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Non Lipschitz triangular canonical form for uniformly observable controlled systems

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Abstract: We study the problem of designing observers for controlled systems which are uniformly observable and differentially observable, but with an order larger than the system state dimension : we have only an injection, and not a diffeomorphism. We establish that they can be transformed into a triangular canonical form but with possibly non locally Lipschitz functions. Since the classical high gain observer is no longer sufficient, we review and propose other observers to deal with such systems, such as a cascade of homogeneous observers.

Keywords: uniform observability, differential observability, canonical observable form, finite-time observers, homogeneous observers, exact differentiators

1. INTRODUCTION

1.1 Context

A lot of attention has been dedicated to the construction of nonlinear observers. Although a general theory has been obtained for linear systems, very few general approaches exist for nonlinear systems. In particular, the theory of high gain (Khalil and Praly (2013) and references therein) and Luenberger (Andrieu and Praly (2013); Andrieu (2014)) observers have been developed for autonomous non linear systems but their extension to controlled systems is not straightforward.

For designing an observer for a system, a preliminary step is often required. It consists in finding a reversible coordinate transformation, allowing us to rewrite the system dynamics in a targeted form more favorable for writing and/or analyzing the observer. In presence of control, two tracks are possible depending on whether we consider the input as a simple time function, making the system time dependent or as a more involved (infinite dimensional) parameter, making the system a family of dynamical system, indexed by the control. Accordingly the transformation mentioned above is simply time-varying or input dependent. Moreover, along the later itself, with the input seen as a parameter, the strength of the input dependence of the transformation may vary.

For example, in (Hammouri and Morales (1990); Besançon et al. (1996)), the transformation can depend arbitrarily on the input with the objective of obtaining a targeted form which is state-affine up to input/output injection, or more generally as in Besançon (1999), a targeted form which has a triangular structure. The dependence may also be on the derivative of the inputs as proposed in Gauthier and Kupka (2001) with the so called phase-variable representation as targeted form.

Alternatively, we may impose the transformation not to depend on the input. This is the context of *uniformly observable systems*. For example Gauthier and Bornard (1981); Gauthier et al. (1992) propose this track to obtain, as targeted form, a so-called triangular canonical form for which a high-gain observer can be built. More precisely, as detailed below, it is known that this observer can be built when, together with the uniform (in the control) observability of the system (see (Gauthier and Kupka, 2001, Definition I.2.1.2) or Definition 2 below), the transformation, obtained from the strong differential observability (see (Gauthier and Kupka, 2001, Definition I.2.4.2) or Definition 1 below), is a diffeomorphism.

In this paper we study the case where we have uniform observability and strong differential observability, but the latter with an order larger than the system state dimension, implying that the transformation is at most an injective immersion, and not a diffeomorphism as above. We shall see that, in this case, the system dynamics can still be described by a triangular canonical form but with functions which may be non locally Lipschitz. This leads us to study observers able to cope with such an extreme context and, in particular, to propose a new observer made of a cascade of homogeneous observers.

1.2 Definitions

Consider a controlled system of the form :

$$\dot{x} = f(x) + g(x)u \quad , \qquad y = h(x) \tag{1}$$

where x is the state in $\mathbb{R}^n$, u is an input in $\mathbb{R}^p$ and y is a measured output in $\mathbb{R}$. Given an input time function $t \mapsto u(t)$, we denote $X(x,t)$ a solution of (1) going through x at time 0. We are interested in estimating $X(x,t)$ knowing y and u but only as long as $(X(x,t),u(t))$ is in a given compact set $\mathcal{C} \times U$. Let $\mathcal{S}$ be an open subset of $\mathbb{R}^n$ containing $\mathcal{C}$. We will use the following two notions of observability defined in Gauthier and Kupka (2001) :

Definition 1. (Differential observability). (See (Gauthier and Kupka, 2001, Definition I.2.4.2).) The system (1) is differentially observable of order m on $\mathcal{S}$ if the function :

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$$\mathbf{H}_m(x) = \begin{pmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{m-1} h(x) \end{pmatrix}$$

is injective on $\mathcal{S}$. If it is also an immersion, the system is called strongly differentially observable. ■

Definition 2. (Uniform observability). (See (Gauthier and Kupka, 2001, Definition I.2.1.2).) The system (1) is uniformly observable on $\mathcal{S}$ if for any pair (x_a, x_b) in $\mathcal{S}^2$ with $x_a \neq x_b$, there is no C^1 function $u : [0,T) \to U$ such that

$$h(X(x_a,t)) = h(X(x_b,t))$$

for all $t \leq T$ such that $(X(x_a,s), X(x_b,s)) \in \mathcal{S}^2$ for all $s \leq t$. ■

In the case where $m = n$, i.e. $\mathbf{H}_n$ is a diffeomorphism, we have :

Proposition 1. (See Gauthier and Bornard (1981); Gauthier et al. (1992)) If the system (1) is uniformly observable and strongly differentially observable of order $m = n$, it can be transformed into the following triangular canonical form :

$$\begin{aligned} \dot{z}_1 &= z_2 + \mathfrak{g}_1(z_1)\, u \\ &\vdots \\ \dot{z}_i &= z_{i+1} + \mathfrak{g}_i(z_1,\ldots,z_i)\, u \qquad (2) \\ &\vdots \\ \dot{z}_n &= \varphi_n(z) + \mathfrak{g}_n(z)\, u \ , \end{aligned}$$

where the functions $\mathfrak{g}_i$ are locally Lipschitz. ■

Such a triangular form named Lipschitz triangular form, with Lipschitz nonlinearities is fortunately the nominal case for the high gain paradigm.

