157}$$

where γ_1 and γ_2 are class $\mathcal{K}$ functions and λ is a strictly positive real number such that (see (33) and (45)) :

$$\gamma_1(2|x|) \;+\; \gamma_2(8x^2) \;\leq\; (1-\varepsilon)\,\lambda\, V(x) \qquad \forall x \in \mathbb{R} \tag{158}$$

for some strictly positive real number ε. Theorem 145 says that it is possible to find such functions V and ϕ. So let us design them.

We start by observing that it is sufficient to have :

$$\gamma_1(s) \;=\; a\, s^4 \quad , \qquad \gamma_2(s) \;=\; b\, s^2 \quad , \qquad V(x) \;\geq\; c\, x^4 \tag{159}$$

for some real numbers a, b, c. So let us take the CLF :

$$V(x) \;=\; \tfrac{1}{4}\, x^4 \;. \tag{160}$$

With Young's inequality (605), we get :

$$\begin{aligned} \overbrace{V(x)}^{\cdot} \;&=\; x^3\, u \;+\; x^3\, d_2 \;+\; x^4\, d_1 \;, & (161)\\ &\leq\; x^3\, u \;+\; \left(x^6 + \frac{d_2^2}{4}\right) \;+\; \left(\frac{3(x^4)^{\frac{4}{3}}}{4} + \frac{d_1^4}{4}\right) \;, & (162)\\ &\leq\; x^3 \left(u + x^3 + \frac{3x^{\frac{7}{3}}}{4}\right) \;+\; \frac{d_2^2}{4} \;+\; \frac{d_1^4}{4} \;. & (163) \end{aligned}$$

So by picking :

$$u \;=\; \phi(x) \;=\; -\frac{\lambda}{4}\, x \;-\; \left(x^3 + \frac{3x^{\frac{7}{3}}}{4}\right) \;, \tag{164}$$

we get the inequality (157) as :

$$\overbrace{V(x)}^{\cdot} \;\leq\; -\lambda\, V(x) \;+\; \frac{d_2^2}{4} \;+\; \frac{d_1^4}{4} \;. \tag{165}$$

This says :

$$\gamma_1(s) \;=\; \frac{s^4}{4} \quad , \qquad \gamma_2(s) \;=\; \frac{s^2}{4} \;. \tag{166}$$

So now we can write the constraint (158) explicitly as :

$$4\, x^4 \;+\; \frac{81}{4}\, x^4 \;\leq\; (1-\varepsilon)\, \frac{\lambda}{4}\, x^4 \qquad \forall x \;. \tag{167}$$

This shows that our objective is met by taking :

$$\lambda \;>\; 97 \tag{168}$$

and therefore, for instance, the control :

$$u \;=\; \phi(x) \;=\; -25\, x \;-\; \left(x^3 + \frac{3x^{\frac{7}{3}}}{4}\right) \;. \tag{169}$$

To summarize the key points of this example are :

- The very specific choice of the function V in (160). We used (159) to arrive to this choice a priori. More generally, the idea is to leave V undefined and to proceed formally up to the end, collecting all the constraints this function has to satisfy. The expression of V is then chosen at the end.
- Handling the disturbance terms by inequalities $\bullet$

To summarize this paragraph, we have established that, to solve the asymptotic stabilization problem or the disturbance attenuation problem, it is sufficient to look for C^1 Lyapunov functions satisfying :

$$\{x \neq 0\,,\ |L_g V(x)| = 0\} \qquad \Longrightarrow \qquad L_f V(x) < 0 \tag{170}$$

and maybe another extra condition close to the origin. The next paragraph is devoted to explicit construction of such functions for systems which can be written in an appropriate recurrent triangular form, called the feedback form.

3.2 Systems in feedback form and backstepping

3.2.1 CLF via reduction or extension

Consider a system whose dynamics can be written in the following triangular form :

$$\left\{ \begin{array}{rcl} \dot{x} & = & f(x,y)\ , \\ \dot{y} & = & h(x,y) + u\ , \end{array} \right. \tag{171}$$

with x in $\mathbb{R}^n$ and y and u in $\mathbb{R}$. We want to know when, knowing a CLF for the full order system (171), we can get one for the reduced order system :

$$\dot{x} = f(x,u) \tag{172}$$

and conversely.

3.2.1.1 Reduction

Assume that we know a CLF V_y for the full order system (171); i.e.,

$$\left\{(x,y) \neq 0\,,\ \frac{\partial V_y}{\partial y}(x,y) = 0\right\} \qquad \Longrightarrow \qquad \frac{\partial V_y}{\partial x}(x,y)\, f(x,y) < 0\ . \tag{173}$$

We look for a CLf for the reduced order system (173). The condition $\frac{\partial V_y}{\partial y}(x,y) = 0$ is a necessary condition that y must satisfy to be a stationary point of the function $V_y(x,\cdot)$, with x fixed. But, since V_y is C^1 Lyapunov function, for each given x, it has a global minimum and therefore at least one stationary point. Let $\phi_x(x)$ denote one such point. We have :

$$\frac{\partial V_y}{\partial y}(x,\phi_x(x)) = 0\ . \tag{174}$$

Since the origin is a global minimizer of V, we can impose the condition :

$$\phi_x(0) = 0\ . \tag{175}$$

Then we define the function :

$$V_x(x) \; = \; V_y(x, \phi_x(x)) \; . \tag{176}$$

We have :

Lemma 177
Let V_y, ϕ_x and V_x be defined as above. If V_y is C^2 and ϕ_x is ***Hölder continuous*** *of order strictly larger than $\frac{1}{2}$, then V_x is a C^1 CLF for the system (172) and ϕ_x is a continuous global asymptotic stabilizer.*

Remark 178
In general, ϕ_x may not be even continuous. In [9], there is a system in the form (171) which is globally asymptotically stabilizable by continuous feedback whereas its reduced order subsystem (172) is not •

With Lemma 177, we have a sufficient condition, for a system in the triangular form (171), allowing us to express a CLF for the reduced order system (172) from one known for the full order system. This fact is important since it shows that stabilization problems maybe studied via systems with reduced dimensions.

3.2.1.2 Extension

Let us now study the converse question :
If we know a CLF for the reduced order system (172) can we build one for the full order system (171)? To answer this question, we reverse the above arguments :

C1 : For the system (172), let V_x be a CLF and $\phi_x : \mathbb{R}^n \to \mathbb{R}$ be a continuous global asymptotic stabilizer satisfying :

$$\frac{\partial V_x}{\partial x}(x)\, f(x, \phi(x)) \; < \; 0 \qquad \forall x \neq 0 \; . \tag{179}$$

We observe that, for any function ℓ which is C^1 positive definite and proper, and with positive derivative, $\ell(V_x)$ is also a CLF for (172).

C2 : From these data, we want to express a function which will be $\frac{\partial V_y}{\partial y}$ and whose zeros are given by :

$$y \; = \; \phi_x(x) \; . \tag{180}$$

Our motivation is that, in this case, (173) will follow directly from (179). So we introduce the C^1 function $\psi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ such that the function :

$$\Psi(x, y) \; = \; \int_{\phi_x(x)}^{y} \psi(x, s) ds \tag{181}$$

is C^1 and we have :

$$\psi(x, y) \; = \; 0 \qquad \Longleftrightarrow \qquad y \; = \; \phi_x(x) \; , \tag{182}$$

$$\psi(x, y)\, [y - \phi_x(x)] \; > \; 0 \qquad \forall\, (x, y) \; \in \; \mathbb{R}^n \times \mathbb{R} \; : \; y \neq \phi_x(x) \; , \tag{183}$$

$$\lim_{|y| \to \infty} \Psi(x, y) \; = \; +\infty \qquad \forall\, x \; . \tag{184}$$

Theorem 185 ([57])
Under the conditions C1 and C2 above, the function :

$$V_y(x,y) \;=\; \ell(V_x(x)) \;+\; \int_{\phi_x(x)}^{y} \psi(x,s)ds \tag{186}$$

is a C^1 CLF for the system (171). Moreover if we have :

$$\liminf_{|x|+|y|\to 0} \frac{|\psi(x,y)|}{|y-\phi_x(x)|} \;>\; 0 \;, \tag{187}$$

then SCP holds.

Example 188 : Construction of a CLF
Let us come back to the system (92) of Example 91 and see how the CLF (93) can be constructed.

The system is :

$$\left\{\begin{array}{rcl} \dot{x} & = & x\,y \;, \\ \dot{y} & = & -y \;+\; u \;. \end{array}\right. \tag{189}$$

It can be seen as the system :

$$\dot{x} \;=\; x\,u_x \tag{190}$$

which is extended by adding the differentiator :

$$\left\{\begin{array}{rcl} \dot{y} & = & -y \;+\; u \;, \\ u_x & = & y \;. \end{array}\right. \tag{191}$$

Namely to go from the reduced order system (190) to the full order system (189), we need to differentiate its control.

The reduced order system (190) being one dimensional, we can simply take :

$$V_x(x) \;=\; \tfrac{1}{2}\,x^2 \tag{192}$$

as a CLF. We have :

$$\dot{\widehat{V(x)}} \;=\; x^2\,u_x \;. \tag{193}$$

It follows for instance that :

$$u_x \;=\; \phi_x(x) \;=\; -x^2 \tag{194}$$

is a C^1 global asymptotic stabilizer.

Now coming back to the full order system (189), we apply Theorem 185. By picking :

$$\ell(v) \;=\; v \;, \tag{195}$$

$$\psi(x,y) \;=\; y \;-\; \phi_x(x) \;=\; y \;+\; x^2 \;, \tag{196}$$

the formula (186) yields :

$$V_y(x,y) \;=\; \ell(V_x(x)) \;+\; \int_{\phi_x(x)}^{y} \psi(x,s)ds \;, \tag{197}$$

$$\;=\; \tfrac{1}{2}\,x^2 \;+\; \tfrac{1}{2}\,\left(y+x^2\right)^2 \;. \tag{198}$$

This is exactly the expression (93). Also since we have :

$$\frac{|\psi(x,y)|}{|y-\phi_x(x)|} \;=\; 1\;, \tag{199}$$

SCP holds •

Remark 200

1. From the CLF V_y, a global asymptotic stabilizer is obtained by applying one of the formulae ϕ_S or ϕ_F or the L_gV controller ϕ_L of Theorem 98. But we may also choose a cancellation design or a domination design.

2. The existence of a function ψ satisfying C2 is guaranteed as soon as ϕ_x is Hölder continuous (see [9]). Usually ϕ_x is C^1. Then the simplest choice[6] for ψ is :

$$\psi(x,y) \;=\; y \;-\; \phi_x(x)\;. \tag{201}$$

   Taking the identity for the function ℓ, (186) gives the more common formula (see [76]) :

$$V_y(x,y) \;=\; V_x(x) \;+\; \tfrac{1}{2}\left(y-\phi_x(x)\right)^2\;. \tag{202}$$

3. The CLF (202) is arrived at in [32] by another way called nowadays the backstepping technique[7]. It is extensively developed in [37, 35] by combining it with other techniques leading to a rich repertoire of procedures. It goes with the introduction of a new coordinate (see [32]) called the error variable :

$$\mathfrak{y} \;=\; y \;-\; \phi_x(x)\;. \tag{203}$$

   With this coordinate, the system (171) rewrites :

$$\left\{\begin{array}{rcl} \dot{x} &=& f(x,\mathfrak{y}+\phi_x(x))\;,\\ \dot{\mathfrak{y}} &=& \mathfrak{h}(x,\mathfrak{y}) \;+\; u\;, \end{array}\right. \tag{204}$$

   with now :

$$\mathfrak{h}(x,\mathfrak{y}) \;=\; h(x,\mathfrak{y}+\phi_x(x)) \;-\; \frac{\partial\phi_x}{\partial x}(x)f(x,\mathfrak{y}+\phi_x(x))\;. \tag{205}$$

   A property of the x subsystem of (204), exhibited and exploited in [32], is that we have :

$$\dot{\overbrace{V(x)}} \;=\; \frac{\partial V_x}{\partial x}(x)\,f(x,\mathfrak{y}+\phi_x(x)) \;=\; -W_x(x) \;+\; \omega^T\mathfrak{y}\;, \tag{206}$$

   with W_x positive definite and :

$$\omega \;=\; \frac{\partial V_x}{\partial x}(x)\int_0^1 \frac{\partial f}{\partial y}(x,s\mathfrak{y}+\phi_x(x))ds\;. \tag{207}$$

   So the x subsystem is strictly passive with $\mathfrak{y}$ as input and ω as output. The design of a global asymptotic stabilizer for (204) can be seen as making, via u, the $\mathfrak{y}$ subsystem strictly passive with $-\omega$ as input and $\mathfrak{y}$ as output, e.g. with a cancellation design :

$$u \;=\; -\mathfrak{y} \;-\; \mathfrak{h}(x,\mathfrak{y}) \;-\; \omega\;. \tag{208}$$

[6] From a mathematical point of view, may not be the practical one.

[7] In the following, we do not use the technique of error variable but nevertheless we use the name of backstepping for the formulae (186) or (202).

4. After getting from V_y a global asymptotic stabilizer ϕ_y, we are with the system (171) in exactly the same situation as we were with the system (172). This means that we are ready to deal with the further extended system :

$$\left\{ \begin{array}{rcl} \dot{x} &=& f(x,y) \ , \\ \dot{y} &=& h(x,y) \ + \ z \ , \\ \dot{z} &=& k(x,y,z) \ + \ u \ . \end{array} \right. \tag{209}$$

So the procedure we have presented here finds its full power. It allows us to deal by recursion (see Example 342) with the Lyapunov design of global asymptotic stabilizers for systems in the form :

$$\left\{ \begin{array}{rcl} \dot{x}_1 &=& f_1(x_1,x_2) \ , \\ \dot{x}_2 &=& f_2(x_1,x_2) \ + \ g_1(x_1,x_2)\, x_3 \ , \\ &\vdots& \\ \dot{x}_n &=& f_n(x_1,\dots,x_n) \ + \ g_n(x_1,\dots,x_n)\, u \ , \end{array} \right. \tag{210}$$

with all the g_i's of constant sign. This form is called feedback form. It is obtained by adding recursively a differentiator of the coordinate, which can be used for control of the previously defined system, and which is fed back by the previously introduced state components. Actually even more general form can be handled, linearity in x_{i+1} of the $\dot{x}_i$ equation can be relaxed as shown in [9, 77, 43]. See also Example 243.

5. Although, in the formula (186) for the CLF, we have already a design flexibility with the functions ℓ and ψ (which we illustrate below via examples), an even more general formula can be given. Specifically, we know, from Theorem 12, that there exists always a C^1 positive definite function $\delta : \mathbb{R}^n \to \mathbb{R}$ such that any continuous function, with values in $[\phi_x - \delta, \phi_x + \delta]$, is also a global asymptotic stabilizer for (172). It follows that, instead of (182), we can take :

$$\psi(x,y) \ = \ 0 \qquad \Longleftrightarrow \qquad y \in [\phi_x(x) - \delta(x), \phi_x(x) + \delta(x)] \ . \tag{211}$$

For instance, in the case where ϕ_x is C^1, this leads to the following C^1 CLF satisfying SCP :

$$V_y(x,y) \ = \ \ell(V_x(x)) \ + \ \tfrac{1}{2} \max\{|y - \phi_x(x)| - \delta(x)\, , \, 0\}^2 \ . \tag{212}$$

This CLF is "flat" in the y-direction around $\phi_x(x)$; i.e.,its derivative is 0. This property has been exhibited and exploited in [15]. It is useful for dealing with problems where the gradient of the control plays a role like in presence of measurement noise or control rate limitations or with delay in the control (see [16, 18]).

6. We have mentioned above that the backstepping technique can be interpreted within the passivity framework. Another technique, referable to the small gain framework, has been proposed in [67] and exploited for instance in [2] :
For the system (171), let V_x be a CLF and ϕ_x be an associated C^1 global asymptotic stabilizer such that the positive definite function :

$$W(x) \ = \ -\frac{\partial V_x}{\partial x}(x) f(x, \phi_x(x)) \tag{213}$$

is proper. As shown in [67] (see also Remark 150), from these data, we can get a C^1 strictly positive function $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that, by letting (compare with (203)) :

$$\mathfrak{y} \;=\; \frac{y - \phi_x(x)}{\varphi(x)} \ , \tag{214}$$

the system (171) rewrites :

$$\left\{ \begin{array}{rcl} \dot{x} &=& f(x, \varphi(x)\mathfrak{y} + \phi_x(x)) \ , \\ \dot{\mathfrak{y}} &=& \mathfrak{h}(x, \mathfrak{y}) \ + \ \frac{1}{\varphi(x)}\, u \ , \end{array} \right. \tag{215}$$

with some function $\mathfrak{h}$, and where the x subsystem is ISS with input $\mathfrak{y}$. Then, from Theorem 49, global asymptotic stability for (204) is obtained by making the $\mathfrak{y}$ subsystem independent of x and with global asymptotic stability, e.g. with :

$$u \;=\; -\varphi(x)\left[\mathtt{sign}(\mathfrak{y})\, k(\mathfrak{y}) + \mathfrak{h}(x, \mathfrak{y})\right] \ , \tag{216}$$

where k is some continuous positive definite function •

3.2.2 Illustration of backstepping via examples

In this section, we illustrate some of the potentialities of the backstepping technique.

Example 217 : Dealing with singularities (see [42])
Consider the system :

$$\left\{ \begin{array}{rcl} \dot{\theta} &=& \theta \ + \ \omega \ , \\ \dot{\omega} &=& \omega \ + \ (1 - \omega)\, u \ . \end{array} \right. \tag{218}$$

It can be shown that the set $\{(\theta, \omega) \,:\, \theta \leq -1 \text{ or } \omega \geq 1\}$ is invariant whatever u is. So the origin cannot be globally asymptotically stabilized. Let us design a control law making the complement of the above set a basin of attraction of the origin.

