# Francis-Wonham nonlinear viewpoint in output regulation of minimum phase systems

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**Abstract:** The paper deals with the problem of output regulation for nonlinear systems under the assumption of periodic exosystem. We build on the results presented in Astolfi et al. (2015) showing that asymptotic regulation can be achieved with an infinite dimensional regulator embedding a linear internal model copying all the harmonics that are multiple of the one associated to the exosystem. The regulator follows a post-processing structure in which the (infinite dimensional) internal model is driven by the error and a static stabiliser is considered for the cascade extended system. The post-processing structure traces the one proposed by Francis and Wonham into a linear framework. The presented analysis is limited to nonlinear systems that have unitary relative degree and that are minimum-phase.

## 1. INTRODUCTION

The problem of asymptotically offsetting the effect of an exosystem-generated signal on some regulated outputs of a system, typically known as output regulation problem, is fundamental in control theory. A landmark in the field was undoubetly given by the set of papers by Francis and Wonham (1976), Francis (1977), Davison (1976), in which the problem was fully addressed for linear systems and exosystems. In those papers the necessity of internal model-based controllers to achieve asymptotic regulation was for the first time proved and fully characterized also in terms of robustness to possible parametric uncertainties affecting the controlled system, leading to the celebrated *internal model principle*. The results obtained for linear systems then pushed research activities in the nonlinear field with the first pioneering results that came out in Huang and Rugh (1990), Isidori and Byrnes (1990). In those papers the center manifold theory was the key tool to prove the necessity of the internal model whenever the problem of output regulation must be solved in spite of (small) parameter uncertainties. Those papers opened a very rich research period in nonlinear output regulation in which many attempts have been done to make the nonlinear framework even more general and constructive. Milestone contributions within the last 20 years of research in nonlinear output regulation were certainly the paper by Serrani et al. (2001), in which the case of uncertain exosystems was for the first time addressed opening the field of adaptive output regulation, the paper by Byrnes and Isidori (2003), in which a "non-equilibrium theory" for output regulation was laid, the paper by Byrnes and Isidori (2004) in which a clear link between the design of nonlinear internal model-based regulator and nonlinear (high-gain) observers was established, and the work of Marconi et al. (2007) in which a complete theory of nonlinear output regulation without the so-called immersion assumptions was proposed. All the efforts were essentially restricted to consider single input-single error regulated systems with a well defined normal form between the control input and the regulation error and with minimum-phase features The noteworthy research attempts in the last 5 years then shifted the attention to deal with more general classes of multivariable systems without necessarily well defined relative degree and minimum-phase behaviours, with the final goal of extending in its full generality the linear theory. The search of design methods able to cope with more general nonlinear systems immediately revealed a substantial difference between the regulator structure originally proposed in the linear framework by Francis, Wonham and Davison and the ones that implicitly came out in the nonlinear framework. In the former, in fact, the regulator is given by a copy of the exogenous dynamics (acting as internal model) processing the regulation error and by a stabiliser designed to stabilise the extended system given by the regulated system driving the internal model. The structure that was used in all (to the best of authors' knowledge) the nonlinear literature, on the other hand, has a "swapped" structure with an internal model acting on the input of the regulated plant and a stabilisation unit stabilising the cascade of the internal model driving the regulated plant. This justified the terminology of *post-processing* and *pre-processing* internal model-based regulators, used to classify respectively the two previous structures, proposed in Isidori and Marconi (2012). In that work, moreover, it was shown that a post-processing structure is a more natural and general choice to deal with multivariable (not necessarily square) systems. This fact, in turn, justified a number of attempts that have been recently done in nonlinear output regulation to adapt existing (pre-processing) internal model structures by "shifting" from the input to the error the internal model. This is, for instance, the case of Isidori and Marconi (2012) and Astolfi et al. (2013) in which the pre-processing structures respectively presented in Byrnes and Isidori (2004) and Marconi et al. (2007) were shifted on the output by thus obtaining post-processing internal model-based regulators. Those papers, though, are still limited to the class of nonlinear systems having normal forms and minimum-phase. A more general class of regulated systems was addressed in Poulain and Praly (2010) and Astolfi and Praly (2017), under the assumption that the exosystem is a simple integrator (namely that the desired steady state is an equilibrium point). This framework was then extended in Astolfi et al. (2015) by considering a linear *periodic* exosystem and showing how *practical* regulation can be achieved (under the assumption of "small" exogenous signals) by means of a post-processing regulator structure having an internal model copying the exosystem frequencies and a stabiliser stabilising the nonlinear system and the internal model via forwarding. The remarkable (as far as the present work is concerned) result proved in Astolfi et al. (2015) is that the steady state error is a periodic signal (with the same period of the exosystem) and that its harmonic components at the frequencies that are copied in the internal model are in fact zero. Practical and not asymptotic regulation is, however, the only result that can be claimed for general nonlinear systems due to the fact that the steady state signals to cope with have in general an infinite number of harmonics. To the best knowledge of the authors there are not in literature systematic results for designing post-processing *asymptotic* regulators for fairly general classes of regulated systems and exosystems. In this respect, in fact, the difficulties showing up are summarised in Bin et al. (2018) and pertain an intertwining between the design of the internal model and the design of the stabiliser that makes very difficult the systematic design and lead to a "chicken-egg" dilemma, by using the terminology of Bin and Marconi (2018), in the regulator construction.

This paper is strongly motivated by the results obtained in Astolfi et al. (2015) and, specifically, by the fact that, in that "periodic" framework, a post-processing regulator embedding a finite-dimensional internal model copying the harmonics of the exosystem generates a steady state error whose Fourier coefficients associated at that harmonics are precisely zero. We are thus naturally interested to answer the question: can we succeed in achieving *asymptotic* regulation in the nonlinear case if the internal model is infinite dimensional and containing all the harmonics multiple of the basic one associated to the exosystem period? The answer is affirmative and the goal of the paper is to present the main technicalities to prove it still in the case the system has a well defined relative degree and it is minimum-phase.

*Notation* $\mathbb{R}$ is the set of real numbers, $\mathbb{R}_{>0} := (0,\infty)$, $\mathbb{R}_{\geq 0} := [0,\infty)$, $\mathbb{N}$ the set of non-negative integer numbers and $\mathbb{N}_{>0}$ the set of strictly positive integer numbers. We denote with $|\cdot|$ the standard Euclidean norm. For $x \in \mathbb{R}^n$, we denote with $x^\top$ its transpose. For $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, we compactly denote $(x,y) := (x^\top, y^\top)^\top$. We define with $\mathbb{P}^1_T(X)$ the set of $C^1$ continuous functions $\mathbb{R} \to X$ which are $T$-periodic. Recall, that by standard results on Fourier theory, if $\varepsilon \in \mathbb{P}^1_T(\mathbb{R})$, then $\varepsilon$ is equal to its Fourier series, namely $\varepsilon(t) = \epsilon_0 + \sum_{k=1}^{\infty} \epsilon^c_k \cos(k\frac{2\pi}{T}t) + \epsilon^s_k \sin(k\frac{2\pi}{T}t)$ where $\epsilon_0 = \frac{1}{T}\int_0^T \varepsilon(t)dt$, $\epsilon^c_k = \frac{2}{T}\int_0^T e(t)\cos\left(k\frac{2\pi}{T}t\right)dt$, $\epsilon^s_k = \frac{2}{T}\int_0^T e(t)\sin\left(k\frac{2\pi}{T}t\right)dt$. Given an infinite dimensional vector of the form $z = (z_0, z_1, \ldots, z_k, \ldots)$ with $z_0 \in \mathbb{R}$ and $z_k \in \mathbb{R}^2$, we define the associated norm $\|\cdot\|_2$ as $\|z\|_2^2 := \sum_{k=0}^{\infty} z_k^\top z_k$, and we define with $\ell^2$ the set of vectors $z$ satisfying $\|z\|_2 < \infty$. The set $\ell^2$ so defined is a Banach space. Finally, throughout the paper we will use the following definitions

$$S = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}, \qquad G = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \omega = \frac{2\pi}{T}. \tag{1}$$

## 2. REVIEW OF LINEAR AND NONLINEAR OUTPUT REGULATION

### 2.1 Nonlinear Byrnes-Isidori Solution

Consider single-input single output systems with unitary relative degree of the form

$$\begin{aligned} \dot{z} &= f(w,z,e) \\ \dot{e} &= q(w,z,e) + u \end{aligned} \tag{2}$$

where $(z,e) \in \mathbb{R}^n \times \mathbb{R}$ is the state, $u \in \mathbb{R}$ is the control input. The initial conditions of (2) range in a compact set $Z \times E \subset \mathbb{R}^n \times \mathbb{R}$. By following Byrnes and Isidori (2004), $w \in \mathbb{R}^{n_w}$ is an exogenous signal which is supposed to be generated by an exosystem of the form