This complement is diffeomorphic to $\mathbb{R}^2$ as exhibited by the following set of (singular) coordinates :

$$\left\{ \begin{array}{rcl} x &=& \log(1 + \theta) \ , \\ y &=& -\log(1 - \omega) \ . \end{array} \right. \tag{219}$$

It maps the set $\{(\theta, \omega) \,:\, \theta > -1 \text{ and } \omega < 1\}$ onto $\mathbb{R}^2$. With these new coordinates the system (218) rewrites :

$$\left\{ \begin{array}{rcl} \dot{x} &=& \dfrac{\exp(x + y) - 1}{\exp(x) + y} \ , \\ \dot{y} &=& [\exp(y) - 1] \ + \ u \ . \end{array} \right. \tag{220}$$

This system is made of the system :

$$\dot{x} \;=\; \frac{\exp(x + u_x) - 1}{\exp(x + u_x)} \tag{221}$$

which is extended by adding the differentiator :

$$\left\{ \begin{array}{rcl} \dot{y} & = & [\exp(y) - 1] \ + \ u \ , \\ u_x & = & y \ . \end{array} \right. \tag{222}$$

This decomposition motivates us for designing a control law in two steps.

Step 1 : We consider the system (221). We can check that :

$$u_x \ = \ -2\,x \tag{223}$$

is a global asymptotic stabilizer associated to the CLF :

$$V_x(x) \ = \ \tfrac{1}{2}\,x^2 \ . \tag{224}$$

Step 2 : We consider the system (220). The formula (202) yields :

$$V_y(x,y) \ = \ V_x(x) \ + \ \tfrac{1}{2}\,(y+2x)^2 \ = \ \tfrac{1}{2}\,x^2 \ + \ \tfrac{1}{2}\,(y+2x)^2 \tag{225}$$

as a CLF. Precisely, we get :

$$\dot{V}_y \ = \ x\,\frac{\exp(x+y)-1}{\exp(x+y)} \ + \ (y+2x)\left([\exp(y)-1]+u+2\frac{\exp(x+y)-1}{\exp(x+y)}\right) \ , \tag{226}$$

$$= \ x\,(1-\exp(x)) \tag{227}$$

$$+ \ (y+2x)\left(x\,\exp(-(x+y))\,\frac{\exp(y+2x)-1}{y+2x}[\exp(y)-1]+u+2\frac{\exp(x+y)-1}{\exp(x+y)}\right) \ .$$

Then a cancellation design gives the global asymptotic stabilizer :

$$u \ = \ -\left(x\,\exp(-(x+y))\,\frac{\exp(y+2x)-1}{y+2x}+[\exp(y)-1]+2\frac{\exp(x+y)-1}{\exp(x+y)}\right) \ - \ (y+2x) \ . \tag{228}$$

To summarize, the key points of this example are :

- The singular change of coordinates mapping the desired basin of attraction onto the full Euclidean space. We shall see in Examples 299 and (418) another technique with a singular Lyapunov function.
- The design in two steps of the stabilizer. This follows the structure in feedback form of the system.
- The use of the formula (202) to construct a CLF when the differentiator is added •

Example 229 : Dealing with input constraints (see [18])
Consider the linear system :

$$\left\{ \begin{array}{rcl} \dot{x} & = & y \ , \\ \dot{y} & = & u \ . \end{array} \right. \tag{230}$$

We look for a C^1 global asymptotic stabilizer ϕ_y satisfying :

- a dead zone effect with :

$$\frac{\partial \phi_y}{\partial x}(0,0) \;=\; \frac{\partial \phi_y}{\partial y}(0,0) \;=\; 0 \ , \tag{231}$$

- the magnitude limit :

$$|\phi_y(x,y)| \;\leq\; 5 \ . \tag{232}$$

To solve this stabilization problem under input constraints, we proceed recursively by dealing first, and under the same constraints, with the reduced order system :

$$\dot{x} \;=\; u_x \ . \tag{233}$$

Then we consider the full order system.

Step 1 : We consider the system (233). To meet the dead zone constraint, it is sufficient to choose $\phi_x(x)$ of order strictly larger than 1 around zero. To meet the magnitude constraint, it is sufficient to choose $\phi_x(x)$ bounded. This motivates for the function :

$$\phi_x(x) \;=\; -\tfrac{1}{2}\,\frac{x^3}{(1+|x|^3)} \ . \tag{234}$$

For V_x, we take simply :

$$V_x(x) \;=\; \tfrac{1}{2}\,x^2 \ . \tag{235}$$

Step 2 : We consider the full order system and apply the formula (186) with :

$$\begin{aligned}
\ell(v) \;&=\; \tfrac{8}{9}\,\frac{v^{\frac{3}{2}}}{1+2v} \ , && (236)\\
\psi(x,y) \;&=\; (y-\phi_x(x)) \;+\; (y|y|-\phi_x(x)|\phi_x(x)|) \ . && (237)
\end{aligned}$$

These non trivial expressions follow from a careful analysis of what is involved in (239) below. This gives the CLF satisfying SCP :

$$V_y(x,y) \;=\; \ell(V_x(x)) \;+\; \int_{\phi_x(x)}^{y} (s-\phi_x(x)) + (s|s|-\phi_x(x)|\phi_x(x)|)\, ds \tag{238}$$

Its derivative is :

$$\begin{aligned}
\dot{\overbrace{V_y(x,y)}} \;=&\; \ell'(V_x(x))x\,\phi_x(x) && (239)\\
&+\;[y-\phi_x(x)]\left[\ell'(V_x(x))x + \left(1+\frac{\frac{\partial V_y}{\partial y}(x,y)}{y-\phi_x(x)}\right)u - (1+2|\phi_x(x)|)\;\phi_x'(x)\,y\right] \ ,
\end{aligned}$$

where we have :

$$\begin{aligned}
\frac{\frac{\partial V_y}{\partial y}(x,y)}{y-\phi_x(x)} \;&=\; |\phi_x(x)|+|y| \quad \text{if} \quad \phi_x(x)\,y>0 \ ,\\
&=\; \frac{\phi_x(x)^2+y^2}{|\phi_x(x)|+|y|} \quad \text{if} \quad \phi_x(x)\,y\leq 0 \ .
\end{aligned} \tag{240}$$

A cancellation design leads to the global asymptotic stabilizer :

$$\phi_y(x,y) \;=\; \frac{(1+2|\phi_x(x)|)\,\phi_x'(x)\,y - \ell'(V_x(x))x}{1+\dfrac{\frac{\partial V_y}{\partial y}(x,y)}{y-\phi_x(x)}} - \mathtt{sat}(y-\phi_x(x))^3 \tag{241}$$

where $\mathtt{sat}$ is the standard saturation function :

$$\mathtt{sat}(s) = \max\{-1\,,\,\min\{1,s\}\}\ . \tag{242}$$

It can be checked, with the choice made, in step 1, for ϕ_x and, in (236) and (237), for ℓ and ψ, that ϕ_y meets the given specifications.

To summarize the key point of this example is the very specific expressions (236) and (237) we have chosen for the functions for ℓ and ψ in the formula (186). Also, we have solved the problem by imposing at each steps that the constraints be satisfied. This may not be the only nor the best way to proceed. For instance, here, it is sufficient to let $|\phi_x(x)|$ to grow as $\sqrt{|x|}$ (and $\ell(v)$ as $v^{\frac{1}{2}}$) when $|x|$ is large. This is related to a control rate limitation, i.e. $|\phi_x(x)|$ is not bounded not but $\overbrace{\phi_x(x)}^{\cdot}$ is •

Example 243 : When ϕ_x is not C^1 (see [9, 77, 43])
Consider the system :

$$\left\{\begin{array}{rcl}\dot{x} &=& x \;-\; y^3\ ,\\ \dot{y} &=& u\ .\end{array}\right. \tag{244}$$

The first order approximation of this system is not stabilizable. Therefore it has no C^1 asymptotic stabilizer. To design a C^0 global asymptotic stabilizer we proceed in two steps.

Step 1 : Consider the system :

$$\dot{x} \;=\; x \;-\; u_x^3\ . \tag{245}$$

The following function :

$$V_x(x) \;=\; \tfrac{1}{2}\,x^2 \tag{246}$$

is a CLF to which we can associate the non C^1 global asymptotic stabilizer :

$$\phi_x(x) \;=\; (2x)^{1/3}\ . \tag{247}$$

Step 2 : We consider now the full order system (244). A hint for the choice of a function ψ appropriate to be used in the formula (186) is that $\phi_x(x)$ is a solution of the polynomial equation :

$$\phi_x^3 \;-\; 2\,x \;=\; 0\ . \tag{248}$$

So ψ will be C^1 if we choose it as :

$$\psi(x,y) \;=\; y^3 \;-\; 2\,x\ . \tag{249}$$

But, in this case, the condition (187) is not satisfied. Nevertheless, for this particular case, we can still get SCP with an appropriate choice of the function ℓ (see [9]). With homogeneity arguments, such a choice is (see [57]) :

$$\ell(v) \;=\; 9\,(2v)^{2/3}\ . \tag{250}$$

This yields the function :

$$V_y(x,y) \;=\; \tfrac{1}{4}\,y^4 \;-\; 2\,x\,y + \tfrac{3(1+32^{2/3})}{4}(2x)^{4/3} \tag{251}$$

as a CLF for the system (244). With this function, a domination design leads to the following continuous global asymptotic stabilizer :

$$\phi_y(x,y) \;=\; \left(7\,y \;-\; 18\,x^{1/3}\right)^{1/3} \;. \tag{252}$$

To summarize the key point in this example is the expressions (250) and (249) of the function ℓ and ψ used in the formula (186) •

For a system with several controls, it is more appropriate to deal with only one at a time.

Example 253 : Multi-input systems (see [63])
Consider the 2-input system :

$$\left\{\begin{array}{rcl} \dot{x}_1 &=& x_2\;, \\ \dot{x}_2 &=& u_1\;, \\ \dot{x}_3 &=& u_2\;, \\ \dot{x}_4 &=& x_3\,(1-u_1)\;. \end{array}\right. \tag{254}$$

To design a global asymptotic stabilizer, we proceed in three steps[8].

Step 1 : We consider the linear system :

$$\left\{\begin{array}{rcl} \dot{x}_1 &=& x_2\;, \\ \dot{x}_2 &=& u_1\;. \end{array}\right. \tag{255}$$

A CLF is :

$$V_1(x_1,x_2) \;=\; \tfrac{1}{2}\left(x_1^2 + (x_1+x_2)^2\right) \;. \tag{256}$$

It corresponds the global asymptotic stabilizer :

$$u_1 \;=\; \phi_1(x_1,x_2) \;=\; -2\,(x_1+x_2) \;. \tag{257}$$

Step 2 : We consider the system :

$$\left\{\begin{array}{rcl} \dot{x}_1 &=& x_2\;, \\ \dot{x}_2 &=& -2\,(x_1+x_2)\;\;, \\ \dot{x}_4 &=& v\,[1+2\,(x_1+x_2)] \end{array}\right. \tag{258}$$

with control v. In view of its structure and after step 1, we can propose the Lyapunov function :

$$\begin{aligned} V_2(x_1,x_2,x_4) \;&=\; V_1(x_1,x_2) \;+\; \tfrac{1}{2}\,x_4^2\;, &(259)\\ &=\; \tfrac{1}{2}\left(x_1^2 + (x_1+x_2)^2 + x_4^2\right) \;. &(260)\end{aligned}$$

[8] Actually a more direct way is possible by replacing x_4 by $y_4 \;=\; x_4 \;+\; x_2\,x_3$.

It is a CLF since the derivative :

$$\overbrace{V_2(x_1, x_2, x_4)}^{\cdot} = -x_1^2 - (x_1 + x_2)^2 + x_4 \left[1 + 2\,(x_1 + x_2)\right] v \tag{261}$$

is made negative definite by taking :

$$v = \phi_2(x_1, x_2, x_4) = -x_4 \left[1 + 2\,(x_1 + x_2)\right] . \tag{262}$$

Step 3 : We consider the full order system (254). It is obtained from (258) by adding the differentiator :

$$\left\{ \begin{array}{rcl} \dot{x}_3 & = & u_2 \;, \\ v & = & x_3 \;. \end{array} \right. \tag{263}$$

The formula (186), with :

$$\ell(v) = v \;, \tag{264}$$

$$\psi(x_1, x_2, x_4, s) = s - \phi_2(x_1, x_2, x_4) \;, \tag{265}$$

yields the CLF :

$$V_3(x_1, x_2, x_4, x_3) = V_2(x_1, x_2, x_4) + \tfrac{1}{2}\left(x_3 + x_4 \left[1 + 2\,(x_1 + x_2)\right]\right)^2 \;, \tag{266}$$

$$= \tfrac{1}{2}\left(x_1^2 + (x_1 + x_2)^2 + x_4^2 + \left(x_3 + x_4 \left[1 + 2\,(x_1 + x_2)\right]\right)^2\right) . \tag{267}$$

Then a cancellation design for instance leads to the following global asymptotic stabilizer :

$$u_2 = \phi_3(x_1, x_2, x_4, x_3) \;, \tag{268}$$

$$= -\left(x_4 + x_3 \left[1 + 2\,(x_1 + x_2)\right]^2 + 2x_4[x_2 - 2\,(x_1 + x_2)]\right) - (x_3 + x_4[1 + 2\,(x_1 + x_2)]) \tag{269}$$

To summarize the key point of this example is the decomposition of the design in three steps :

1. In step 1, we look at the (x_1, x_2) subsystem and design u_1.
2. In step 2, we look at the (x_1, x_2, x_4) subsystem with x_3 as control. This is made possible by the expression of u_1 in terms of (x_1, x_2).
3. In step 3, we reach the full order system by adding a differentiator •

Example 270 : Arbitrary pole assignment (see [13])
Consider a single-input controllable linear system whose dynamics are written as :

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & x_2 + a_{11}\, x_1 \;, \\ & \vdots & \\ \dot{x}_i & = & x_{i+1} + \sum_{j=1}^{i} a_{ij} x_j \;, \\ & \vdots & \\ \dot{x}_n & = & u + \sum_{j=1}^{n} a_{nj} x_j \;, \end{array} \right. \tag{271}$$

or in compact form :

$$\dot{x} = \mathcal{A}x + \mathcal{B}u . \tag{272}$$

Let a linear asymptotic stabilizer be given in the form :

$$u = -\mathcal{K}x . \tag{273}$$

Let also $\mathcal{P}$ and $\mathcal{Q}$ be given positive definite matrices related by the following Lyapunov equation :

$$(\mathcal{A}-\mathcal{B}\mathcal{K})^T\mathcal{P} + \mathcal{P}(\mathcal{A}-\mathcal{B}\mathcal{K}) = \mathcal{Q} . \tag{274}$$

We want to show that the same controller can be obtained by applying the backstepping technique and even more that the constructed CLF and its derivative are quadratic forms given by $\mathcal{P}$ and $\mathcal{Q}$ respectively.

To do this, we rewrite (271) with appropriate coordinates. We consider the Choleski decomposition of $\mathcal{Q}$ with a matrix $\mathcal{L}$ which is lower triangular with 1's the main diagonal :

$$\mathcal{Q} = \mathcal{L}^T\mathtt{diag}(q_i)\,\mathcal{L} . \tag{275}$$

We denote :

$$B := \mathcal{L}\mathcal{B} = \mathcal{B} \quad , \qquad A := \mathcal{L}\mathcal{A}\mathcal{L}^{-1} . \tag{276}$$

This matrix A can be decomposed as :

$$A = \begin{pmatrix} \star & 1 & 0 & \dots & \dots & 0 \\ \star & \star & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \star & \dots & \dots & \star & 1 & 0 \\ \star & \dots & \dots & \dots & \star & 1 \\ \star & \dots & \dots & \dots & \dots & \star \end{pmatrix} = \begin{pmatrix} \overline{A} & b \\ a^T & c \end{pmatrix} , \tag{277}$$

with the matrix $\overline{A}$ of dimension $(n-1)\times(n-1)$ and with the same structure as A. Then we introduce the other notations :

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathcal{L}x \quad , \qquad P = \mathcal{L}^{-T}\mathcal{P}\mathcal{L}^{-1} \quad , \qquad \begin{pmatrix} \alpha^T & \gamma \end{pmatrix} = K^T = \mathcal{K}^T\mathcal{L}^{-1} \tag{278}$$

and, as for A, we decompose P and $\mathtt{diag}(q_i)$ in blocks as :

$$P = \begin{pmatrix} \overline{P}+\pi pp^T & \pi p \\ \pi p^T & \pi \end{pmatrix} \quad , \qquad \mathtt{diag}(q_i) = \mathtt{diag}(\overline{Q}, q_n) \tag{279}$$

With these notations, we observe that, since P is positive definite, the same holds for $\overline{P}$. Also (274) gives the following three equalities :

$$\overline{P}(\overline{A}-bp^T) + (\overline{A}-bp^T)^T\overline{P} = -\overline{Q} - q_n pp^T , \tag{280}$$

$$\frac{1}{\pi}\overline{P}b + a + \overline{A}^T p = \alpha + \frac{q_n}{2\pi}p , \tag{281}$$

$$p^T b + c = \gamma - \frac{q_n}{2\pi} . \tag{282}$$

Finally the system (271) rewrites :

$$\left\{\begin{array}{rcl}\dot{x} &=& \overline{A}\,x \;+\; b\,y\;,\\ \dot{y} &=& a^T x \;+\; c\,y \;+\; u\;.\end{array}\right. \tag{283}$$

By picking :

$$V_x(x) \;=\; \tfrac{1}{2}\,x^T\,\overline{P}\,x \quad , \qquad \phi_x(x) \;=\; -p^T x\;, \tag{284}$$

we get a CLF and an associate global asymptotic stabilizer for the x subsystem of (283). In particular, with (280), we have :

$$\dot{\overbrace{V_x(x)}} \;=\; -\tfrac{1}{2}\,x^T\overline{Q}\,x \;-\; \tfrac{q_n}{2}\,(p^T x)^2\;. \tag{285}$$

Now, for the full order system (283), the formula (186) yields the CLF :

$$\begin{aligned} V_y(x,y) \;&=\; V_x(x) \;+\; \tfrac{\pi}{2}\,(y-\phi_x(x))^2\;, && (286)\\ &=\; \tfrac{1}{2}\,x^T\overline{P}\,x \;+\; \tfrac{\pi}{2}\,(y+p^T x)^2\;, && (287)\\ &=\; \tfrac{1}{2}\,x^T\mathcal{P}\,x\;. && (288)\end{aligned}$$

Its derivative is :

$$\begin{aligned} \dot{\overbrace{V_y(x,y)}} \;&=\; x^T\overline{P}(\overline{A}x+by) \;+\; \pi\,(y+p^T x)\,(a^T x+cy+u+p^T[\overline{A}x+by])\;, && (289)\\ &=\; -\tfrac{1}{2}\,x^T(\overline{Q}+q_n pp^T)\,x \;+\; \pi\,(y+p^T x)\left(\frac{x^T\overline{P}b}{\pi}+a^T x+cy+u+p^T[\overline{A}x+by]\right)\;. && (290)\end{aligned}$$

So the feedback transformation :

$$u \;=\; -\left(\frac{x^T\overline{P}b}{\pi}+a^T x+cy+p^T[\overline{A}x+by]\right) \;+\; v \tag{291}$$

yields :

$$\dot{\overbrace{V_y(x,y)}} \;=\; -\tfrac{1}{2}\,x^T\overline{Q}\,x \;-\; \tfrac{q_n}{2}\,(p^T x)^2 \;+\; \pi\,(y+p^T x)\,v\;. \tag{292}$$

It follows that, by picking :

$$v \;=\; -\tfrac{q_n}{2\pi}\,(y-p^T x)\;, \tag{293}$$

we get :

$$\begin{aligned} \dot{\overbrace{V_y(x,y)}} \;&=\; -\tfrac{1}{2}\,x^T\overline{Q}\,x \;-\; \tfrac{q_n}{2}\,y^2\;, && (294)\\ &=\; -\tfrac{1}{2}\,x^T\mathcal{Q}\,x\;. && (295)\end{aligned}$$

Finally, from (291) and (293), and by using (281), (282) and (278), we see that the controller is :

$$\begin{aligned} u \;&=\; -\left(x^T\left[\frac{1}{\pi}\,\overline{P}\,b+a+\overline{A}^T p\right] \;+\; \left[p^T b+c\right]y\right) \;-\; \tfrac{q_n}{2\pi}\,(y-p^T x)\;, && (296)\\ &=\; -x^T\alpha \;-\; \gamma\,y\;, && (297)\\ &=\; -\mathcal{K}\,x\;. && (298)\end{aligned}$$

So the Lyapunov design has allowed us to reach the given controller with the given CLF and derivative.