$$\dot{w} = s(w). \tag{3}$$

The exosystem (3) is assumed to be Poisson stable (see Byrnes and Isidori (2003)), and in particular, its solutions live in some compact set $W \subset \mathbb{R}^{n_w}$ for all time. Next, we suppose the existence of a differentiable function $\pi : \mathbb{R} \to \mathbb{R}^n$ solution of

$$L_s\pi(w) = f(w, \pi(w), 0) \qquad \forall w \in W\,.$$

With the map so defined, it can be verified that the set $\mathcal{A} \subset W \times \mathbb{R}^n$ defined as $\mathcal{A} := \{(w,z) \in W \times \mathbb{R}^n : z = \pi(w)\}$ is invariant for the *zero dynamics* of system (2). System (11) is also supposed to *strongly minimum-phase* according to the following assumption.

*Assumption 1. The set $\mathcal{A}$ is asymptotically and locally exponentially stable for system (2) with a domain of attraction of the form $W \times \mathcal{D}$ where $\mathcal{D}$ is an open set of $\mathbb{R}^n$ such that $Z \subset \mathcal{D}$.*

Previous assumption allows to properly define the *friend* $\psi$, namely the steady-state input which is needed to steer constantly to zero the output, as

$$\psi(t) := -q(w(t), \pi(t), 0), \quad \forall\, t \geq 0. \tag{4}$$

As a matter of fact, when selecting $u = \psi$, it can be verified that $z = \pi(w)$ and $e = 0$ is an equilibrium of (2) for any $w \in W$. As a consequence, in order to build a dynamical system which is able to generate the function $\psi$ in (4) (the *internal model unit*), we review the solution proposed in Byrnes and Isidori (2004). The starting point is to suppose the existence of a regression law characterizing the friend $\psi$, as stated by the next assumption.

*Assumption 2. There exist a number $d \in \mathbb{N}_{>0}$ and a locally Lipschitz function $\varphi : \mathbb{R}^d \mapsto \mathbb{R}$, so that*

$$\psi^{(d)}(t) = \varphi(\psi(t), \psi^{(1)}(t), \ldots, \psi^{(d-1)}(t)), \quad \forall\, t \geq 0. \tag{5}$$

Then, the proposed dynamical regulator solving the problem of output regulation is given by

$$\begin{aligned} \dot\eta &= \Phi(\eta) - \sigma D_\kappa \Gamma e \\ u &= -\sigma e + C\eta \end{aligned} \tag{6}$$

where $\eta \in \mathbb{R}^d$ is the state with initial conditions ranging in a compact set $\Xi \subset \mathbb{R}^d$,

$$\Phi(\eta) = \begin{pmatrix} \eta_2 \\ \vdots \\ \eta_d \\ \varphi_s(\eta) \end{pmatrix}, \quad \Gamma = \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_d \end{pmatrix}, \quad C = (1 \;\; 0 \; \cdots \; 0),$$

$(\gamma_1, \ldots, \gamma_d)$ are selected so that the polynomial $\lambda^d + \gamma_1 \lambda^{d-1} + \ldots + \gamma_{d-1}\lambda + \gamma_d$ is Hurwitz, $D_\kappa = \mathrm{diag}(\kappa, \ldots, \kappa^d)$, $\varphi_s$ is a globally Lipschitz function that agrees with $\varphi$ on the final attractor, and $\sigma, \kappa \in \mathbb{R}_{>0}$ are high-gain parameters that, if chosen large enough, ensure asymptotic output regulation as stated in the following theorem.

*Theorem 1.* (Byrnes and Isidori (2004)). *There exists $\kappa^\star > 0$, and for any $\kappa \geq \kappa^\star$ there exists $\sigma^\star > 0$, such that for any $\sigma \geq \sigma^\star$ the closed-loop trajectories* (2), (3), (6) *starting in $Z \times E \times W \times \Xi$ are bounded for all $t \geq 0$ and moreover* $\lim_{t\to\infty} e(t) = 0$.

The main drawbacks of such approach can be summarized as follows.

- In practical applications, the function $\varphi$ in (5) can be very difficult to obtain in closed form.
- The regression law (5) implies smoothness of the function $q$, which has to be at least $C^d$ with respect to his arguments.
- As shown in Bin et al. (2018), this solution is not robust with respect to $C^1$ perturbations of the nominal function $q$ since the function $\varphi$ may vary according to such perturbations. This changes are not easy to map in $\varphi_s$ and, in practice, only practical regulation is obtained, see Astolfi et al. (2017).
- Since the lower bounds $\kappa^\star$ and $\sigma^\star$ depend on the Lipschitz constants of $q, \varphi$, the values of $\kappa$ and $\sigma$ can be very large. However, high-gain controller may be undesired in practical applications.

Note that also the alternative solution proposed in Marconi et al. (2007) presents similar drawbacks. The main limitation in particular is the design of the regulator in practical applications as it is based on the exact solution of a PDE. An alternative version, in which such PDE does not need to be solved, has been proposed in Marconi and Praly (2008), although only practical regulation is obtained and similar drawbacks hold.

### *2.2 The Linear Lesson*

As explained in the introduction, the solution proposed in Byrnes and Isidori (2004) for the class of nonlinear system (2)-(3) can be considered as a *pre-processing* design which differs from classical *post-processing* solution proposed in linear output regulation. In order to highlight this fact, we recall here the main results of Francis and Wonham (1976); Francis (1977); Davison (1976) by specializing their results to the class of minimum-phase systems with unitary relative degree. Although such restriction is not necessary, it is instrumental in this context to highlight the main differences between linear and nonlinear output regulation frameworks. In particular, consider a linear system of the form

$$\begin{aligned} \dot z &= Fz + Ne + Pw \\ \dot e &= Lz + He + Qw + u \end{aligned} \tag{7}$$

with $z \in \mathbb{R}^n$, $e \in \mathbb{R}$, $u \in \mathbb{R}$ and for some matrices $F, N, P, L, H, Q$ of suitable dimension. In such context, also the signal $w$ is supposed to be linear, which is therefore generated by a neutrally stable system. In order to simplify the rest of this section, we restrict here to the case in which $w$ is a sinusoidal signal determined by only one frequency, namely it can be though as generated by an autonomous systems of the form

$$\dot w = Sw \tag{8}$$

where $w \in \mathbb{R}^2$, $S$ is defined in (1) for some $\omega \in \mathbb{R}_{>0}$. The more general case in which various frequencies and constant signals contribute to $w$ can be easily dealt by superimposition of the effects. By following Francis and Wonham (1976); Francis (1977); Davison (1976), when the matrix $F$ is Hurwitz, the output regulation problem can be addressed for system (7)-(8) with the following two step *procedure*:

P1) Extend the system (7) with the following internal model in which $\eta \in \mathbb{R}^2$ and $S, G$ selected as in (1),

$$\dot\eta = S\eta + Ge\,. \tag{9}$$

P2) Select $\sigma, K$ so that the matrix

$$A := \begin{pmatrix} F & N & 0 \\ L & H - \sigma & K \\ 0 & G & S \end{pmatrix}$$

is Hurwitz and select the regulator

$$u = -\sigma e + K\eta\,. \tag{10}$$

In order to show that output regulation can be achieved, the following two *arguments* can be used.

A1) Since the matrix $A$, which characterizes the unforced (namely $w = 0$) closed-loop system (7), (9), (10), is Hurwitz, the forced closed-loop system (7)-(10) has an asymptotically stable steady-state solution $(\bar z, \bar e, \bar\eta)$ which is fully characterized the Sylvester equation

$$\Pi S = A\Pi + B$$

where $B = (P, Q, GQ)$. In particular, $(\bar z, \bar e, \bar\eta) = (\Pi_z w, \Pi_e w, \Pi_\eta w)$, where $\Pi = (\Pi_z, \Pi_e, \Pi_\eta)$.

A2) Geometric arguments, briefly reviewed in Byrnes et al. (2012), can be used to exploit the properties of the internal model (9) to show that since $S, G$ is a controllable pair, then $\Pi_\eta S = S\Pi_\eta + G\Pi_e$ implies necessarily $\Pi_e = 0$, which implies $\bar e = 0$, namely asymptotic output regulation is achieved.