The key points to get the result are :

1. The choice of coordinates which make Q diagonal while preserving the lower triangular structure of A.
2. The decomposition of P as (279) and the three relations (280)-(282) which follow.
3. The choice of V_x and ϕ_x in (284).

Actually we have done only the last step of the backstepping technique, but the recursive form of A allows us to conclude that we would have obtained the same result by applying the same steps recursively one state component after the other. Namely, the same technique applies to the x subsystem of (283) with y as control and with (284) and (285) as control objective.

The above fact must be put together with the following remark :
For a general system in feedback form, the terms of order strictly larger than 1 at the origin do not contribute to the first order approximation of the final controller we obtain by recursively applying the backstepping technique with say cancellation designs.

It follows that the backstepping technique allows us to design global asymptotic stabilizers with imposed local behavior. This property has been exploited for instance in [13] (and generalized in [55]) to design a global asymptotic stabilizer with local H_∞ properties •

We conclude this section by presenting an example demonstrating the potentialities of Lyapunov design. It is directly inspired by a solution to the problem of global stabilization of bifurcations in a model of jet engine surge and stall proposed in [36]. It illustrates how, when facing real world problems, we have to combine several techniques.

Example 299 : Combining techniques (see [36])
Consider the system :

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & -(x_1+1)(x_1+x_2^2) \ , \\ \dot{x}_2 & = & x_3 \ - \ x_2 f(x_2) \ + \ x_1 \ , \\ \dot{x}_3 & = & u \ , \end{array} \right. \tag{300}$$

where f is an unknown C^1 function with known lower bound F :

$$F \ \leq \ \inf_s f(s) \ . \tag{301}$$

We are looking for a controller, linear in the coordinates (x_1, x_2, x_3), and making the origin asymptotically stable with basin of attraction :

$$\Omega \ = \ \{(x_1, x_2, x_3) \,|\, x_1 > -1\} \ . \tag{302}$$

The system (300) is made of the system :

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & -(x_1+1)(x_1+x_2^2) \ , \\ \dot{x}_2 & = & u_1 \ - \ x_2 f(x_2) \ + \ x_1 \ , \end{array} \right. \tag{303}$$

to which is added the differentiator :

$$\left\{ \begin{array}{rcl} \dot{x}_3 & = & u \ , \\ u_1 & = & x_3 \ . \end{array} \right. \tag{304}$$

So we design the controller in two steps.

Step 1 : We consider the system (303). Since we want to restrict our attention to the set Ω, we choose for the x_1 subsystem a Lyapunov function, positive definite, C^2 but defined on this open set only :

$$V_1(x_1) \;=\; \int_0^{x_1} \frac{s}{s+1} ds \;=\; x_1 - \log(x_1+1) \; . \tag{305}$$

Its interest is in the implication :

$$V_1(x_1) \;\leq\; c \qquad \Longrightarrow \qquad -1 \;<\; x_1 \; , \tag{306}$$

for all non negative real number c. We obtain the derivative :

$$\begin{aligned} \overbrace{V_1(x_1)}^{\cdot} \;&=\; -x_1^2 \;-\; x_1\, x_2^2 \; , &(307)\\ &\leq\; -x_1^2 \;+\; x_2^2 \qquad \forall x_1 > -1 \; . &(308)\end{aligned}$$

Then let V be a C^2 Lyapunov function. In order to keep some flexibility for handling the second step and to meet possibly the constraint of linearity of the control, at this stage, we do not specify what the function V is. We let :

$$V_2(x_1,x_2) \;=\; V_1(x_1) \;+\; V(x_2) \; . \tag{309}$$

For all $x_1 > -1$, we get the following bounds for the derivative :

$$\begin{aligned} \overbrace{V_2(x_1,x_2)}^{\cdot} \;&=\; -V_1'(x_1)\,(x_1+1)\,(x_1+x_2^2) \;+\; V'(x_2)\,[u_1 - x_2 f(x_2) + x_1] \; , &(310)\\ &\leq\; -x_1^2 \;+\; x_2^2 \;-\; V'(x_2)\,x_2\,f(x_2) \;+\; V'(x_2)\,[u_1+x_1] \; , &(311)\\ &\leq\; -\tfrac{1}{2}\,x_1^2 \;+\; V'(x_2)\left[-x_2 f(x_2) + \frac{x_2^2}{V'(x_2)} + u_1 + V'(x_2)\right] \; . &(312)\end{aligned}$$

So we see that, if we impose the following constraint[9] on V :

$$(k-1) \;+\; f(x_2) \;\geq\; \frac{x_2}{V'(x_2)} + \frac{V'(x_2)}{x_2} \; , \tag{313}$$

for some real number k, and we choose the control :

$$u_1 \;=\; \phi_2(x_1,x_2) \;=\; -k\,x_2 \; , \tag{314}$$

the derivative satisfies, for all $x_1 > -1$,

$$\overbrace{V_2(x_1,x_2)}^{\cdot} \;\leq\; -\tfrac{1}{2}\,x_1^2 \;-\; V'(x_2)\,x_2 \; . \tag{315}$$

---

[9] Since V is a Lyapunov function, we have :

$$V'(x_2)\,x_2 \;>\; 0 \qquad \forall x_2 \;\neq\; 0.$$

It follows that the constraint (313) is met for instance by :

$$V(x_2) \;=\; \tfrac{1}{2}\,x_2^2 \quad , \qquad k \;\geq\; 3 \;-\; F \; .$$

This proves that ϕ_2 is an asymptotic stabilizer on the set $\{(x_1,x_2) \,|\, x_1 > -1\}$.

Step 2 : We consider the full order system (300) and apply the formula (186) with :

$$\begin{aligned} \ell(v) &= v \ , & (316) \\ \psi((x_1,x_2),x_3) &= a\,(x_3 + kx_2) \ , & (317) \end{aligned}$$

where a is a strictly positive real number to be specified later on. This yields the CLF :

$$V_3(x_1,x_2,x_3) \;=\; V_1(x_1) \;+\; V(x_2) \;+\; \tfrac{a}{2}\,(x_3 + kx_2)^2 \ . \tag{318}$$

For all $x_1 > -1$, its derivative satisfies :

$$\begin{aligned} \dot{\overbrace{V_3(x_1,x_2,x_3)}} &= -V_1'(x_1)\,(x_1+1)\,(x_1+x_2^2) \;+\; V'(x_2)\,[u_1 - x_2 f(x_2) + x_1] \\ &\quad + a\,(x_3+kx_2)\,[u + k\,(x_3 - x_2 f(x_2) + x_1)] \ , & (319) \\ &\le -\tfrac{1}{2}\,x_1^2 \;-\; V'(x_2)\,x_2 \\ &\quad + a\,(x_3+kx_2)\left[\frac{V'(x_2)}{a} + u + k\,(x_3 - x_2 f(x_2) + x_1)\right] \ , & (320) \\ &\le -\tfrac{1}{4}\,x_1^2 \;-\; V'(x_2)\,x_2 \\ &\quad + a\,(x_3+kx_2)\left[\frac{V'(x_2)}{a} + u + k\,(x_3 - x_2 f(x_2) + ak(x_3+kx_2))\right] \ , & (321) \end{aligned}$$

where we have used (315) to write (320) and completed the squares to write (321). A cancellation design gives the following asymptotic stabilizer on the set Ω :

$$u \;=\; \phi_3(x_1,x_2,x_3) \;=\; -\left[\frac{V'(x_2)}{a} + k\,(x_3 - x_2 f(x_2) + ak(x_3+kx_2))\right] - (x_3+kx_2) \ . \tag{322}$$

It provides, for all $x_1 > -1$,

$$\dot{\overbrace{V_3(x_1,x_2,x_3)}} \;\le\; -\tfrac{1}{4}\,x_1^2 \;-\; V'(x_2)\,x_2 \;-\; a\,(x_3+kx_2)^2 \ . \tag{323}$$

This establishes asymptotic stability of the origin with Ω as domain of attraction.

Now to meet the linearity constraint on u, defined in (322), we should have :

$$\frac{V'(x_2)}{a} \;-\; k\,x_2\,f(x_2) \;=\; b\,x_2 \ , \tag{324}$$

with some real number b. Specifically, in this case the controller is simply :

$$\phi_3(x_1,x_2,x_3) \;=\; -[k + (b + ak^2)]\,x_2 \;-\; [1 + k(1+a)]\,x_3 \ , \tag{325}$$

with the three parameters a, b and k. From our construction, it is appropriate if we can find a C^2 Lyapunov function V satisfying (313) and (324). But the constraint (324) imposes readily that V be :

$$V(x_2) \;=\; a \int_0^{x_2} [bs + ksf(s)]\,ds \ . \tag{326}$$

Fortunately this function is positive definite and proper on Ω if b satisfies :

$$b \;>\; -k\,F \ . \tag{327}$$

Then with (326), the constraint (313) reads :

$$(k-1) \;+\; f(x_2) \;\geq\; \frac{1}{a(b+kf(x_2))} + a(b+kf(x_2)) \;. \tag{328}$$

So to conclude, the problem is solved if (327) and (328) hold. This is the case for instance if we choose the controller parameters as :

$$k \;\geq\; 2 \;+\; c \quad , \qquad a \;=\; \frac{1}{k} \quad , \qquad b \;=\; c\,k \tag{329}$$

where :

$$c \;\geq\; 1 \;+\; \max\{0, -F\} \;. \tag{330}$$

To conclude, if we want to emphasize only one point in this example, it is the idea of not choosing V a priori but only at the end, once all the constraints this functions must satisfy are known •

3.3 Backstepping with disturbances

When disturbances are acting on systems with feeback form, the other tool to be used with the backstepping technique is the inequalities as those found for instance in [22]. As an application, we can prove that the ISS property can be propagated through differentiators, by using the simplest CLF formula (202) and the following inequality. See also [28, 27] and [35, Lemma 2.20].

Lemma 331
Let $F : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ and $G : \mathbb{R}^n \to \mathbb{R}$ be continuous functions. Assume we have :

$$F(x, 0) \;=\; 0 \qquad \forall x \in \mathbb{R}^n \;. \tag{332}$$

Then there exist two continuous functions $\delta_x : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and $\delta_d : \mathbb{R}^p \to \mathbb{R}_{\geq 0}$ satisfying :

$$\delta_d(0) \;=\; 0 \;, \tag{333}$$

$$|G(x)\, F(x, d)| \;\leq\; G(x)^2\, \delta_x(|x|) \;+\; \delta_d(|d|) \qquad \forall (x, d) \in \mathbb{R}^x \times \mathbb{R}^p \;. \tag{334}$$

Precisely, consider the following disturbed version of (171) :

$$\left\{ \begin{array}{rcl} \dot{x} &=& f(x, y, d_x) \;, \\ \dot{y} &=& h(x, y, d_y) \;+\; g(x, y, d_y)\, u \;, \end{array} \right. \tag{335}$$

where the functions f, h and g are continuous and we have :

$$g_u(x, y, d_y) \;\geq\; \eta(x, y) \qquad \forall\, (x, y, d_y) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{p_y} \;, \tag{336}$$

where $\eta : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}_{>0}$ is continuous. We have :

Theorem 337
If there exists a C^1 Lyapunov function V_x, a class $\mathcal{K}^\infty$ function a_x, a class $\mathcal{K}$ function γ_x and a C^1 function ϕ_x such that we have :

$$\frac{\partial V_x}{\partial x}(x)\, f(x, \phi_x(x), d_x) \;\leq\; -a_x(V_x(x)) \;+\; \gamma_x(|d_x|) \qquad \forall (x, d_x) \in \mathbb{R}^n \times \mathbb{R}^{p_x} \;, \tag{338}$$

then there exists a C^1 Lyapunov function V_y, a class $\mathcal{K}^\infty$ function a_y, a class $\mathcal{K}$ function γ_y and a C^0 function ϕ_y such that we have :

$$\begin{aligned}\frac{\partial V_y}{\partial x}(x,y)\, f(x,y,d_x) \;+\; \frac{\partial V_y}{\partial y}(x,y)\; [h(x,y,d_y) + g(x,y,d_y)\phi_y(x,y)]& \\ \leq\; -a_y(V_y(x,y)) \;+\; \gamma_y(|d_x| + |d_y|) \;,&\end{aligned} \tag{339}$$

for all (x, y, d_x, d_y) in $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{p_x} \times \mathbb{R}^{p_y}$.

Remark 340
This Theorem says that, if the x subsystem can be made ISS via a C^1 stabilizer, then the extended system can be made ISS via a C^0 stabilizer. Then, if this derived stabilizer can actually be taken C^1, we can propagate the property and deal with disturbed feedback systems of the type :

$$\left\{\begin{array}{rcl} \dot{x}_1 &=& f_1(x_1, x_2, d_1) \;, \\ \dot{x}_2 &=& f_2(x_1, x_2, d_2) \;+\; g_2(x_1, x_2, d_2)\, x_3 \;, \\ &\vdots& \\ \dot{x}_n &=& f_2(x_1, \ldots, x_n, d_n) \;+\; g_n(x_1, \ldots, x_n, d_n)\, u \;. \end{array}\right. \tag{341}$$

Example 342 : Taking care of uncertainties via disturbance attenuation
Consider the system :

$$\left\{\begin{array}{rcl} \dot{x}_1 &=& x_2 \;+\; x_1^2\, f(x_1, x_2, x_3, u, t) \;, \\ \dot{x}_2 &=& x_3 \;, \\ \dot{x}_3 &=& u \;, \end{array}\right. \tag{343}$$

where f is an unknown continuous function taking values in $[-1, 1]$. We want to design a global asymptotic stabilizer.

Since f depends on x_3 and u, (not to write that it is unknown,) we cannot apply directly the backstepping technique. So instead, for the design, we consider the following system :

$$\left\{\begin{array}{rcl} \dot{x}_1 &=& x_2 \;+\; x_1^2\, d \;, \\ \dot{x}_2 &=& x_3 \;, \\ \dot{x}_3 &=& u \;, \end{array}\right. \tag{344}$$

where d is any function in $L^\infty_{loc}([0, \infty), [-1, 1])$. Its first interest lies in the fact that any solution of (343) is a solution of (344). So if we find a solution for the global asymptotic stabilization problem for the system (344), this same solution will be appropriate for the system (343). The second interest of the system (344) is that it has a disturbed feedback form with two differentiators added. So we go for a design in three steps.

Step 1 : We consider the system :

$$\dot{x}_1 \;=\; u_1 \;+\; x_1^2\, d\ . \tag{345}$$

An appropriate CLF is simply :

$$V_1(x_1) \;=\; \tfrac{1}{2}\, x_1^2\ . \tag{346}$$

It corresponds the global asymptotic stabilizer :

$$\phi_1(x_1) \;=\; -x_1 \;-\; x_1^3\ . \tag{347}$$

Specifically, by completing the squares, we get :

$$\begin{aligned} \overbrace{V_1(x_1)}^{\cdot} \;&=\; -x_1^2 \;-\; x_1^4 \;+\; x_1^3\, d \qquad \forall (x_1, d) \in \mathbb{R}^2\ , && (348)\\ &\leq\; -\tfrac{1}{2}\, x_1^2 \;-\; \tfrac{1}{2}\, x_1^4 \qquad \forall (x_1, d) \in \mathbb{R} \times [-1, 1]\ . && (349) \end{aligned}$$

Step 2 : We consider the system :

$$\left\{ \begin{aligned} \dot{x}_1 \;&=\; x_2 \;+\; x_1^2\, d\ ,\\ \dot{x}_2 \;&=\; u_2\ . \end{aligned} \right. \tag{350}$$

The formula (202) yields :

$$V_2(x_1, x_2) \;=\; \tfrac{1}{2}\, x_1^2 \;+\; \tfrac{1}{2}\, (x_2 + x_1 + x_1^3)^2 \tag{351}$$

as a CLF. Using (349), we check that its derivative satisfies :

$$\overbrace{V_2(x_1, x_2)}^{\cdot} \;\leq\; -\tfrac{1}{2}\, x_1^2 \;-\; \tfrac{1}{2}\, x_1^4 \;+\; [x_2 + x_1 + x_1^3]\, \left[x_1 + u_2 + (1 + 3x_1^2)(x_2 + x_1^2 d)\right]\ , \tag{352}$$

for all (x_1, x_2, d) in $\mathbb{R}^2 \times [-1, 1]$. Then, by completing the squares, we get :

$$(x_2 + x_1 + x_1^3)\, (1 + 3x_1^2)\, x_1^2\, d \;\leq\; \tfrac{1}{4}\, x_1^4 \;+\; (x_2 + x_1 + x_1^3)^2 (1 + 3x_1^2)^2\ , \tag{353}$$

for all (x_1, x_2, d) in $\mathbb{R}^2 \times [-1, 1]$. We conclude that :

$$\begin{aligned} &\overbrace{V_2(x_1, x_2)}^{\cdot} \;\leq && (354)\\ &\quad -\tfrac{1}{2}\, x_1^2 \;-\; \tfrac{1}{4}\, x_1^4 \;+\; [x_2 + x_1 + x_1^3]\, \left[x_1 + u_2 + [1 + 3x_1^2][x_2 + (x_2 + x_1 + x_1^3)(1 + 3x_1^2)]\right]\ . \end{aligned}$$

A cancellation design leads to the following stabilizer :

$$\begin{aligned} \phi_2(x_1, x_2) \;&=\; -\left(x_1 + [1 + 3x_1^2][x_2 + (x_2 + x_1 + x_1^3)(1 + 3x_1^2)]\right) \;-\; [x_2 + x_1 + x_1^3]\ , && (355)\\ &=\; -\left(3x_1 + 3x_2 + 2x_1^3 + 9x_1^2 x_2 + 6x_1^3 + 15x_1^5 + 9x_1^7 + 18x_1^4 x_2\right)\ . && (356) \end{aligned}$$

It gives :

$$\overbrace{V_2(x_1,x_2)}^{\cdot} \leq -\tfrac{1}{2}\,x_1^2 \;-\; \tfrac{1}{4}\,x_1^4 \;-\; [x_2+x_1+x_1^3]^2 \qquad \forall (x_1,x_2,d)\in\mathbb{R}^2\times[-1,1]\ . \tag{357}$$

Step 3 : We consider the full order system (343). The formula (202) gives :

$$\begin{aligned} V_3(x_1,x_2,x_3) \;=\; & \tfrac{1}{2}\,x_1^2 \;+\; \tfrac{1}{2}\,(x_2+x_1+x_1^3)^2 \\ & + \tfrac{1}{2}\,\left(x_3+\left[3x_1+3x_2+2x_1^3+9x_1^2x_2+6x_1^3+15x_1^5+9x_1^7+18x_1^4x_2\right]\right)^2\ . \end{aligned} \tag{358}$$

Its derivative satisfies, for all (x_1,x_2,x_3,d) in $\mathbb{R}^3\times[-1,1]$,

$$\begin{aligned} \overbrace{V_3(x_1,x_2,x_3)}^{\cdot} \leq\; & -\tfrac{1}{2}\,x_1^2 \;-\; \tfrac{1}{4}\,x_1^4 \;-\; [x_2+x_1+x_1^3]^2 \\ & + \left(x_3+[3x_1+3x_2+2x_1^3+9x_1^2x_2+6x_1^3+15x_1^5+9x_1^7+18x_1^4x_2]\right)\times \\ & \times\ \Big([x_2+x_1+x_1^3]+u \\ & \quad +\left(3+6x_1^2+18x_1x_2+18x_1^2+75x_1^4+63x_1^6+72x_1^3x_2\right)\left(x_2+x_1^2d\right) \\ & \quad +\left(3+9x_1^2+18x_1^4\right)x_3\Big)\ . \end{aligned} \tag{359}$$

So, going on exactly as in the previous steps, we get the stabilizer :

$$\begin{aligned} \phi_3(x_1,x_2,x_3) \;=\; & \\ & -\left(x_2+x_1+x_1^3\right) - \left(3+6x_1^2+18x_1x_2+18x_1^2+75x_1^4+63x_1^6+72x_1^3x_2\right)x_2 \\ & -\left(3+9x_1^2+18x_1^4\right)x_3 - \left(x_3+\left[3x_1+3x_2+2x_1^3+9x_1^2x_2+6x_1^3+15x_1^5+9x_1^7+18x_1^4x_2\right]\right) \\ & -\left(3+6x_1^2+18x_1x_2+18x_1^2+75x_1^4+63x_1^6+72x_1^3x_2\right)^2\ . \end{aligned} \tag{360}$$

It gives the derivative :

$$\begin{aligned} \overbrace{V_3(x_1,x_2,x_3)}^{\cdot} \leq\; & -\tfrac{1}{2}\,x_1^2 \;-\; [x_2+x_1+x_1^3]^2 \\ & - \left(x_3+[3x_1+3x_2+2x_1^3+9x_1^2x_2+6x_1^3+15x_1^5+9x_1^7+18x_1^4x_2]\right)^2\ , \end{aligned} \tag{361}$$

for all (x_1,x_2,x_3,d) in $\mathbb{R}^3\times[-1,1]$.

To summarize the key point in this example is the combination of :

- the recursive use of the formula (202) to handle each step where a differentiator is added,
- the completion of squares to manipulate the terms with disturbances •

We have seen in Theorem 145 that, when a matching condition is satisfied, we can arbitrarily attenuate the action of the disturbance on the state. This property, although weakened, extends to systems in feedback form. Namely, what can be arbitrarily attenuated, is the action of the disturbance on the first state of this form; i.e., x_1 in (210) :
Consider the following disturbed system :

$$\left\{\begin{array}{rcl} \dot{x} &=& f(x) \;+\; g(x)\,(y+d_x)\ , \\ \dot{y} &=& h(x,y)+u+d_y\ , \end{array}\right. \tag{362}$$

where the functions f and g are C^{k+1} and the function h is C^k. We have (see also [28, 37, 27, 35]) :

Theorem 363 ([58])
Assume the existence of a C^{k+2} Lyapunov function V_x, a strictly positive real number λ, and a C^{k+1} function ϕ_x satisfying :

$$\overbrace{V_x(x)}^{\cdot} \;=\; L_f V_x(x) \;+\; L_g V_x(x)\,\phi_x(x) \;\leq\; -\lambda\, V_x(x) \qquad \forall x \in \mathbb{R}^n \;. \tag{364}$$

Let α be a class $\mathcal{K}^\infty$ function satisfying :

$$\alpha(|x|) \;\leq\; V_x(x) \;. \tag{365}$$

Let also γ_d be any non negative function such that we can find a strictly positive real number s_0 and a non negative real number μ to satisfy[10] *:*

$$\gamma_d \circ \alpha^{-1}(s^2) \;\leq\; \mu\, s \qquad \forall s \in [0, s_0] \;, \tag{366}$$

Under these conditions, there exist a C^{k+1} Lyapunov function V_y satisfying :

$$(\gamma_d(|x|))^2 \;\leq\; (1-\varepsilon)\,\lambda\, V_y(x,y) \qquad \forall (x,y) \;, \tag{367}$$

for some $\varepsilon > 0$, and a C^k function $\phi_y : \mathbb{R}^n \to \mathbb{R}$ such that we have :

$$\overbrace{V_y(x,y)}^{\cdot} \;\leq\; -\lambda\, V_y(x,y) \;+\; |(d_x, d_y)|^2 \qquad \forall (x, y, d_x, d_y) \;. \tag{368}$$

Remark 369

1. The inequalities (367) and (368) are exactly in the form needed to apply the small gain Theorem 78. In particular the fact that, for any given function γ_d, we can match the inequality (367) says that the above Theorem embeds actually a nonlinear L^∞ gain assignment result. Note however that the argument of γ_d in (367) is x and not (x, y). This says that the nonlinear L^∞ from (d_x, d_y) to x only can be assigned.

2. As Theorem 337, Theorem 363 can be used recursively to deal with systems in the form :

$$\left\{ \begin{array}{rcl} \dot{x}_1 &=& f_1(x_1) \;+\; g_1(x_1)\,(x_2 + d_1) \;, \\ \dot{x}_2 &=& f_2(x_1, x_2) \;+\; g_2(x_1, x_2)\,(x_3 + d_2) \;, \\ &\vdots& \\ \dot{x}_n &=& f_n(x_1, \dots, x_n) \;+\; g_n(x_n, \dots, x_n)\,(u + d_n) \;, \end{array} \right. \tag{370}$$

   and to design a controller which makes the closed loop system ISS and attenuates arbitrarily the action of $(d_1, \dots, d_n)$ on x_1 •

[10] The condition (366) is needed to guarantee the existence of a function V_y which is C^1 around the origin. A hint is that (365) and (366) give, for $|x|$ small enough, the inequality

$$(\gamma_d(|x|))^2 \;\leq\; (\gamma_d \circ \alpha^{-1}(V_x(x)))^2 \;\leq\; \mu^2\, V_x(x)$$

to be compared with (367).

Example 371 : Gain assignment and partial state feedback
Consider the system :

$$\left\{\begin{array}{rcl} \dot{x}_1 & = & -x_1^3 + x_4^3 \,, \\ \dot{x}_2 & = & -x_2 + x_3 + x_1 x_4 \,, \\ \dot{x}_3 & = & -x_2 \,, \\ \dot{x}_4 & = & x_5 + x_3 + x_1 x_4 \,, \\ \dot{x}_5 & = & u + x_2 \,. \end{array}\right. \tag{372}$$

We look for a continuous global asymptotic stabilizer which depends only on (x_4, x_5).

To solve this problem, we observe that this system (372) is nothing but the system (155) to which is added a disturbed differentiator. So we continue along the lines of Example 154.

From the inequalities (33) and (45) obtained for the solutions of the system (28); i.e., the (x_1, x_2, x_3) subsystem with x_4 as input, and from Theorem 78, we know that the problem can be solved by finding, for the system :

$$\left\{\begin{array}{rcl} \dot{x} & = & y + d_2 + d_1 y \,, \\ \dot{y} & = & u + d_3 \,, \end{array}\right. \tag{373}$$

a CLF V_y and a continuous function ϕ_y such that, with the control :

$$u = \phi_y(x, y) \,, \tag{374}$$

the following inequality holds for the derivative :

$$\dot{\overbrace{V_y(x,y)}} \leq -\lambda V_y(x,y) + \gamma_1(|d_1|) + \gamma_2\left(\sqrt{d_2^2 + d_3^2}\right) \,, \tag{375}$$

where γ_1 and γ_2 are class $\mathcal{K}$ functions and λ is a strictly positive real number satisfying :

$$\gamma_1(2|x|) + \gamma_2(8x^2) \leq (1-\varepsilon)\,\lambda\, V_y(x,y) \qquad \forall (x,y) \in \mathbb{R}^2 \tag{376}$$

for some strictly positive real number ε.

The system (373) is made of the system (see (156)) :

$$\dot{x} = u_x + d_2 + d_1 x \,, \tag{377}$$

to which is added the disturbed differentiator :

$$\left\{\begin{array}{rcl} \dot{y} & = & u + d_3 \,, \\ u_x & = & y \,. \end{array}\right. \tag{378}$$

From Example 154, we know that, for the reduced order system (377), the CLF :

$$V_x(x) = \tfrac{1}{4} x^4 \tag{379}$$

and the control :

$$u_x = \phi_x(x) = -\frac{\lambda}{4} x - \left(x^3 + \frac{3x^{\frac{7}{3}}}{4}\right) \,, \tag{380}$$

yield :

$$\overbrace{V_x(x)}^{\cdot} \;\leq\; -\lambda\, V_x(x) \;+\; \frac{d_2^2}{4} \;+\; \frac{d_1^4}{4} \;. \tag{381}$$

For the full order system (373), the formula (202) gives the CLF :

$$V_y(x,y) \;=\; V_x(x) \;+\; \tfrac{1}{2}\,(y-\phi_x(x))^2 \;. \tag{382}$$

By using (381), we get :

$$\begin{aligned}
\overbrace{V_y(x,y)}^{\cdot} \;&\leq\; -\lambda\, V_x(x) \;+\; \frac{d_2^2}{4} \;+\; \frac{d_1^4}{4} && (383)\\
&\quad + \;(y-\phi_x(x))\,[V_x'(x)+u+d_3-\phi_x'(x)\,(y+d_2+d_1x)] \;\;, \\
&\leq\; -\lambda\, V_x(x) \;+\; \frac{d_2^2}{4} \;+\; \frac{d_1^4}{4} \;+\; (y-\phi_x(x))\,\left[x^3+u-\phi_x'(x)y\right] && (384)\\
&\quad + \left(\frac{(y-\phi_x(x))^2}{2}+\frac{d_3^2}{2}\right) \;+\; \left((y-\phi_x(x))^2\phi_x'(x)^2+\frac{d_2^2}{4}\right)\\
&\quad + \left(\frac{3(y-\phi_x(x))^{\frac{4}{3}}\phi_x'(x)^{\frac{4}{3}}x^{\frac{4}{3}}}{4}+\frac{d_1^4}{4}\right) \;,\\
&\leq\; -\lambda\, V_x(x) \;+\; \frac{d_2^2}{4} \;+\; \frac{d_1^4}{4} \;+\; \frac{d_3^2}{2} \;+\; \frac{d_2^2}{4} \;+\; \frac{d_1^4}{4} && (385)\\
&\quad + \;(y-\phi_x(x))\left[x^3+u-\phi_x'(x)y+\frac{y-\phi_x(x)}{2}+(y-\phi_x(x))\phi_x'(x)^2\right.\\
&\qquad\qquad\qquad\qquad\qquad\qquad\left. +\frac{3(y-\phi_x(x))^{\frac{1}{3}}\phi_x'(x)^{\frac{4}{3}}x^{\frac{4}{3}}}{4}\right] \;,
\end{aligned}$$

where we have completed the squares and used Young's inequality to get (384). From there a cancellation design leads to the control :

$$\begin{aligned}
u \;&=\; \phi_y(x,y) \;, && (386)\\
&=\; -\frac{\lambda}{2}\,(y-\phi_x(x)) && (387)\\
&\quad - \left[x^3-\phi_x'(x)y+\frac{y-\phi_x(x)}{2}+(y-\phi_x(x))\phi_x'(x)^2+\frac{3(y-\phi_x(x))^{\frac{1}{3}}\phi_x'(x)^{\frac{4}{3}}x^{\frac{4}{3}}}{4}\right] \;.
\end{aligned}$$

It gives :

$$\overbrace{V_y(x,y)}^{\cdot} \;\leq\; -\lambda\, V_y(x,y) \;+\; \frac{d_3^2+d_2^2}{2} \;+\; \frac{d_1^4}{2} \;. \tag{388}$$

This shows that we have obtained (375) with the functions :

$$\gamma_1(s) \;=\; \frac{s^4}{2} \quad , \qquad \gamma_{23}(s) \;=\; \frac{s^2}{2} \;. \tag{389}$$

It follows that the constraint (376) rewrites :

$$8\,x^4 \;+\; 32\,x^4 \;\leq\; (1-\varepsilon)\,\tfrac{\lambda}{4}\,x^4 \qquad \forall x \;. \tag{390}$$

Hence global asymptotic stability holds if we choose :

$$\lambda > 160 . \tag{391}$$

To summarize, the key point of this example is once again the combination of using the formula (202) to handle the addition of a differentiator and the inequalities to handle the terms with disturbances •

4 C^1 dissipative systems and application to systems in feedforward form

In practice, the models with justified nonlinear dynamics we are able to write can often be obtained via a variational formulation like within the Euler-Lagrange formalism. For such systems a natural candidate Lyapunov function is say the total energy. Indeed this quantity is either preserved or dissipated and gives usually a norm information on the state. By combining such systems in series or parallel, we get quite involved structures. The technique we are presenting now is dedicated to some such structures.

4.1 C^1 dissipative systems

Definition 392
The system :

$$\dot{x} \;=\; f(x) \;+\; g(x,u)\,u \tag{393}$$

is said C^1 dissipative if there exists a C^1 Lyapunov function V, called the storage function, satisfying :

$$L_f V(x) \;\leq\; 0 \qquad \forall x \in \mathbb{R}^n \ . \tag{394}$$

Remark 395
From Theorem 16, for a C^1 dissipative system, with the control at the origin, the origin is a globally stable solution. Unfortunately, the converse is not true (see [3, Example V.4.11]). It follows that, even if we know that the origin is globally stable, when u is at the origin, we still need to exhibit a C^1 Lyapunov function to establish C^1 dissipativity •

Example 396 : The cart pendulum system
Consider the celebrated cart-pendulum system. Let :
– (M,x) be mass and position of the cart which is moving horizontally,
– (m,l,θ) be mass, length and angular deviation from the upward position for the pendulum which is pivoting around a point fixed on the cart,
– finally F be a horizontal force acting on the cart and considered here as control.
The associated kinetic and potential energies are :

$$\begin{aligned} E_k(\dot{x},\dot{\theta}) &= \tfrac{1}{2}(M+m)\dot{x}^2 \;+\; \tfrac{1}{2}m\,l^2\,\dot{\theta}^2 \;+\; m\,l\,\cos(\theta)\,\dot{x}\,\dot{\theta}\ , & (397)\\ E_p(\theta) &= m\,l\,g\,(\cos(\theta)+1)\ . & (398)\end{aligned}$$

It follows from the Euler-Lagrange equation that the dynamics are :

$$\left\{\begin{aligned} (M+m)\ddot{x} + ml\cos(\theta)\ddot{\theta} &= ml\dot{\theta}^2\sin(\theta) + F\ ,\\ \ddot{x}\cos(\theta) + l\ddot{\theta} &= g\sin(\theta)\ . \end{aligned}\right. \tag{399}$$

We restrict here our attention to the three coordinates $(\theta,\dot{x},\dot{\theta})$ leaving in the manifold $\mathbb{S}^1\times\mathbb{R}^2$.