Opposed to the nonlinear case reviewed in Section 2.1, such approach is *post-processing*[1] In particular, it is readily seen that the design of the regulator (9) is a copy of the exosystem (8) driven by the regulated output $e$. This design is selected independently of the *friend* (4), namely, no matter what the right steady-state input $q$ is needed,

[1] We refer also to Bin and Marconi (2018) for a more detailed discussion about differences between *pre* and *post-processing* solutions.

the internal model is not changed. This is a substantial difference between linear and nonlinear output frameworks which allows to conclude also robustness of the regulator (9), (10), with respect to parameter uncertainties of the matrices $(F, N, P, L, H, Q)$, see Chapter 1.3 in Byrnes et al. (2012) and Bin et al. (2018). The main obstructions that prevent the extension of this linear paradigm to the nonlinear framework (2) can be therefore summarized by the following two *questions*:

Q1) Can we mimic the two-step procedure P1) - P2) in the nonlinear framework (2)-(3), in which the design of the internal model is not affected by the selection of the stabilizer and does not depend on the friend $\psi$?

Q2) Can we find alternative arguments to the geometric analysis in A2) that can be extended to the nonlinear framework (2)-(3) to conclude that output regulation is achieved, once boundedness of trajectories is established?

## 3. MAIN RESULT

### *3.1 Main Idea*

In order to try to reply to the second question Q2) of previous section, we recall the main result established in Astolfi et al. (2015), which motivates this work.

*Lemma 1.* (Astolfi et al. (2015)). *Consider system*

$$\dot{\eta} = S\eta + Ge$$

*where $S, G$ are defined in (1), $\eta \in \mathbb{R}^2$, $e \in \mathbb{R}$, and suppose it admits a solution $\bar{\eta} \in \mathbb{P}^1_T(\mathbb{R}^2)$, $\bar{e} \in \mathbb{P}^0_T(\mathbb{R})$, with $T = 2\pi/\omega$. Then, the Fourier coefficients of $\bar{e}$ associated to $\omega$ are zero, namely*

$$\frac{2}{T}\int_0^T \bar{e}(t)\cos(\omega t)\,dt = \frac{2}{T}\int_0^T \bar{e}(t)\sin(\omega t)\,dt = 0.$$

Lemma 1 establishes that if $(\eta, e)$ are $T$-periodic bounded functions, then necessarily the Fourier coefficients of $e$ associated to the frequency $\omega = 2\pi/T$ must necessarily be zero. This fact has a certain number of interesting consequences/properties that can be highlighted.

The first consequence is that if the spectral content of $e$ coincides with one single frequency component, then $e$ must be necessarily zero. In particular, consider again the linear framework of Section 2.2. In light of (8), and by definition of $S$ in (1), the signal $w$ is a $T$-periodic trajectory containing only frequency contents at $\omega = 2\pi/T$. As a consequence, in light of stability of the matrix $A$ and by using linearity, it can be proved that the forced closed-loop system (7)-(10), admits a unique $T$-periodic solution $(\bar{z}, \bar{e}, \bar{\eta}) \in \mathbb{P}^1_T(\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^2)$ containing frequency components only at $\omega$. As a consequence, direct application of Lemma 1 allows to conclude that $\bar{e} = 0$.

An second property of Lemma 1 to be highlighted is that such results hold as long as $\bar{e}, \bar{\eta}$ are bounded $T$-periodic functions, no matter how they are generated. This means that such result can be used also in the framework of nonlinear systems. In other words, Lemma 1, provide an answer to the question Q2) of Section 2.2 for the specific case of periodic exogenous systems. In particular, consider the nonlinear system (2), extended with the internal model (9), and suppose that we are able to design a controller of the form $u = \phi(e, \eta)$ able to guarantee the existence of an attractive and stable $T$-periodic solution $(\bar{z}, \bar{e}, \bar{\eta})$ for the closed-loop system when $w$ is present. As a consequence, for such periodic solution, we can apply Lemma 1, to establish that the Fourier coefficient of $\bar{e}$ corresponding to $\omega$ is zero. As shown in Astolfi et al. (2015), however, such result is not enough to conclude that $\bar{e} = 0$, since $\bar{e}$ could contain higher order Fourier coefficients which are caused by higher order deformations induced by the nonlinear functions $f, q$.

Finally, a third property of Lemma 1 is that superposition of effects can be used in the in which many systems of the form $\dot{\eta}_k = kS\eta_k + Ge$ with $k$ ranging in $\{1, \ldots, N\}$, are considered, for which a unique solution $(\bar{\eta}_1, \ldots, \bar{\eta}_k, \bar{e}) \in \mathbb{P}^1_T(\mathbb{R}^2 \times \cdots \times \mathbb{R}^2 \times \mathbb{R})$ exists. In this case, it can be shown, see (Astolfi et al., 2015), that all the Fourier coefficients of $\varepsilon$ corresponding to $\omega, \ldots, N\omega$, are zero.