We consider the function :

$$\mathcal{V}(\theta,\dot{x},\dot{\theta}) \;=\; E_k(\dot{x},\dot{\theta}) \;+\; E_p(\theta)\ ; \tag{400}$$

i.e., the total energy. It is C^1 and proper on $\mathbb{S}^1\times\mathbb{R}^2$. It has two stationary points, a global minimum at $(\pi,0,0)$ and a saddle point at $(0,0,0)$. So it is a C^1 Lyapunov function for the point $(\pi,0,0)$. Also, either by invoking the Euler-Lagrange formalism or by direct computation, we get :

$$\dot{\overbrace{\mathcal{V}(\theta,\dot{x},\dot{\theta})}} \;=\; F\,\dot{x}\ . \tag{401}$$

This shows that the cart-pendulum system, restricted to the coordinates $(\theta,\dot{x},\dot{\theta})$, is C^1 dissipative and actually passive with $\dot{x}$ as output.

To summarize the key point of this example is, for this mechanical system, the use of the total energy as storage function for establishing the C^1 dissipativity property •

From its definition, a C^1 dissipative system is a passive system for the particular output function :

$$h(x,u) \;=\; \frac{\partial V}{\partial x}(x)\,g(x,u)\ . \tag{402}$$

If the system is also zero-state detectable, it follows from Theorem 20 that global asymptotic stability of the origin is provided by the control obtained as solution of :

$$u \;=\; -h(x,u) \;=\; -\frac{\partial V}{\partial x}(x)\,g(x,u)\ , \tag{403}$$

when it makes sense. This result can be generalized as follows :

Theorem 404 ([8, Corollary 1.6])
Assume the system (393) is C^1 dissipative and zero-state detectable with output function $\left(\; W(x) \quad \frac{\partial V}{\partial x}(x)g(x,0)\;\right)$. Then, for any real number $\overline{u}$ in $(0,+\infty]$, there exists a continuous global asymptotic stabilizer strictly bounded in norm by $\overline{u}$.

Remark 405
The controller mentioned in this Theorem is any continuous function ϕ satisfying :

$$|\phi(x)| \;<\; \overline{u} \qquad \forall x\in\mathbb{R}^n\ , \tag{406}$$

$$\left|\frac{\partial V}{\partial x}(x)g(x,0)\right| \neq 0 \qquad \Longrightarrow \qquad \frac{\partial V}{\partial x}(x)g(x,\phi(x))\,\phi(x) \;<\; 0\ . \tag{407}$$

For instance, when g does not depend on u, we can take :

$$\phi(x) \;=\; -\min\left\{\frac{\overline{u}}{|L_gV(x)|}\,,\,1\right\}\,L_gV(x)^T\ . \tag{408}$$

When g depends on u, it is more difficult to give an expression, but instead we can propose the dynamic controller (see [48]) :

$$\left\{\begin{array}{rcl} \dot{\chi} &=& -\left[1-\frac{|\chi|^2}{\overline{u}^2}\right]\left[\frac{\partial V}{\partial x}(x)\,g(x,\chi)\right]^\top \;-\; \chi\ ,\\[2mm] u &=& \chi\ . \end{array}\right. \tag{409}$$

It is designed by applying the backstepping technique with the CLF :

$$V_\chi(x,\chi) \;=\; V(x) \;-\; \frac{\overline{u}^2}{2}\,\log\left(1-\frac{|\chi|^2}{\overline{u}^2}\right)\ . \tag{410}$$

Example 411 : Dealing with input constraints (other solution)
Consider the system :

$$\left\{\begin{array}{rcl} \dot{x}_1 & = & u \ , \\ \dot{x}_2 & = & x_3^3 \ , \\ \dot{x}_3 & = & -x_2 \ + \ u \ . \end{array}\right. \tag{412}$$

We look for a continuous global asymptotic stabilizer ϕ satisfying :

$$|\phi(x_1, x_2, x_3)| \ \leq \ 1 \ . \tag{413}$$

The system (412) is C^1 dissipative. Indeed the C^1 Lyapunov function :

$$V(x_1, x_2, x_3) \ = \ \tfrac{1}{2}\, x_1^2 \ + \ \tfrac{1}{2}\, x_2^2 \ + \ \tfrac{1}{4}\, x_3^4 \tag{414}$$

gives :

$$\overbrace{V(x_1, x_2, x_3)}^{\cdot} \ = \ (x_1 + x_3^3)\, u \ . \tag{415}$$

This derivative is made non positive by taking :

$$u \ = \ \phi(x_1, x_2, x_3) \ = \ -\mathtt{sat}(x_1 + x_3^3) \ , \tag{416}$$

with the function `sat` defined in (242). This control law guarantees the global asymptotic stability of the origin since this point is the only solution of the following two sets of equations :

$$\left\{\begin{array}{rcl} \dot{x}_1 & = & 0 \\ \dot{x}_2 & = & x_3^3 \\ \dot{x}_3 & = & -x_2 \end{array}\right. \quad , \qquad x_1 \ + \ x_3^3 \ = \ 0 \ \bullet \tag{417}$$

Example 418 : Orbit transfer with weak but continuous thrust
The Gauss equations describe the dynamics of a point mass satellite subject to a thrust. In appropriate coordinates and for the case of a two dimensional thrust, these equations are (see [7]) :

$$\left\{\begin{array}{rcl} \dot{p} & = & 2\, p\, S \ , \\ \dot{\varepsilon} & = & -j\, \varpi(p, \varepsilon)\, \varepsilon + [\varepsilon + (2 + \mathrm{Re}(\varepsilon))]\, S \ , \\ \dot{\eta} & = & -j\, [\varpi(p, \varepsilon) - \mathrm{Im}(\eta) W]\, \eta + \frac{1}{2}(1 + |\eta|^2)\, W \ , \end{array}\right. \tag{419}$$

where :
– the state variables (p, ε, η) in $\mathbb{R} \times \mathbb{C}^2$, called the orbital parameters, are, with $j^2 = -1$,

$$\left\{\begin{array}{rcl} p & = & a\,(1 - e^2) \ , \\ \varepsilon & = & e\, [\cos(\omega + \Omega) + j \sin(\omega + \Omega)]\, [\cos(L) - j \sin(L)] \ , \\ \eta & = & \tan(i/2)\, [\cos(\Omega) + j \sin(\Omega)]\, [\cos(L) - j \sin(L)] \ , \\ L & = & \omega \ + \ \Omega \ + \ v \ , \end{array}\right. \tag{420}$$

where a is the semi-major axis, e is the eccentricity, i is the inclination to the equator, Ω is the right ascension of the ascending node, ω is the angle between the ascending node and the perigee, v is the true anomaly. Note that, from their definitions, p and ε satisfy :

$$|\varepsilon| < 1 \quad , \qquad p > 0 \ . \tag{421}$$

– S is the component of the thrust which is colinear with the kinetic momentum and W is the component orthogonal to S and to the earth-satellite axis.
– Re and Im denote the real and imaginary part and $\varpi(p, \varepsilon)$ is a C^1 function.

We look for a continuous control law for (S, W) satisfying :

$$S^2 \ + \ W^2 \ \leq \ \gamma_{\max}^2 \tag{422}$$

for a given strictly positive real number $\gamma_{\max}$, and such that the orbit whose parameters are $(\bar{p}, 0, 0)$ is made an asymptotically stable attractor.

To get a solution, we note that, by definition, without any thrust, the orbit and therefore each orbital parameter is unchanged. So an appropriate storage function is given by the sum of functions of one parameter only. With in mind the constraints (421), we let :

$$V(p, \varepsilon, \eta) \ = \ \tfrac{1}{2} \left[\log\left(\frac{p}{\bar{p}}\right)\right]^2 \ - \ \tfrac{1}{2} \log\left(1 - |\varepsilon|^2\right) \ + \ |\eta|^2 \ . \tag{423}$$

This yields :

$$\overbrace{V(p, \varepsilon, \eta)}^{\cdot} \ = \ \left(2 \log\left(\frac{p}{\bar{p}}\right) + \frac{|\varepsilon|^2 + \mathrm{Re}(\varepsilon)(2 + \mathrm{Re}(\varepsilon))}{1 - |\varepsilon|^2}\right) S \ + \ \mathrm{Re}(\eta)\,(1 + |\eta|^2)\, W \ . \tag{424}$$

This establishes the C^1 dissipativity property. Also $\overbrace{V(p, \varepsilon, \eta)}^{\cdot}$ is made non negative by picking the vector $\begin{pmatrix} S \\ W \end{pmatrix}$ as a Lipschitz continuous function of the state, colinear to the vector :

$$-\begin{pmatrix} 2 \log\left(\frac{p}{\bar{p}}\right) + \frac{|\varepsilon|^2 + \mathrm{Re}(\varepsilon)(2 + \mathrm{Re}(\varepsilon))}{1 - |\varepsilon|^2} \\ \mathrm{Re}(\eta)\,(1 + |\eta|^2) \end{pmatrix}$$

and with any non zero norm, satisfying (422), as long as this latter vector is non zero. With such a control, asymptotic stability of the desired orbital parameters can be established with the help of Theorem 16, considering the full order system (419). Actually, by studying the linearization of the closed-loop system, we can check that local exponential stability holds also •

4.2 C^1 dissipative systems via reduction or extension

Consider a system whose dynamics can be written in the following triangular form :

$$\left\{\begin{array}{rcl} \dot{y} & = & h(x) \ + \ h_u(x, y, u)\, u \ , \\ \dot{x} & = & f(x) \ + \ f_u(x, u)\, u \ , \end{array}\right. \tag{425}$$

with x in $\mathbb{R}^n$ and y in $\mathbb{R}^q$ and where f and h are C^1 and zero at the origin. We want to study when the C^1 dissipativity of the full order system (425) implies the C^1 dissipativity of the reduced order x subsystem, and conversely.

4.2.1 Reduction

Assume the system (425) is C^1 dissipative; i.e., we have a C^1 Lyapunov function V_y satisfying, when u is at the origin,

$$\overset{\cdot}{\overbrace{V_y(x,y)}} = \frac{\partial V_y}{\partial x}(x,y)\, f(x) + \frac{\partial V_y}{\partial y}(x,y)\, h(x) \leq 0 . \tag{426}$$

We are looking for a C^1 Lyapunov function V_x satisfying :

$$\frac{\partial V_x}{\partial x}(x)\, f(x) \leq 0 . \tag{427}$$

Clearly if there exists a function $\mathcal{M}$ satisfying :

$$\frac{\partial V_y}{\partial y}(x, \mathcal{M}(x)) = 0 , \tag{428}$$

it is sufficient to take :

$$V_x(x) = V_y(x, \mathcal{M}(x)) . \tag{429}$$

But, since V_y is C^1 Lyapunov function, for each given x, it has a global minimum and therefore at least one stationary point reached at say $y = \mathcal{M}(x)$. So this function $\mathcal{M}$ satisfying (428) does exist. We can also impose :

$$\mathcal{M}(0) = 0 . \tag{430}$$

Lemma 431
If V_y is C^2 and $\mathcal{M}$ is Hölder continuous of order strictly larger than $\frac{1}{2}$ then the x subsystem of (425) is C^1 dissipative with storage function :

$$V_x(x) = V_y(x, \mathcal{M}(x)) . \tag{432}$$

Moreover, if the function $\mathcal{M}$ is C^1 and, for each point x in $\mathbb{R}^n \setminus \{0\}$, each unit vector v in $\mathbb{R}^q$, and each positive real number k, we can find a point $\mathfrak{y}$ in $\mathbb{R}^q$ satisfying[11] *:*

$$k \left| \frac{\partial V_\mathfrak{y}}{\partial x}(x, \mathfrak{y}) \right| < \frac{\partial V_\mathfrak{y}}{\partial \mathfrak{y}}(x, \mathfrak{y})\, v , \tag{433}$$

where we have let :

$$\mathfrak{y} = y - \mathcal{M}(x) \quad , \qquad V_\mathfrak{y}(x, \mathfrak{y}) = V_y(x, \mathfrak{y} + \mathcal{M}(x)) , \tag{434}$$

then we have :

$$\overset{\cdot}{\overbrace{\mathcal{M}(x)}} = \frac{\partial \mathcal{M}}{\partial x}(x)\, f(x) = h(x) \tag{435}$$

and $V_x(x) + \frac{1}{2}|y - \mathcal{M}(x)|^2$ is another storage function for the full order system (425).

[11] The condition (433) says that the growth of $\frac{\partial V_\mathfrak{y}}{\partial \mathfrak{y}}$ with respect to $\mathfrak{y}$ dominates the one of $\frac{\partial V_\mathfrak{y}}{\partial x}$ in all the directions.

Remark 436
The equation (435) implies that for each solution $X(x,t;0)$ of the reduced order x subsystem with u at the origin, we have :

$$\mathcal{M}(X(x,T;0)) \;-\; \mathcal{M}(x) \;=\; \int_0^T h(X(x,t;0))dt \qquad \forall T \geq 0 \;. \tag{437}$$

So, if the origin is a globally asymptotically stable solution of the reduced order x subsystem with u at the origin, we get, with (430),

$$\mathcal{M}(x) \;=\; -\lim_{T\to+\infty} \int_0^T h(X(x,t;0))dt \;\bullet \tag{438}$$

4.2.2 Extension

Let us study now how to establish C^1 dissipativity by extension. The idea to tackle this problem is to find conditions under which Lemma 431 applies.

First, we assume that, when u is at the origin, the origin is a globally asymptotically stable solution of the reduced order x subsystem of the system (425). With (590), this implies there exist two class $\mathcal{K}^\infty$ functions α_1 and α_2 such that, for any solution $X(x,t;0)$ of the x subsystem with u at the origin, we have :

$$\alpha_1(|X(x,t;0)|) \;\leq\; \alpha_2(|x|)\exp(-t) \qquad \forall t \geq 0 \;. \tag{439}$$

Second, to guarantee the existence of the limit in (438), we assume that the function $|h|$ is sufficiently "flat" around the origin so that we have[12] :

$$\limsup_{|x|\to 0} \frac{|h(x)|}{\alpha_1(|x|)} \;<\; +\infty \;. \tag{440}$$

Indeed, with (439), this inequality implies the existence of a class $\mathcal{K}^\infty$ function α_3 such that we have :

$$\int_0^\infty |h(X(x,t;0))|dt \;\leq\; \alpha_3(|x|) \qquad \forall x \in \mathbb{R}^n \;. \tag{441}$$

Example 442 :
Consider the system :

$$\dot{x} \;=\; -x^3 \;. \tag{443}$$

Its solutions are :

$$X(x,t) \;=\; \frac{x}{\sqrt{1+2tx^2}} \;. \tag{444}$$

When the function h is :

$$h(x) \;=\; x^2 \;, \tag{445}$$

we get :

$$\int_0^T h(X(x,t))dt \;=\; \tfrac{1}{2}\log(1+2Tx^2) \;. \tag{446}$$

[12] In the case where the origin is locally exponentially stable, we can always get $\alpha_1(s) = s$ for s small, so that (440) holds when h is C^1 and zero at the origin.

As T goes to infinity, this integral goes to $+\infty$ for all non zero x. On the other hand, when we have the "flatter" function :

$$h(x) \;=\; x^4 \;, \tag{447}$$

we get :

$$\int_0^T h(X(x,t))dt \;=\; \frac{x^2}{2}\left(1 - \frac{1}{1+2Tx^2}\right) \;. \tag{448}$$

This integral converges to $\frac{x^2}{2}$. We conclude that, for the given system (443), the function $h(x) = x^2$ is not "flat" enough whereas the function $h(x) = x^4$ is "flat" enough. It is interesting to relate this fact with the following ones :

- The origin of the system :

$$\left\{ \begin{array}{rcl} \dot{y} &=& x^2 \;+\; u \;, \\ \dot{x} &=& -x^3 \;+\; u \;, \end{array} \right. \tag{449}$$

  is not asymptotically stabilizable by a continuous controller.

- The origin of the system :

$$\left\{ \begin{array}{rcl} \dot{y} &=& x^4 \;+\; u \;, \\ \dot{x} &=& -x^3 \;+\; u \;, \end{array} \right. \tag{450}$$

  is asymptotically stabilizable by the continuous controller :

$$u \;=\; -x \;-\; (1+x)\left(y + \tfrac{1}{2}x^2\right) \tag{451}$$

  associated with the Lyapunov function :

$$V_y(x,y) \;=\; \tfrac{1}{2}\,x^2 \;+\; \tfrac{1}{2}\,\left(y + \tfrac{1}{2}x^2\right)^2 \quad \bullet \tag{452}$$

When (441) holds, from [11, Théorème (3.149)] for instance, we know that the following function $\mathcal{M}$ is well defined and continuous on $\mathbb{R}^n$:

$$\mathcal{M}(x) \;=\; -\int_0^\infty h(X(x,t;0))dt \;. \tag{453}$$

And, as it can be checked "by hand", we have :

$$\dot{\overbrace{\mathcal{M}(x)}} \;=\; h(x) \qquad \forall x \in \mathbb{R}^n \;. \tag{454}$$

With this function, the system (425) gives, when u is at the origin,

$$\left\{ \begin{array}{rcl} \dot{\overbrace{y - \mathcal{M}(x)}} &=& 0 \;, \\ \dot{x} &=& f(x) \;. \end{array} \right. \tag{455}$$

It follows that the origin is a globally stable solution of (425) when u is at the origin. If $\mathcal{M}$ is not only continuous but also C^1, then the system (425) is C^1 dissipative. Specifically, with the help of Theorem 22, our assumption of global asymptotic stability implies the existence of a C^1 Lyapunov function V_x such that the function :

$$W_x(x) \;=\; -L_f V_x(x) \tag{456}$$

is positive definite. Then by letting :

$$V_y(x,y) \;=\; V_x(x) \;+\; \tfrac{1}{2}\,|y - \mathcal{M}(x)|^2 \;, \tag{457}$$

we get, for the system (425), when u is at the origin,

$$\dot{\overbrace{V_y(x,y)}} \;=\; -W_x(x)\;. \tag{458}$$

This shows that V_y is a storage function.

To summarize the possibility of going from global asymptotic stability to C^1 dissipativity while extending the reduced order x subsystem into the full order system (425) relies on the following two properties :

1. The function $|h|$ is "flat" enough so that (440) holds. This guarantees the existence and the continuity of a function $\mathcal{M}$ satisfying (454).

2. The function $\mathcal{M}$ is actually C^1. This guarantees that the system (425) is C^1 dissipative.

4.2.3 Application

With the storage function (457), we are ready to apply Theorem 404 and possibly get a global asymptotic stabilizer if the zero-state detectability assumption holds for the system (425). So, with the two properties above, this detectability condition and the stabilizer we can get, we are with the closed loop full order system (425) in exactly the same situation as we were with the open loop reduced order x subsystem. This means that we are ready to deal with the further extended system :

$$\left\{\begin{array}{rcl} \dot z &=& k(x,y) \;+\; k_u(x,y,z,u)\,u\;,\\ \dot y &=& h(x) \;+\; h_u(x,y,u)\,u\;,\\ \dot x &=& f(x) \;+\; f_u(x,u)\,u\;,\end{array}\right. \tag{459}$$

which is obtained by adding the integrator :

$$\dot z \;=\; k(x,y) \;+\; k_u(x,y,z,u)\,u\;. \tag{460}$$

Here the new state z integrates functions of all the previously introduced state components. So we maybe able by recursion to do a Lyapunov design of global asymptotic stabilizers for systems in the form :

$$\left\{\begin{array}{rcl} \dot x_n &=& f_n(x_1,\ldots,x_{n-1}) \;+\; g_n(x_1,\ldots,x_n,u)\,u\;,\\ &\vdots& \\ \dot x_2 &=& f_2(x_1) \;+\; g_1(x_1,x_2,u)\,u\;,\\ \dot x_1 &=& f_1(x_1) \;+\; g_1(x_1,u)\,u\;.\end{array}\right. \tag{461}$$

This form is called feedforward form. It is obtained by adding recursively integrators.

In the following paragraphs, we study this technique in more details. But before closing this section, an important remark has to be made.

Remark 462

1. As for the backstepping technique, the Lyapunov function V_y defined in (457) has received an interpretation in terms of a new of coordinate. It corresponds to the forwarding technique. It has been introduced and developed in [48]. The change of coordinate is :

$$\mathfrak{y} \;=\; y \;-\; \mathcal{M}(x) \; . \tag{463}$$

   Its existence relies on the existence of the vector valued function $\mathcal{M}$. It yields to the Lyapunov function :

$$V_{\mathfrak{y}}(x,\mathfrak{y}) \;=\; V_y(x,y) \;=\; V_x(x) \;+\; \tfrac{1}{2}\,|\mathfrak{y}|^2 \; . \tag{464}$$

2. The Lyapunov function V_y can also be written as :

$$V_y(x,y) \;=\; V_x(x) \;+\; \tfrac{1}{2}\,|y|^2 \;+\; S(x,y) \tag{465}$$

   with :

$$S(x,y) \;=\; -y^T\mathcal{M}(x) \;+\; \tfrac{1}{2}\,|\mathcal{M}(x)|^2 \; . \tag{466}$$

   Namely V_y is made of the sum of three terms :
   – the Lyapunov function for the reduced order x subsystem,
   – the Lyapunov function for the extending y subsystem, which would be appropriate if x were at the origin,
   – a cross term S.

   This point of view with a cross term has been introduced and developed in [26]. It applies to a broader class of systems than the forwarding technique (see [64]), the existence of a C^1 scalar cross term S holding under weaker conditions than a C^1 vector function $\mathcal{M}$. In the following, we deal only with the change of coordinates, leaving to the reader to consult [64] to get more information about the cross term technique •

4.3 The forwarding technique with an exact change of coordinates

4.3.1 The technique

The forwarding technique with an exact change of coordinates we have introduced in the previous section applies in fact to a larger class of systems than (425) (and therefore, by recursion, larger than (461)). It is :

$$\left\{\begin{array}{rcl} \dot{y} &=& h_y(y) \;+\; h_x(x,y)\,x \;+\; h_u(x,y,u)\,u \; , \\ \dot{x} &=& f(x) \;+\; f_u(x,y,u)\,u \; , \end{array}\right. \tag{467}$$

where all the functions are C^1 and with still the assumption of global asymptotic stability of the origin of the x subsystem when u is at the origin. More precisely :

H1 : There exists a C^1 Lyapunov function V_x such that the function :

$$W_x(x) \;=\; -L_f V_x(x) \tag{468}$$

is positive definite.

The difference between (425) and (467) is in the fact that the function h can actually depend on y and is then decomposed into the sum $h_y + h_x x$. However this dependence on y is restricted by the following assumption :

H2 : There exists a C^1 Lyapunov function V_y such that the function :

$$W_y(y) \;=\; -L_{h_y} V_y(y) \tag{469}$$

is non positive.

This implies that, when u and x are at the origin, the origin is a globally stable solution of the y subsystem.

For the system (467), we assume the knowledge of a function Ψ which is C^1 and satisfies the properties :

P1 : We have :

$$\Psi(0, y) \;=\; y \ . \tag{470}$$

P2 : For each non negative real number c, the set $\{y \,:\, \exists x \in \mathbb{R}^n \,:\, |x| \leq c\,,\ |\Psi(x, y)| \leq c\}$ is bounded.

P3 : Ψ is a solution of the partial differential equation :

$$\frac{\partial \Psi}{\partial x}(x, y)\, f(x) \;+\; \frac{\partial \Psi}{\partial y}(x, y)\, (h_y(y) + h_x(x, y) x) \;=\; h_y(\Psi(x, y)) \ . \tag{471}$$

The fact of needing to find a solution to (471) implies that finding an expression for Ψ may not be an easy task. But assuming we have it satisfying the properties P1, P2 and P3, we introduce a new "coordinate"[13] :

$$\mathfrak{y} \;=\; \Psi(x, y) \ . \tag{472}$$

Then, for the $(x, \mathfrak{y})$ "coordinates", the dynamics (467) give :

$$\left\{ \begin{array}{rcl} \dot{\mathfrak{y}} & = & h_y(\mathfrak{y}) \;+\; \left(\frac{\partial \Psi}{\partial x}(x, y) f_u(x, y, u) + \frac{\partial \Psi}{\partial y}(x, y) h_u(x, y, u) \right) u \ , \\ \dot{x} & = & f(x) \;+\; f_u(x, y, u)\, u \ . \end{array} \right. \tag{473}$$

We have (see [48]) :

[13] $\mathfrak{y}$ is abusively called a coordinate since we do not impose a bijection between (x, y) and $(x, \mathfrak{y})$. In fact the analysis goes by picking a solution in the (x, y) coordinates, study its properties with $(x, \mathfrak{y})$ and infer properties in the (x, y) coordinates.

Theorem 474
Under the assumptions H1 and H2, if there exists a function Ψ satisfying the properties P1, P2 and P3, the system (467) is C^1 dissipative with storage function :

$$V_y(x,y) \;=\; V_x(x) \;+\; V_y(\Psi(x,y)) \; . \tag{475}$$

Moreover, for any real number $\overline{u}$ in $(0,+\infty]$, there exists a continuous global asymptotic stabilizer strictly bounded in norm by $\overline{u}$ if the system (467) is zero-state detectable with output function :

$$\begin{pmatrix} \frac{\partial V_x}{\partial x}(x) f_u(x,y,0) + \frac{\partial V_y}{\partial y}(\Psi(x,y)) \left(\frac{\partial \Psi}{\partial x}(x,y) f_u(x,y,0) + \frac{\partial \Psi}{\partial y}(x,y) h_u(x,y,0)\right) \\ W_x(x) \\ W_y(\Psi(x,y)) \end{pmatrix}^T$$

or, in the case where W_x is positive definite, if the y subsystem is zero-state detectable with input (x,u) and output function :

$$\begin{pmatrix} \frac{\partial V_y}{\partial y}(y) \frac{\partial \Psi}{\partial x}(0,y) f_u(0,y,0) + \frac{\partial V_y}{\partial y}(y) h_u(0,y,0) & \quad W_y(y) \end{pmatrix} \; .$$

Remark 476

1. For the storage function (475), it is sufficient for V_x to be a C^1 Lyapunov function such that W_x given by (468) is non negative (and not positive definite as imposed by H1).

2. An expression of the stabilizer can be obtained as explained in Remark 405. It does require an expression for the function Ψ •

Example 477 :
Consider the system :

$$\left\{ \begin{array}{rcl} \dot{y} &=& x_1 \;+\; x_2^2 \; , \\ \dot{x}_1 &=& x_2 \; , \\ \dot{x}_2 &=& -x_1 \;-\; x_2 \;+\; u \; . \end{array} \right. \tag{478}$$

Assumption H1 holds with the functions :

$$V_x(x_1,x_2) \;=\; x_2^2 \;+\; x_2 x_1 \;+\; x_1^2 \; , \tag{479}$$

$$W_x(x_1,x_2) \;=\; V_x(x_1,x_2) \; . \tag{480}$$

Also, the origin is an exponentially stable solution of the (x_1,x_2) subsystem when u is zero.
Assumption H2 holds with the functions :

$$V_y(y) \;=\; \tfrac{1}{2} y^2 \quad , \qquad W_y(y) \;=\; 0 \; . \tag{481}$$

Since the solutions $(X_1(x_1,x_2;t), X_2(x_1,x_2;t))$ of the (x_1,x_2) subsystem are exponentially converging to zero when u is zero, the function :

$$\mathcal{M}(x_1,x_2) \;=\; -\int_0^\infty \left[X_1(x_1,x_2;s) + X_2(x_1,x_2;s)^2\right] ds \; , \tag{482}$$

is well defined, C^1 (see [11, Théorème 3.150]) and, as can be checked "by hand" satisfies :

$$\overbrace{\mathcal{M}(x_1, x_2)}^{\cdot} \;=\; x_1 \;+\; x_2^2 \;, \tag{483}$$

$$\mathcal{M}(0,0) \;=\; 0 \;. \tag{484}$$

Finally, the function Ψ defined as :

$$\Psi(x_1, x_2, y) \;=\; y \;-\; \mathcal{M}(x_1, x_2) \tag{485}$$

satisfies P1, P2 and P3.

To get an expression for $\mathcal{M}$, and therefore Ψ, we note that it is a solution of :

$$\frac{\partial \mathcal{M}}{\partial x_1}(x_1, x_2)\, x_2 \;-\; \frac{\partial \mathcal{M}}{\partial x_2}(x_1, x_2)\,(x_1 + x_2) \;=\; \left(x_1 \;+\; x_2^2\right) \;. \tag{486}$$

By taking a solution in the form of a polynomial function of order 2 in (x_1, x_2) and by equating the coefficients, we get :

$$\mathcal{M}(x_1, x_2) \;=\; -\left(x_1 + x_2 + \tfrac{1}{2}[x_1^2 + x_2^2]\right) \;. \tag{487}$$

With this expression at hand, we introduce the new coordinate :

$$\mathfrak{y} \;=\; \Psi(x, y) \;=\; y \;+\; \left(x_1 + x_2 + \tfrac{1}{2}[x_1^2 + x_2^2]\right) \;. \tag{488}$$

The system (478) rewrites :

$$\left\{\begin{array}{rcl} \dot{\mathfrak{y}} & = & (1 + x_2)\, u \;, \\ \dot{x}_1 & = & x_2 \;, \\ \dot{x}_2 & = & -x_1 \;-\; x_2 \;+\; u \;. \end{array}\right. \tag{489}$$

Then, with (479), we propose the Lyapunov function :

$$V_{\mathfrak{y}}(x_1, x_2, \mathfrak{y}) \;=\; \left(x_2^2 + x_2 x_1 + x_1^2\right) \;+\; \tfrac{1}{2}\,\mathfrak{y}^2 \;. \tag{490}$$

It gives :

$$\overbrace{V_{\mathfrak{y}}(x_1, x_2, \mathfrak{y})}^{\cdot} \;=\; (2[1 + x_2]\mathfrak{y} + 2[2x_2 + x_1])\, u \;-\; \left(x_2^2 + x_2 x_1 + x_1^2\right) \;. \tag{491}$$

Since the function :

$$W_{\mathfrak{y}}(x_1, x_2, \mathfrak{y}) \;=\; \left(x_2^2 + x_2 x_1 + x_1^2\right) \;+\; (2[1 + x_2]\mathfrak{y} + 2[2x_2 + x_1])^2 \tag{492}$$

is positive definite, we conclude that :

$$\phi(x_1, x_2, \mathfrak{y}) \;=\; -\,(2[1 + x_2]\mathfrak{y} + 2[2x_2 + x_1]) \tag{493}$$

is a global asymptotic stabilizer which, thanks to (487), we can write explicitly in :

$$\phi(x_1, x_2, \mathfrak{y}) \;=\; -\left(2[1 + x_2]\left[y + \left(x_1 + x_2 + \tfrac{1}{2}[x_1^2 + x_2^2]\right)\right] + 2[2x_2 + x_1]\right) \;. \tag{494}$$

To summarize the key point in this example is to make sure that the conditions of existence of the change of "coordinate" are satisfied and then find it via the solution of the partial differential equation (486) •

4.3.2 About the change of "coordinates"

We have seen that the forwarding technique with an exact change of coordinates relies mainly on the existence and the knowledge of an expression of the C^1 function Ψ satisfying the three properties P1, P2 and P3. To get a better idea of what this function Ψ is, we set u at the origin and we observe that the mapping between the solutions $(X(x,y,t), Y(x,y,t))$ of the system (467) and those $\mathfrak{Y}(\mathfrak{y},t)$ of the system (473) yields, for all t in the respective domain of existence,

$$\Psi(X(x,y,t), Y(x,y,t)) \;=\; \mathfrak{Y}(\Psi(x,y);t) \; . \tag{495}$$

Since h_y is C^1, the solutions $\mathfrak{Y}(\mathfrak{y},t)$ are unique and we have :

$$\mathfrak{Y}(\mathfrak{Y}(\mathfrak{y},t);-t) \;=\; \mathfrak{y} \; . \tag{496}$$

So (495) rewrites :

$$\Psi(x,y) \;=\; \mathfrak{Y}(\Psi(X(x,y,t), Y(x,y,t));-t) \; . \; . \tag{497}$$

If the corresponding solutions exist on $[0,+\infty)$ and $(-\infty,0]$ respectively, this implies :

$$\Psi(x,y) \;=\; \lim_{t\to+\infty} \mathfrak{Y}(\Psi(X(x,y,t), Y(x,y,t));-t) \; . \tag{498}$$

Now, we remark that we have, from H1 and (470) in P1,

$$\lim_{t\to+\infty} X(x,y,t) \;=\; 0 \quad , \qquad \Psi(0,y) \;=\; y \; . \tag{499}$$

So if "everything works fine" when the limit "enters" the function $\mathfrak{Y}$ in (497), the function Ψ is given by :

$$\Psi(x,y) \;=\; \lim_{t\to+\infty} \mathfrak{Y}(Y(x,y,t);-t) \; . \tag{500}$$

To be fully rigorous, let us restrict our attention to the case where the function h_y is linear; i.e.,

$$h_y(y) \;=\; H\, y \; . \tag{501}$$

In this case, we get :

$$\mathfrak{Y}(\mathfrak{y},t) \;=\; \exp(Ht)\,\mathfrak{y} \; , \tag{502}$$

$$Y(x,y,t) \;=\; \exp(Ht)\,y \;+\; \int_0^t \exp(H(t-s))\,h_x(X(x,y,s), Y(x,y,s))\,X(x,y,s)\,ds \; . \tag{503}$$

So (500) rewrites :

$$\Psi(x,y) \;=\; y \;+\; \int_0^\infty \exp(-Hs)\,h_x(X(x,y,s), Y(x,y,s))\,X(x,y,s)\,ds \; . \tag{504}$$

With a proof mimicking the arguments of [64, Sections 5.2.1 and 5.2.2], we have the following existence result :

Lemma 505
Assume H1 and H2 hold with $h_y(y) = Hy$. If we have :

1. $$\max\left\{\mathrm{Re}\left(\textit{eigen value}\left(\tfrac{\partial f}{\partial x}(0)\right)\right)\right\} \;<\; \min\left\{0\,,\, \mathrm{Re}\left(\textit{eigen value}\,(H)\right)\right\} \;, \tag{506}$$

2. the function (of x) $\sup_y \frac{|h_x(x,y)|}{1+|y|}$ is locally bounded,

then the function Ψ given by (504) is well defined, C^1, satisfies the properties P1, P2 and P3 and is a solution of (471) which, in the present case, is :

$$\frac{\partial \Psi}{\partial x}(x,y)\, f(x) \;+\; \frac{\partial \Psi}{\partial y}(x,y)\, [Hy + h_x(x,y)x] \;=\; H\, \Psi(x,y) \ . \tag{507}$$

Remark 508
Since the condition (506) implies the local exponential stability of the origin of the x subsystem of (467), the "flatness" constraint discussed in section 4.2 is always satisfied •

Knowing with Lemma 505 that the function Ψ exists, as already observed in Example 477, we can get an expression for it :

- either by solving the partial differential equation (471) ,
- or, for each (x,y), by computing the solutions $(X(x,y,t), Y(x,y,t))$ of the system (467) with u at the origin, those $\mathfrak{Y}(\mathfrak{y},t)$ of the system (473) and then by evaluating the limit (500) (or the integral (504)) (method of characteristics).

For any real world application, this program seems to be out of range. Nevertheless, it may still be possible to get an expression for Ψ. The idea is that, by definition, we should have, when $h_y(y) = 0$:

$$\dot{\overbrace{\Psi(x,y) - y}} \;=\; h_x(x,y)\, x \ . \tag{509}$$

This says that we look for a function of (x,y) whose derivative is $h_x(x,y)x$. To make such a search fruitful, it maybe opportune to get prepared while dealing before with the x subsystem.

Example 510 : The cart pendulum system (continued) (see [71])
Let us come back to the cart-pendulum of Example 396. Our ultimate goal is to make the upward position of the pendulum and the zero position of the cart asymptotically stable with a basin of attraction as large as possible. As a step toward this goal, we study here the possibility of asymptotically stabilizing the homoclinic orbit[14] of the pendulum and the zero position of the cart. The motivation is that, if such an objective is met, all the solutions arrive in finite time in the neighborhood of the point to be made asymptotically stable. In this situation, we can switch the controller to a linear controller locally stabilizing this point.