### *3.2 Francis-Wonham Nonlinear Viewpoint*

The main idea of this work, is therefore that of building an internal model unit composed by an infinite number of oscillator of the form (9), so that we can apply Lemma 1 to any of the Fourier coefficients characterizing the Fourier developments of a periodic steady-state solution of (2). A first observation however is that in doing so, the model of the exosystem (3) generating $w$ is no more needed. As a matter of fact, in view of previous consideration, we only need that the steady-state solution of the zero-dynamics of (2) are periodic, and so is the friend (4). For this, in the following, all the instances of $w$ are replaced by $t$ to point out that we only need functions which are periodic. In particular, the following system is considered

$$\begin{aligned} \dot{z} &= f(t, z, e) \\ \dot{e} &= q(t, z, e) + u \end{aligned} \tag{11}$$

where $z \in \mathbb{R}^n$, $e \in \mathbb{R}$, $u \in \mathbb{R}$. As in Section 2.1, we suppose that the initial conditions of (11) range in some given compact sets $Z \times E \subset \mathbb{R}^n \times \mathbb{R}$, and the following assumptions, which replace Assumptions 1 and 2, are stated.

*Assumption 3. The functions $f$ is continuous with respect to time and locally Lipschitz with respect to $(z, e)$. The function $q$ is $C^1$ with respect to $(t, z, e)$. The functions $f, q$ are $T$-periodic in the argument $t$, namely $f(t + T, z, e) = f(t, z, e)$ and $q(t+T, z, e) = q(t, z, e)$ for any $t \geq 0$, $z \in \mathbb{R}^n$ and $e \in \mathbb{R}$.*

*Assumption 4. The zero-dynamics $\dot{z} = f(t, z, 0)$ admits a unique solution $\bar{z} \in \mathbb{P}^1_T(\mathbb{R}^n)$ which is asymptotically (and locally exponentially) stable with domain of attraction an open set $\mathcal{A} \subseteq \mathbb{R}^n$ containing $Z$.*

Under previous assumptions, by following the two step procedure for linear systems in Section 2.2, we propose here a two-step procedure for the nonlinear system (11):

P1) Design the internal model as a bunch of an infinite number of oscillators, namely

$$\begin{aligned} \dot{\eta}_0 &= e & \eta_0 &\in \mathbb{R} \\ \dot{\eta}_k &= kS\eta_k + Ge & \eta_k &\in \mathbb{R}^2 \end{aligned} \tag{12}$$

where $k \in \mathbb{N}_{>0}$, $\eta = (\eta_0, \eta_1, \ldots) = (\eta_k)_{k \in \mathbb{N}}$ is the state with initial condition $\eta(0)$ ranging in some given closed set $\Xi \subset \ell^2$, and $S, G$ are defined in (1).

Note that $\eta_0$ is an integral action needed to remove constant bias, see Astolfi and Praly (2017).

P2) Select the following regulator

$$u = -\sigma e + \sigma \sum_{k=0}^{\infty} M_k^\top (\eta_k - M_k e) \tag{13}$$

where $\sigma > 0$ is a parameter to be chosen, and $M_0 \in \mathbb{R}$, $M_k \in \mathbb{R}^2$, with $k \in \mathbb{N}_{>0}$, are defined so that

$$\begin{aligned} -\sigma M_0 &= 1, \\ -\sigma M_k &= k\, S + G, \qquad k \in \mathbb{N}_{>0}. \end{aligned} \tag{14}$$

Similarly to the nonlinear framework in Section 2.1, by using the compact notation

$$\begin{aligned} \Phi &= \text{blckdiag}(0, \omega S, 2\omega S, \ldots), \\ \Gamma &= (1, G, G, \ldots), \\ M &= (M_0, M_1, M_2, \ldots), \end{aligned} \tag{15}$$

the internal model unit (12) and the control law (13) can be compactly rewritten as

$$\dot{\eta} = \Phi\eta + \Gamma e, \tag{16a}$$

$$u = -\sigma(1 + M^\top M)e + \sigma M^\top \eta, \tag{16b}$$

where in this case the regulator is linear although of infinite dimension. It is readily seen that in the control law (16b) we have two terms: the first, $-\sigma e$, is the stabilizing law, while the second, $M^\top(\eta - Me)$ is a term obtained by using the forwarding approach [2] and applied to the specific cascade system (11), (12). This term is therefore needed to stabilize the internal model unit (12). Note that with respect to the standard forwarding procedure, reviewed for instance in Poulain and Praly (2010), the control law (13) is here obtained by ignoring the term $q(t, z, e)$ in the equation (11). By solving (14) we can also explicitly compute the matrices $M_k$ as follows

$$M_0 = -\frac{1}{\sigma}, \qquad M_k = \frac{1}{(k\omega)^2 + \sigma^2} \begin{pmatrix} k\omega \\ -\sigma \end{pmatrix}. \tag{17}$$

The following lemma state that the term $M \in \ell^2$, namely the term $-\sigma M^\top M e$ in the control law (16b) is not diverging.