To meet our objective and simplify the computations, we modify the cart-pendulum system into another one by changing the control into :

$$F \;=\; m \sin(\theta) \left(g\cos(\theta) - l\dot{\theta}^2 \right) \;+\; \frac{g}{l}\, \left(M + m\sin(\theta)^2 \right) u \ , \tag{511}$$

[14] The one which makes just one turn in infinite time.

with u as new control. Also, in order to simplify the equations, we change the coordinates and time as follows :

$$\left\{\begin{array}{rcl} x & = & \frac{1}{l}\,\mathcal{X}\ , \\ v & = & \frac{1}{\sqrt{gl}}\,\dot{\mathcal{X}}\ , \\ \theta & = & \theta\ , \\ \omega & = & \sqrt{\frac{l}{g}}\,\dot{\theta}\ , \\ \tau & = & \sqrt{\frac{g}{l}}\,t\ . \end{array}\right. \tag{512}$$

Then still denoting by " $\dot{}$ " the derivation with respect to the new time τ, the new system is :

$$\left\{\begin{array}{rcl} \dot{x} & = & v\ , \\ \dot{v} & = & u\ , \\ \dot{\theta} & = & \omega\ , \\ \dot{\omega} & = & \sin(\theta)\ -\ u\cos(\theta)\ . \end{array}\right. \tag{513}$$

We view this system as made of a first subsystem with state variables (v, θ, ω) which is extended by adding the integrator giving the position. So we proceed in two steps :

Step 1 : We consider the (v, θ, ω) subsystem. We look for a controller making asymptotically stable the following set with an as large as possible domain of attraction :

$$\mathcal{S}\ =\ \left\{(v,\theta,\omega):\ E(\omega,\theta) = \tfrac{1}{2}\omega^2 + \cos(\theta) = 1\ ,\ v = 0\right\}\ . \tag{514}$$

E is actually the total mechanical energy of the pendulum alone. When $E = 1$, the pendulum is on its homoclinic orbit. To meet the stabilization objective of this step, it is sufficient to find a C^1 positive definite and proper function V in the variable $(E - 1, v)$ and to make its derivative non positive on an as large as possible domain. Since we have :

$$\dot{E}\ =\ -\cos(\theta)\,\omega\,u\quad ,\qquad \dot{v}\ =\ u\ , \tag{515}$$

a good candidate for V is :

$$V(E-e,v)\ =\ V_E(E-1)\ +\ \tfrac{k_v}{2}\,v^2\ , \tag{516}$$

where k_v is a strictly positive real number and V_E is a C^2 function defined at least on $[-2, +\infty)$ where it is proper and satisfies[15] :

$$V_E(s)\ =\ 0\quad \Rightarrow s\ =\ 0\ , \tag{517}$$

$$V_E'(s)\ =\ 0\quad \Rightarrow s\ =\ 0\ , \tag{518}$$

$$\lim_{s\to+\infty} V_E(s)\ =\ +\infty\ . \tag{519}$$

At this stage we do not specify what V_E is. As in Example 299, we want to keep some flexibility for handling the second step. We get :

$$\dot{V}\ =\ [-V_E'(E-1)\,\omega\,\cos(\theta)\ +\ k_v\,v]\,u\ . \tag{520}$$

[15] The last property and the fact that θ lives in $\mathbb{S}^1$ imply that if, E is bounded, so is ω.

Hence an appropriate controller is :

$$\phi(v,\theta,\omega) \;=\; m(v,\theta,\omega)\;(V_E'(E-1)\,\omega\,\cos(\theta) \;-\; k_v\,v) \tag{521}$$

where m is any strictly positive, Lipschitz continuous function.

Step 2 : We consider now the full order system (512). To meet our stabilization objective it remains to asymptotically stabilize x at zero. We observe that the system (512) is obtained from the (v,θ,ω) subsystem by adding the integrator :

$$\dot{x} \;=\; v\;. \tag{522}$$

So we apply the forwarding technique with an exact change of coordinates. It leads us to look for a C^1 function $\mathcal{M}(v,\theta,\omega)$ such that, when u is given by (521), we have :

$$\overset{\cdot}{\overbrace{\mathcal{M}(v,\theta,\omega)}} \;=\; v\;. \tag{523}$$

To find an expression for this function, we try to express v as the derivative of a function of (v,θ,ω). We remark that, with (521), the $\dot{v}$ equation in (513) rewrites as :

$$u \;=\; \dot{v} \;=\; m(v,\theta,\omega)\;(V_E'(E-1)\,\omega\,\cos(\theta) \;-\; k_v\,v)\;\;, \tag{524}$$

or :

$$k_v\,v \;=\; \frac{\dot{v}}{m(v,\theta,\omega)} \;+\; V_E'(E-1)\,\omega\,\cos(\theta)\;. \tag{525}$$

For the last term, with the help of (515), we get :

$$\begin{aligned} V_E'(E-1)\,\omega\,\cos(\theta) \;&=\; \overset{\cdot}{\overbrace{V_E'(E-1)\,\sin(\theta)}} \;+\; V_E''(E-1)\,\sin(\theta)\,\omega\,\cos(\theta)\,u\;, && (526)\\ &=\; \overset{\cdot}{\overbrace{V_E'(E-1)\,\sin(\theta)}} \;+\; V_E''(E-1)\,\sin(\theta)\,\omega\,\cos(\theta)\,\dot{v}\;. && (527) \end{aligned}$$

This yields :

$$k_v\,v \;=\; \overset{\cdot}{\overbrace{V_E'(E-1)\sin(\theta)}} \;-\; \frac{1-m(v,\theta,\omega)V_E''(E-1)\,\sin(\theta)\,\omega\,\cos(\theta)}{m(v,\theta,\omega)}\,\dot{v}\;. \tag{528}$$

Also, for any C^1 function q, we have :

$$\overset{\cdot}{\overbrace{q(v)}} \;=\; q'(v)\,\dot{v}\;. \tag{529}$$

Collecting all the above relations, we get :

$$k_v\,v \;=\; -\overset{\cdot}{\overbrace{q(v)}}+\overset{\cdot}{\overbrace{V_E'(E-1)\sin(\theta)}} \;-\; \frac{1-m(v,\theta,\omega)\,[V_E''(E-1)\,\sin(\theta)\,\omega\,\cos(\theta)+q'(v)]}{m(v,\theta,\omega)}\,\dot{v} \tag{530}$$

This shows that, if we choose the function m satisfying :

$$\frac{1-m(v,\theta,\omega)\,[V_E''(E-1)\,\sin(\theta)\,\omega\,\cos(\theta)+q'(v)]}{m(v,\theta,\omega)} \;=\; 1\;, \tag{531}$$

or in other words :

$$m(v,\theta,\omega) \;=\; \frac{1}{1+q'(v)+V_E''(E-1)\sin(\theta)\cos(\theta)\omega}\;, \tag{532}$$

then we have simply :

$$k_v\, v \;=\; -\dot{\overbrace{q(v)}}+\dot{\overbrace{V_E'(E-1)\sin(\theta)}}\;-\;\dot{v}\;. \tag{533}$$

By comparing to (523), we see that we have obtained the expression we were looking for :

$$\mathcal{M}(v,\theta,\omega) \;=\; \frac{-q(v)+V_E'(E-1)\sin(\theta)-v}{k_v}\;. \tag{534}$$

Before going on, we have to make sure that the function m given by (532) is appropriate; i.e., for all (v,θ,ω), we have :

$$1+q'(v)+V_E''(E-1)\sin(\theta)\cos(\theta)\omega \;>\; 0\;. \tag{535}$$

But, since the definition of E in (514) gives :

$$|\omega| \;\leq\; \sqrt{2(E+1)}\;, \tag{536}$$

we conclude that it is sufficient to impose that $q'(v)$ is non negative for all v and :

$$V_E''(s)\,\sqrt{2(s+2)} \;\leq\; \eta \;<\; 1 \qquad \forall s\in[-2,+\infty)\;. \tag{537}$$

Let us note also that if the functions[16] $|V_E'(s)|\sqrt{2(s+2)}$ and $\frac{|v|}{q'(v)}$ are bounded then so is the control ϕ in (521).

Let us now come back to our design. We follow the forwarding technique with an exact change of coordinates and let :

$$\mathfrak{y} \;=\; x \;-\; \frac{q(v)-V_E'(E-1)\sin(\theta)-v}{k_v}\;, \tag{538}$$

$$u \;=\; u_{\mathfrak{y}} \;+\; \frac{V_E'(E-1)\omega\cos(\theta)-k_v v}{1+q'(v)+V_E''(E-1)\sin(\theta)\cos(\theta)\omega}\;. \tag{539}$$

This gives :

$$\dot{\mathfrak{y}} \;=\; \frac{1+q'(v)+V_E''(E-1)\sin(\theta)\cos(\theta)\omega}{k_v}\,u_{\mathfrak{y}}\;. \tag{540}$$

Then, with (516), we take :

$$V(x,v,\theta,\omega) \;=\; V_E(E-1)\;+\;\frac{k_v}{2}\,v^2\;+\;V_{\mathfrak{y}}(\mathfrak{y}) \tag{541}$$

where $V_{\mathfrak{y}}$ is any C^1 Lyapunov function. The stationary points of V are all on the homoclinic orbit we want to asymptotically stabilize. We get :

$$\dot{V} \;=\; [-V_E'\omega\cos(\theta)+k_v v]\left[u_{\mathfrak{y}}\;+\;\frac{V_E'\omega\cos(\theta)-k_v v}{1+q'+V_E''\sin(\theta)\cos(\theta)\omega}\right] \tag{542}$$

[16]We can take for instance : $q(v)=v|v|$ and $V_E(s)=(1+s^2)^{\frac{1}{4}}$.

$$+ \frac{1 + q' + V_E'' \sin(\theta)\cos(\theta)\omega}{k_v} V_{\mathfrak{y}}' u_{\mathfrak{y}} \ .$$

This yields to the possible choice :

$$u_{\mathfrak{y}} \ = \ - \left[-V_E'(E-1)\omega\cos(\theta) + k_v v\right] - \frac{1 + q'(v) + V_E''(E-1)\sin(\theta)\cos(\theta)\omega}{k_v} V_{\mathfrak{y}}'(\mathfrak{y}) \ . \quad (543)$$

Actually we can choose $u_{\mathfrak{y}}$, V_E, q and $V_{\mathfrak{y}}$ in such a way that :

- The final controller is bounded by any a priori given bound,
- We have, using (535),

$$\dot{V} \ = \ 0 \qquad \Rightarrow \qquad \left[V_E'(E-1)\omega\cos(\theta) - k_v v\right] \ = \ V_{\mathfrak{y}}'(\mathfrak{y}) \ = \ 0 \ . \quad (544)$$

$V_{\mathfrak{y}}$ being a Lyapunov function, with (543), we have also :

$$\dot{V} \ = \ 0 \qquad \Rightarrow \qquad \left[V_E'(E-1)\omega\cos(\theta) - k_v v\right] \ = \ \mathfrak{y} \ = \ u \ = \ 0 \ . \quad (545)$$

Then, by successive derivations, we can check that any solution of the closed loop system which satisfies :

$$\left[V_E'(E-1)\omega\cos(\theta) - k_v v\right] \ = \ \mathfrak{y} \ = \ u \ = \ 0 \quad (546)$$

satisfies also either :

$$E \ = \ 1 \quad , \qquad x \ = \ v \ = \ 0 \ , \quad (547)$$

or :

$$\theta \ \in \ \{0, \pi\} \quad , \qquad \omega \ = \ x \ = \ v \ = \ 0 \ . \quad (548)$$

From Theorem 16, we conclude that all the solutions of the closed loop system converge either to the desired set $\left\{(x, v, \theta, \omega) : \ E(\omega, \theta) = \frac{1}{2}\omega^2 + \cos(\theta) = 1 \ , \ x = v = 0\right\}$ or one of the two the equilibrium points $(\theta \in \{0, \pi\}, \omega = x = v = 0)$. By looking at the linearization of the dynamics at these points, it can be seen that they have a stable manifold and an unstable manifold. From [38], we conclude that the set of points in $\mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{R}$ and not belonging to the domain of attraction of the desired set is of measure zero. And the solutions issued from such points all converge to the equilibrium points $(\theta = k\pi, x = v = \omega = 0)$ •

4.4 The forwarding technique with an approximate change of coordinates

We have mentioned that, from a practical point of view, the main difficulty in applying the forwarding technique with an exact change of coordinates is to find an expression for the function Ψ. So, one may ask if we could use an approximation. We address this question now (see [48]).

Let Ψ_a be an approximation of Ψ to which we impose to be C^1, and to satisfy P1 and P2. It allows us to introduce the new "coordinate" :

$$\mathfrak{y} \ = \ \Psi_a(x, y) \ . \quad (549)$$

The system (467) rewrites as :

$$\left\{\begin{array}{rcl} \dot{\mathfrak{y}} & = & h_y(\mathfrak{y}) \;+\; \mathfrak{h}_x(x,y)\, x \;+\; \mathfrak{h}_u(x,y,u)\, u \;, \\ \dot{x} & = & f(x) \;+\; \mathfrak{f}_u(x,\mathfrak{y},u)\, u \;, \end{array}\right. \tag{550}$$

where in particular the function $\mathfrak{h}_x$ is given by :

$$\mathfrak{h}_x(x,y)\, x \;=\; \frac{\partial \Psi_a}{\partial x}(x,y)\, f(x) \;+\; \frac{\partial \Psi_a}{\partial y}(x,y)\, (h_y(y) + h_x(x,y)x) \;-\; h_y(\Psi_a(x,y)) \;. \tag{551}$$

As opposed to (471), Ψ_a being only an approximation, the term $\mathfrak{h}_x\, x$ is not zero. Concerning f and h_y, we keep the assumptions H1 and H2 considered for the case where we have the exact Ψ. Concerning $\mathfrak{h}_x x$, we assume[17] :

P3' : There exists a continuous function ℓ which is proper and with a continuous strictly positive derivative ℓ' defined on $(0,+\infty)$ such that $\ell'(V_x(x))\frac{\partial V_x}{\partial x}(x)$ has a continuous extension at the origin and we have :

$$\left|\frac{\partial V_y}{\partial y}(\Psi_a(x,y))\, \mathfrak{h}_x(x,y)\, x\right| \;\leq\; \ell'(V_x(x))\, W_x(x)\, (1 + V_y(\Psi_a(x,y))) \qquad \forall (x,y)\;. \tag{552}$$

In the case where the origin is locally exponentially stable for the x subsystem of (550) with u at the origin, the above inequality reduces to :

$$\left|\frac{\partial V_y}{\partial y}(\Psi_a(x,y))\, \mathfrak{h}_x(x,y)\right| \;\leq\; |x|\, \gamma(|x|)\, (1 + V_y(\Psi_a(x,y))) \qquad \forall (x,y) \tag{553}$$

with γ some non decreasing, non negative continuous function.

Remark 554

1. While P3 was leading to the fact that, with the exact change of "coordinate", $\mathfrak{h}_x x$ was zero, P3' imposes only a magnitude limitation on this term for x small and $\Psi_a(x,y)$ large. W_x in the right hand side of (552) quantifies how much $\mathfrak{h}_x x$ should be "flat" for x close to the origin with respect to the strength of attractiveness of the origin of the x subsystem. In particular, in a generic situation, (552) or (553) implies (with P1) :

$$\mathfrak{h}_x(0,y) \;=\; 0\;. \tag{555}$$

This is mainly saying that the transformed $\mathfrak{h}_x x$ should be "flatter" around the origin than the original $h_x x$.

2. With P1, (551) and (555) give :

$$0 \;=\; \frac{\partial \Psi_a}{\partial x}(0,y)\,\frac{\partial f}{\partial x}(0) \;+\; \frac{\partial^2 \Psi_a}{\partial x \partial y}(0,y) \odot h_y(y) \;+\; h_x(0,y) \;-\; \frac{\partial h_y}{\partial y}(y)\,\frac{\partial \Psi_a}{\partial x}(0,y)\;. \tag{556}$$

So instead of the partial differential equation (471) in the (x,y) variables, we have now a partial differential equation in the y variable only (see Example 560) •

[17] The meaning and implications of the inequalities (552) or (553) are given in [48].