*Lemma 2. Let $M$ be given by (17). Then, for any $\sigma \geq 1$, $\|M\|_2 \leq \frac{1}{\sigma}\sqrt{1 + \frac{\pi\sigma}{2\omega}}$.*

**Proof.** The result follows by computing

$$\begin{aligned} \|M\|_2^2 &= \sum_{k=0}^{\infty} M_k^\top M_k = \frac{1}{\sigma^2} + \sum_{k=1}^{\infty} \frac{1}{(k\omega)^2 + \sigma^2} \\ &\leq \frac{1}{\sigma^2} + \int_1^\infty \frac{1}{(s\omega)^2 + \sigma^2} ds \leq \frac{1}{\sigma^2} + \frac{\pi}{2\omega\sigma}. \end{aligned}$$

□

Finally, we can state the following theorem, which is the main result of this work, claiming that the regulator (16) solves the robust asymptotic output regulation problem for system (11).

*Theorem 2. For any pair of functions $f, q$ satisfying Assumptions 3-4 hold. there exist a $\sigma^\star \geq 1$ such that, for any $\sigma > \sigma^\star$, any trajectory of the the closed-loop system (11), (16), starting inside $Z \times E \times \Xi$, is such that $(z(t), e(t))$ is bounded for all $t \geq 0$, $\eta(t) \in \ell^2$ for all $t \geq 0$, and moreover $\lim_{t\to\infty} e(t) = 0$.*

**Proof.** For reason of spaces the proof of this theorem is here only sketched. In particular, the first observation is that for any $\psi \in \mathbb{P}_T^1(\mathbb{R})$, and any $\sigma \geq 1$, there exists a choice of initial conditions $\bar{\eta} \in \ell^2$ so that the following

$$\dot{\eta} = \Phi\eta, \quad \eta(0) = \bar{\eta}, \qquad \psi(t) = \sigma M^\top \eta(t)$$

holds for all $t \geq 0$. By letting $\psi(t) = -q(t, \bar{z}(t), 0)$, in which $\bar{z}$ is given by Assumption 4, and with the $\bar{\eta}$ previously defined we can then define the following change of coordinates

$$z \mapsto \zeta := z - \bar{z}, \qquad \eta \mapsto \xi := \eta - Me - \bar{\eta}, \tag{18}$$

by which we obtain the following closed-loop system

$$\begin{aligned} \dot{\zeta} &= F(t, \zeta) + N(t, \zeta, e) \\ \dot{e} &= p(t, \zeta, e) - \sigma\, e + \sigma M^\top \xi \\ \dot{\xi} &= (\Phi - \sigma M M^\top)\xi - M p(t, \zeta, e) \end{aligned} \tag{19}$$

where we used the following notation

$$\begin{aligned} F(t, \zeta) &:= f(t, \zeta + \bar{z}, 0) - f(t, \bar{z}, 0), \\ N(t, \zeta, e) &:= f(t, \zeta + \bar{z}, e) - f(t, \zeta + \bar{z}, 0), \\ p(t, \zeta, e) &:= q(t, \zeta + \bar{z}, e) - q(t, \bar{z}, 0). \end{aligned}$$

Assumption 4 and standard results on converse Lyapunov theorems (see Marconi et al. (2007)) allow to establish the existence of a function $V(t, \zeta)$ satisfying a certain number of properties, see Marconi et al. (2007),

$$\underline{\alpha}(|\zeta|) \leq V(t, \zeta) \leq \bar{\alpha}(|\zeta|), \quad \lim_{\zeta \to \partial\mathcal{A}} \underline{\alpha}(|\zeta|) = +\infty$$

$$\frac{\partial V(t, \zeta)}{\partial \zeta} F(t, \zeta) + \frac{\partial V(t, \zeta)}{\partial t} \leq -\alpha(|\zeta|)$$

for all $t \in \mathbb{R}_{\geq 0}$ and $\zeta \in \mathcal{A}$, with $\alpha, \underline{\alpha}, \bar{\alpha}$, class $\mathcal{K}$ functions which are quadratic around the origin (for this local exponential stability is needed). By analyzing the derivative of the function $U(t, x, e, \eta) := V(t, \zeta) + e^2 + \xi^\top \xi$, we can finally select a value of $\sigma$ large enough to conclude that its derivative is always non-positive. Boundedness (in $\ell^2$ sense) of the initial conditions and linearity of the change of coordinate (18) allows to conclude boundedness (in $\ell^2$ sense) of the trajectories of the closed-loop system (19). Finally, application of the of Barbalat's lemma, allows to conclude that $\lim_{t\to\infty} |\zeta(t)| = 0$ and $\lim_{t\to\infty} |e(t)| = 0$. □

### 3.3 Discussion of the Main Result

We discuss now the result of Theorem 2. First of all, we remark that asymptotic regulation is obtained robustly to any (small) $C^1$ model perturbation [3] of the functions $f, q$ as long as the resulting closed-loop trajectories of (11), (16), are bounded and $T$-periodic. As a matter of fact, the design of the regulator (16) does not make any explicit use of the functions $f, q$, since we only need to select the parameter $\sigma$ large enough to achieve asymptotic regulation. His magnitude, in particular, has to be selected larger than the Lipschitz constants of $q$ on the compact set of interest and so that stabilization of the zero-dynamics is achieved [4].

A second remark is that obviously the controller (16) is not implementable since its state is infinite. Nevertheless, from a practical point of view, a finite version of the internal model (12), namely with $k \in \{1, \ldots, N\}$ for some $N < \infty$, is still very interesting if the closed-loop trajectories are bounded and $T$-periodic, since the result of Lemma 1 holds for these solutions, as showed in Astolfi

[2] See Mazenc and Praly (1996) and Poulain and Praly (2010), Astolfi and Praly (2017), for a review of forwarding approach.

[3] In the sense defined in Astolfi and Praly (2017), Bin et al. (2018).

[4] In the same spirit of semi-global stabilization problems of nonlinear minimum-phase systems, see Chapter 9.3 in Isidori (1995).

et al. (2015). Moreover, recall that higher frequencies are naturally attenuated by low-pass behavior of system (11).

Furthermore, it is worth also noticing that asymptotic regulation could be obtained by means of a finite dimensional regulator if we are able to identify the spectral content of friend $\psi(t)$ in (4). In particular, by developing the Fourier series of $\psi$ as $\psi(t) = \psi_0 + \sum_{k=1}^{\infty} \psi_k^c \cos(k\omega t) + \psi_k^s \sin(k\omega t)$, suppose that we know the existence of a finite set of integers $\mathcal{I} \subset \mathbb{N}$ so that $\psi_k^c = \psi_k^s = 0$ for all $k \in \mathbb{N} \subset \mathcal{I}$. In such case, we may design the regulator (16) with a finite number of states $\eta_k$ $k \in \mathcal{I}$. Asymptotic regulation can be then proved by combining the technicalities of Lemma 1, Theorem 2, and the results presented in Astolfi et al. (2015).

We remark also the following main differences with respect to the methodology proposed by Byrnes and Isidori (2004) and reviewed in Section 2.1: we do not need the exact knowledge of the friend $\psi$ in (4) and the regression law $\varphi$ in Assumption 2; the function $q$ is allowed to be only $C^1$ with respect to its arguments, while Assumption 2 needs the signals to be $C^d$; we restrict to the framework of $T$-periodic exosignals.

Finally, note that the proposed solution, based on an infinite dimensional regulator, is, to some extents, similar to repetitive control schemes (see, e.g., Califano et al. (2018)). However, differently from such approach, the use of full state information of $\eta$ allows, in our context, to handle systems of relative degree larger than 0.

## 4. CONCLUSIONS

We showed that robust asymptotic output regulation can be obtained by means of a infinite dimensional internal-model based regulator which follows the linear paradigm proposed by Francis, Wonham and Davison in the 70's, when periodic exosignals are considered. This work can be viewed as a first step to be developed in the following directions: a full characterization of its finite-dimensional version in the case of minimum phase systems by combining the results developed here with Astolfi et al. (2015); the use of the proposed approach for more general classes of nonlinear systems, with a particular focus on the non-minimum phase case; the use of adaptive or identification techniques to address the case in which the period $T$ is unknown.

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