We have :

Theorem 557 ([48])
Under the assumptions H1 and H2, if there exists a function Ψ_a satisfying the properties P1, P2 and P3', the system (467) is C^1 dissipative with storage function :

$$V_{\mathfrak{y}}(x, \Psi_a(x,y)) \;=\; 2\,\ell(V_x(x)) \;+\; \log(1 + V_y(\Psi_a(x,y)))\;. \tag{558}$$

Moreover, for any real number $\overline{u}$ in $(0,+\infty]$, there exists a continuous global asymptotic stabilizer strictly bounded in norm by $\overline{u}$ when the y subsystem of (467) is zero-state detectable with input (x,u) and output function :

$$\left(\; \frac{\partial V_y}{\partial x}(y)\frac{\partial \Psi_a}{\partial x}(0,y) f_u(0,y,0) + \frac{\partial V_y}{\partial y}(y) h_u(0,y,0) \qquad W_y(y) \;\right)$$

Remark 559

1. As opposed to (475), in (558), V_x must be the function given by assumption $H1$ with corresponding W_x positive definite (see Remark 476.1).
2. In (558), if not given, the function ℓ is to be designed such that (552) holds •

Example 560 : Example 477 continued
Let us come back to the system (478) and work with an approximate change of coordinate. For this, we restrict our attention to an approximating function Ψ_a of the form :

$$\Psi_a(x_1,x_2,y) \;=\; y \;-\; \mathcal{M}_a(x_1,x_2) \tag{561}$$

where $\mathcal{M}_a$ is to be designed so that :

$$\mathcal{M}_a(0,0) \;=\; 0\;. \tag{562}$$

In this case, we get, from (551) and (478),

$$\mathfrak{h}_x\left(x_1,x_2,y-\mathcal{M}_a(x_1,x_2)\right) \;=\; -\frac{\partial \mathcal{M}_a}{\partial x_1}(x_1,x_2)\,x_2 \;+\; \frac{\partial \mathcal{M}_a}{\partial x_2}(x_1,x_2)\,(x_1+x_2) \;+\; (x_1+x_2^2) \tag{563}$$

and

$$V_y(y) \;=\; \tfrac{1}{2}\,y^2\;. \tag{564}$$

Then, the condition (552) of P3' is equivalent to :

$$\begin{aligned} |y-\mathcal{M}_a(x_1,x_2)| &\left| -\tfrac{\partial \mathcal{M}_a}{\partial x_1}(x_1,x_2)\,x_2 \;+\; \tfrac{\partial \mathcal{M}_a}{\partial x_2}(x_1,x_2)\,(x_1+x_2) \;+\; (x_1+x_2^2)\right| \\ &\qquad\qquad \leq\; (x_1^2+x_2^2)\,\gamma(\sqrt{x_1^2+x_2^2})\;\left(1+\tfrac{1}{2}\,|y-\mathcal{M}_a(x_1,x_2)|^2\right) \end{aligned} \tag{565}$$

for all (x_1,x_2,y). This is implied in particular by :

$$\left| -\frac{\partial \mathcal{M}_a}{\partial x_1}(x_1,x_2)\,x_2 \;+\; \frac{\partial \mathcal{M}_a}{\partial x_2}(x_1,x_2)\,(x_1+x_2) \;+\; (x_1+x_2^2)\right| \;\leq\; (x_1^2+x_2^2)\,\gamma(\sqrt{x_1^2+x_2^2})\;, \tag{566}$$

for all (x_1, x_2). In its turn this condition says that the left hand side should be of order two at the origin and therefore implies :

$$-\frac{\partial \mathcal{M}_a}{\partial x_1}(0,0)\, x_2 \;+\; \frac{\partial \mathcal{M}_a}{\partial x_2}(0,0)\,(x_1 + x_2) \;+\; x_1 \;=\; 0\ . \tag{567}$$

We get directly :

$$\frac{\partial \mathcal{M}_a}{\partial x_1}(0,0) \;=\; \frac{\partial \mathcal{M}_a}{\partial x_2}(0,0) \;=\; -1\ . \tag{568}$$

Having obtained a constraint only on the derivatives of $\mathcal{M}_a$ at the origin. Let us try if a function $\mathcal{M}_a$ simply linear would be appropriate. We pick :

$$\mathcal{M}_a(x_1, x_2) \;=\; -x_1 \;-\; x_2\ . \tag{569}$$

We get that (566) and therefore (565) hold, with $\gamma(s) = 1$.

With the function $\mathcal{M}_a$ we have found, the change of coordinate is :

$$\mathfrak{y} \;=\; y \;+\; x_1 \;+\; x_2\ , \tag{570}$$

So the system (478) rewrites :

$$\left\{ \begin{array}{rcl} \dot{\mathfrak{y}} & = & x_2^2 \;+\; u\ , \\ \dot{y}_1 & = & x_2\ , \\ \dot{y}_2 & = & -x_1 \;-\; x_2 \;+\; u\ . \end{array} \right. \tag{571}$$

As expected, the term of second order x_2^2 as not been removed from the $\dot{\mathfrak{y}}$ equation (compare with (489)).

Following (558) in Theorem 557, we let[18] :

$$V_{\mathfrak{y}}(x_1, x_2, \mathfrak{y}) \;=\; 2\,\ell(x_2^2 + x_2 x_1 + x_1^2) \;+\; \log\left(1 + \tfrac{1}{2}\mathfrak{y}^2\right)\ , \tag{572}$$

with a function ℓ to be designed. This yields :

$$\overbrace{V_{\mathfrak{y}}(x_1, x_2, \mathfrak{y})}^{\cdot} \;=\; 2\,\ell'(x_2^2 + x_2 x_1 + x_1^2)\,\left[-(x_2^2 + x_2 x_1 + x_1^2) + (2x_2 + x_1)u\right] \;+\; \frac{\mathfrak{y}}{1 + \frac{1}{2}\mathfrak{y}^2}\,(x_2^2 + u)\ . \tag{573}$$

Since we have[19] :

$$\frac{\mathfrak{y}}{1 + \frac{1}{2}\mathfrak{y}^2}\, x_2^2 \;\leq\; \sqrt{2}\,(x_2^2 + x_2 x_1 + x_1^2) \qquad \forall(\mathfrak{y}, x_1, x_2)\ , \tag{574}$$

we choose the function ℓ as :

$$\ell(s) \;=\; \tfrac{1+\sqrt{2}}{2}\, s\ . \tag{575}$$

[18]Actually, for this particular case, a better choice is :

$$V_{\mathfrak{y}}(x_1, x_2, \mathfrak{y}) \;=\; \ell(x_2^2 + x_2 x_1 + x_1^2) \;+\; \left[\sqrt{1 + \mathfrak{y}^2} - 1\right]\ .$$

[19]It is to get such an inequality with the right hand side not depending on $\mathfrak{y}$ that the log is introduced in (572).

Indeed, this yields :

$$\overbrace{V_{\mathfrak{y}}(x_1,x_2,\mathfrak{y})}^{\cdot} \;=\; -(x_2^2+x_2x_1+x_1^2)+\left[\left(1+\sqrt{2}\right)(2x_2+x_1)+\frac{\mathfrak{y}}{(1+\frac{1}{2}\mathfrak{y}^2)}\right]u\;. \tag{576}$$

A candidate for a stabilizer is therefore :

$$\phi_a(x_1,x_2,\mathfrak{y}) \;=\; -\left[\left(1+\sqrt{2}\right)(2x_2+x_1)+\frac{\mathfrak{y}}{(1+\frac{1}{2}\mathfrak{y}^2)}\right]\;. \tag{577}$$

It gives :

$$\overbrace{V_{\mathfrak{y}}(x_1,x_2,\mathfrak{y})}^{\cdot} \;=\; -(x_2^2+x_2x_1+x_1^2)-\left[\left(1+\sqrt{2}\right)(2x_2+x_1)+\frac{\mathfrak{y}}{(1+\frac{1}{2}\mathfrak{y}^2)}\right]^2\;. \tag{578}$$

This implies global asymptotic stability of the origin.

To summarize, the key points of this example are :

1. By comparing (486) and (566), we see that we are asking to the approximation $\mathcal{M}_a$ of $\mathcal{M}$ to solve the partial differential equation (486) only up to the first order around the origin. Namely, we have transformed the problem of solving the partial differential equation (486) into the one of solving the linear system (567).

2. It is important to compare the new stabilizer ϕ_a in (577), obtained with the approximate change of coordinate, with ϕ in (494) obtained with the exact change of coordinate. In particular, we see that, for (x_1,x_2) fixed, ϕ_a is a bounded function of y, although we were not looking for this property. On the contrary, ϕ is not a bounded function of y. Not being able to remove the terms of higher order in (x_1,x_2), the strategy for the new stabilizer is to privilege the (x_1,x_2) components of the reduced order x subsystem at times where they are large without paying attention to what the y component of the integrator is doing at those times. Unfortunately, this latter fact leads typically to poor performance with too big excursions or too slow time response •

Let us recapitulate on the forwarding technique with an approximate change of coordinates :

- The main benefit is that, instead of solving exactly a partial differential equation in (x,y) like (507), it is sufficient to approximate its solution up to the first order in x, for x at the origin. As a consequence, typically, we are left with solving a partial differential equation in y only.

- The losses are :

  1. Instead of a function V_x with a non negative function W_x, we need now an expression of a function V_x with a positive definite W_x. This may generate difficulties when the forwarding technique with an approximate change of coordinates is applied recursively. However, the problem can be overcome some how as shown in [48, Proposition III.3]

2. We have to design the function ℓ by manipulating inequalities.
3. The class of stabilizers that we can reach is poorer. In particular they are typically bounded in the state component of the integrator and give poor performance.

Remark 579
A final remark on the forwarding technique with an exact or an approximate change of coordinate is that the design gives for the closed loop system a Lyapunov function depending on the function Ψ or Ψ_a which is difficult to handle. This prevents us from dealing with the disturbance attenuation problem as easily as in the backstepping technique since the derivative of these functions play an important role. Nevertheless by using small gain arguments as those presented in [72], several results on this problem have been obtained. See [73, 1, 46] for instance $\bullet$

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Glossary and Notations

a' **:** For a C^1 function $a : \mathbb{R} \to \mathbb{R}$, we denote by a' its derivative.

$\overbrace{a(x)}^{\cdot}$ **:** For any solution $X(x,t;d)$ of :

$$\dot{x} \;=\; f(x,d) \quad , \qquad X(x,0;d) \;=\; x \;, \tag{580}$$

defined on $[0,T)$ (see Definition 2) and for any C^1 function $a : \mathbb{R}^n \to \mathbb{R}$, we have :

$$\frac{\partial}{\partial t} a(X(x,t;d)) \;=\; \frac{\partial a}{\partial x}(X(x,t;d))\, f(X(x,t;d),d(t)) \qquad \text{for almost all} \quad t \in [0,T) \;. \tag{581}$$

In other respect, we can define a function $b : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ as :

$$b(x,d) \;=\; \frac{\partial a}{\partial x}(x)\, f(x,d) \;. \tag{582}$$

In view of the similarity of (581) and (582), we adopt the notation :

$$\overbrace{a(x)}^{\cdot} \;=\; \frac{\partial a}{\partial x}(x)\, f(x,d) \;. \tag{583}$$

But we insist on the fact that $\overbrace{a(x)}^{\cdot}$ is a function of (x,d) only and the time t is not even concerned.

$a\big|_S$ **:** For a function $a : \mathbb{R}^n \to \mathbb{R}^m$ and a subset S of $\mathbb{R}^n$, we denote by $a\big|_S$ the restriction of a to S; i.e., the function $a\big|_S : S \to \mathbb{R}^m$.

Adding a differentiator : Given a system :

$$\dot{x} \;=\; f(x,v,d_x) \;, \tag{584}$$

with state x, control v and disturbance d_x, we say that we add a differentiator when we consider the augmented system :

$$\left\{ \begin{array}{rcl} \dot{x} &=& f(x,v,d_x) \;, \\ \dot{v} &=& h(x,v,u,d_y) \;, \end{array} \right. \tag{585}$$

with state (x,v), control u and disturbance (d_x,d_y). This says that the control v for the system (584) is a state component of the system (585). In other words for the latter the control u acts on the derivative of the control of the former.

Adding an integrator : For the system :

$$\dot{x} \;=\; f(x,u,d_x) \tag{586}$$

with state x, control u and disturbance d_x, we say that we add an integrator when we consider the augmented system :

$$\left\{ \begin{array}{rcl} \dot{y} &=& h(y,x,u,d_y) \;, \\ \dot{x} &=& f(x,u,d_x) \;, \end{array} \right. \tag{587}$$

with state (x,y), control u and disturbance (d_x,d_y). This says that the new state component y integrates a function of y and all the other variables already present (x,u,d_y).

Backstepping : See adding a differentiator and Remark 200.3.

Cancellation design : When the derivative of a Lyapunov function V satisfies :

$$\dot{V} \; \leq \; T_- \; + \; L_g V \; \times \; (u \; + \; T\,) \tag{588}$$

where T_- is a non positive term and T is an arbitrary term, a cancellation design consists in choosing the control as :

$$u \; = \; -T \; - \; Q(x)\, L_g V(x)^T \tag{589}$$

where Q is a positive definite matrix.

(Class) C^k **:** A functions is said (of class) C^k if it has continuous partial derivatives up to and including order k. So a continuous function is (of class) C^0.

Class $\mathcal{K}$ **and** $\mathcal{K}^\infty$ **functions :** A function $\alpha : [0, \infty) \to [0, \infty)$ is said of class $\mathcal{K}$ if it is continuous, strictly increasing and $\alpha(0) = 0$. It is of class $\mathcal{K}^\infty$ if it is of class $\mathcal{K}$ and is unbounded.

Class $\mathcal{KL}$ **function :** A function $\beta : [0, \infty) \times [0, \infty) \to [0, \infty)$ is said of class $\mathcal{KL}$ if, for each $t \geq 0$, $\beta(\cdot, t)$ is of class $\mathcal{K}$, and, for each $r > 0$, $\beta(r, \cdot)$ is strictly decreasing and $\lim_{t\to\infty} \beta(r, t) = 0$. From [68, Proposition 7], β is a class $\mathcal{KL}$ function if and only if there exist two class $\mathcal{K}^\infty$ functions α_1 and α_2 satisfying :

$$\alpha_1(\beta(r, t)) \; \leq \; \alpha_2(r)\, \exp(-t) \qquad \forall (r, t) \; . \tag{590}$$

CLF : CLF stands for control Lyapunov function (see Definition 87).

Control Lyapunov function : Control Lyapunov functions are introduced in Definition 87.

Completing squares : See Young's inequality.

Disturbance attenuation : The problem of disturbance attenuation for the system :

$$\dot{x} \; = \; f(x, u, d) \tag{591}$$

with state x, control u and disturbance d consists in finding a controller $u = \phi(x)$ such that all the solutions of the closed loop system depend as less as possible on d. See Remark 46.2.

Domination design : When the derivative of a Lyapunov function V satisfies (588), a domination design consists in choosing the control as :

$$u \; = \; Q(x)\, L_g V(x)^T \tag{592}$$

where Q is a sufficiently large positive definite matrix.

Feedback form : A system is said to be in feedback form if we can find coordinates such that their dynamics are :

$$\left\{\begin{array}{rcl} \dot{x}_1 & = & f_1(x_1, x_2) \ , \\ \dot{x}_2 & = & f_2(x_1, x_2, x_3) \ , \\ & \vdots & \\ \dot{x}_n & = & f_n(x_1, \dots, x_n, u) \ . \end{array}\right. \tag{593}$$

Such a form is obtained by successively adding differentiators.

Feedforward form : A system is said to be in feedforward form if we can find coordinates such that their dynamics are :

$$\left\{\begin{array}{rcl} \dot{x}_n & = & f_n(x_1, \dots, x_n, u) \ , \\ & \vdots & \\ \dot{x}_2 & = & f_2(x_1, x_2, u) \ , \\ \dot{x}_1 & = & f_1(x_1, u) \ . \end{array}\right. \tag{594}$$

Such a form is obtained by successively adding integrators.

Forwarding : See adding an integrator and Remark 462.1.

Gradient controller : See L_gV-controller.

Hölder continuous : A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is Hölder continuous of order α at x if there exist positive real numbers k and δ such that we have :

$$|f(x+h) - f(x)| \ \leq \ k\,|h|^\alpha \qquad \forall |h| \leq \delta \ . \tag{595}$$

ISS : ISS stands for input to state stable (See Definition 24).

L_fV : The notation L_fV is used for the Lie derivative of V in the direction of the vector field f. Specifically if $X(x,t)$ is a solution of :

$$\dot{x} \ = \ f(x) \ , \tag{596}$$

then we have :

$$L_fV(x) \ = \ \lim_{t \to 0_+} \frac{V(X(x,t)) - V(x)}{t} \ . \tag{597}$$

If V is a C^1 function, then we have :

$$L_fV(x) \ = \ \frac{\partial V}{\partial x}(x)\, f(x) \ . \tag{598}$$

When f is actually a matrix field, L_fV is a row vector.

L_gV controller : A controller is said an L_gV controller if it acts in a direction opposed to the one given by the Lie derivative of a Lyapunov function V in the direction of the control (matrix) vector field g. A general expression is :

$$u \;=\; -Q(x)\, L_gV(x)^T \;, \tag{599}$$

where Q is a positive definite matrix.

$L^\infty_{loc}([0,\infty),\mathbb{R}^p)$: $L^\infty_{loc}([0,\infty),\mathbb{R}^p)$ denotes the set of measurable functions $d : [0,\infty) \rightarrow \mathbb{R}^p$ such that, for each compact subset K of $[0,\infty)$, there exists a real number c satisfying

$$|d(t)| \;\leq\; c \qquad \text{for almost all} \quad t \in K \;. \tag{600}$$

Lyapunov function : A function $V : \mathbb{R}^n \rightarrow [0,\infty)$ is said a C^r Lyapunov function if it is r times continuously differentiable, positive definite and proper.

Matching condition : See remark 150.3.

Origin : All along these notes, for the system (1), we assume the existence of a point in $\mathbb{R}^n$, a point in $\mathbb{R}^p$ and a point in $\mathbb{R}^m$, each of them called the origin and denoted 0 such that :

$$f(0,0,0) \;=\; 0 \;. \tag{601}$$

Passive : A system :

$$\left\{\begin{array}{rcl} \dot{x} &=& f(x,u) \\ y &=& h(x,u) \end{array}\right. \tag{602}$$

is said (respectively strictly) passive if there exists a C^1 Lyapunov function V, called the storage function, and a non negative (respectively. positive definite) function W such that, for all x and u we have :

$$\frac{\partial V}{\partial x}(x)\, f(x,u) \;\leq\; -W(x) \;+\; y^T u \;. \tag{603}$$

Negative definite : A function $V : \mathbb{R}^n \rightarrow [0,\infty)$ is said negative definite if $-V$ is positive definite.

Positive definite : A function $V : \mathbb{R}^n \rightarrow [0,\infty)$ is said positive definite if :

$$V(x) \;=\; 0 \qquad \Longrightarrow \qquad x \;=\; 0 \;,$$

Proper function : Let Ω be a subset of $\mathbb{R}^n$, a function $V : \Omega \rightarrow \mathbb{R}$ is said proper on Ω if, for any real numbers c_i and c_s, the set :

$$\{x : \leq c_i \leq V(x) \leq c_s\}$$

is a (maybe empty) compact subset of Ω. See [62]. When Ω is the whole set $\mathbb{R}^n$, this is equivalent to say that we have :

$$\lim_{|x|\rightarrow+\infty} |V(x)| \;=\; +\infty \;. \tag{604}$$

$\mathbb{R}$ **:** $\mathbb{R}$ is the set of real numbers, $\mathbb{R}_{\geq 0}$ is the set of non negative real numbers, $\mathbb{R}_{>0}$ is the set of strictly positive real numbers.

SCP : SCP stands for small control property (see Definition 89).

Young's inequality : For all $p > 1$ and all (a, b)in $\mathbb{R}^2$, we have :

$$a\,b \ \leq\ \tfrac{1}{p}\,a^p \ +\ \tfrac{p-1}{p}\,b^{\frac{p}{p-1}}\ . \tag{605}$$

In the case where $p = 2$, this inequality is known as "completed the squares".