But as we shall see in Section 2, when the system is strongly differentially observable of order $m > n$, triangularity is preserved but Lipschitzness is lost. Hence high gain observers as those presented in Gauthier et al. (1992) can no longer be used.

We thus present in Section 3 possible designs of observers for the triangular canonical form (2) with non-Lipschitz $\mathfrak{g}_i$. Everything is finally illustrated with an example in Section 4.

Notations.

(1) We define the signed power function as

$$\lfloor a \rceil^b = \mathtt{sign}(a)\, |a|^b$$

where b is a nonnegative real number. In the particular case where $b = 0$, $\lfloor a \rceil^0$ is actually any number in the set

$$\mathrm{S}(a) = \begin{cases} \{1\} & \text{if } a > 0 \\ [-1,1] & \text{if } a = 0 \\ \{-1\} & \text{if } a < 0 \end{cases}$$

Namely, writing $c = \lfloor a \rceil^0$ means $c \in \mathrm{S}(a)$. Note that the set valued map $a \mapsto S(a)$ is upper semi-continuous with closed and convex values.

(2) For x in $\mathbb{R}^p$ with $p \geq i$, we denote

$$\mathbf{x}_i = (x_1, ..., x_i) \ .$$

and, for x in $\mathcal{S}$,

$$\mathbf{H}_i(x) = (h(x), \ldots, L_f^{i-1} h(x)) \qquad (3)$$

2. IMMERSION CASE ($m > n$)

The specificity of the triangular canonical form (2) is not so much in its structure but more in the dependence of its functions $\mathfrak{g}_i$ and φ_n. Indeed, for any k, $\mathbf{H}_k(x)$ satisfies always :

$$\overbrace{\mathbf{H}_k(x)}^{\cdot} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \ldots & \ldots & 0 & 1 \\ 0 & \ldots & \ldots & \ldots & 0 \end{pmatrix} \mathbf{H}_k(x) + \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ L_f^k h(x) \end{pmatrix} + L_g \mathbf{H}_k(x) u$$

To get (2), we need further the existence of a sufficiently smooth function φ_k satisfying

$$L_f^k h(x) \ = \ \varphi_k(\mathbf{H}_k(x)) \qquad (4)$$

and of sufficiently smooth functions $\mathfrak{g}_i$ satisfying

$$L_g L_f^{i-1}(x) \ = \ \mathfrak{g}_i(h(x), \ldots, L_f^{i-1} h(x)) \qquad (5)$$

and this at least for all x in $\mathcal{C}$ if not in $\mathcal{S}$.

Let us illustrate via the following elementary example what can occur.

Example 1. Consider the system

$$\dot{x}_1 \ = \ x_2 \ , \quad \dot{x}_2 \ = \ x_3^3 \ , \quad \dot{x}_3 \ = \ 1 + u \ , \quad y \ = \ x_1$$

It is uniformly observable, differentially observable of order 3 and strongly differentially observable of order 5 since

$$\begin{aligned} \mathbf{H}_3(x) &= (h(x), L_f h(x), L_f^2 h(x)) \ = \ (x_1, x_2, x_3^3) \\ \mathbf{H}_5(x) &= (\mathbf{H}_3(x), L_f^3 h(x), L_f^4 h(x)) \ = \ (\mathbf{H}_3(x), 3x_3^2, 6x_3) \end{aligned}$$

where $\mathbf{H}_3$ is a bijection and $\mathbf{H}_5$ is an injective immersion on $\mathbb{R}^3$. For $\mathbf{H}_3$, we have

$$L_f^3 h(x) \ = \ 3x_3^2 \ = \ 3(L_f^2 h(x))^{2/3}$$

Hence there is no Lipschitz function φ_3 satisfying (4). Similarly, for $\mathbf{H}_5$, we have

$$L_g L_f^2 h(x) = 3x_3^2 = L_f^3 h(x) = 3(L_f^2 h(x))^{2/3}$$

so there is no locally Lipschitz function $\mathfrak{g}_3$ satisfying (5).■

Concerning the existence of continuous functions φ_k and $\mathfrak{g}_i$ we have the following results given without proof due to space limitations.

Proposition 2. Suppose the system (1) is differentially observable of order m on an open set $\mathcal{S}$ containing the given compact set $\mathcal{C}$. There exists a continuous function $\varphi_m : \mathbb{R}^m \to \mathbb{R}$ satisfying (4) for all x in $\mathcal{C}$. If the system (1) is strongly differentially observable of order m on $\mathcal{S}$, the function φ_m can be chosen Lipschitz on $\mathbb{R}^m$. ■

Proposition 3. Suppose the system (1) is uniformly observable on an open set $\mathcal{S}$ containing the given compact set $\mathcal{C}$, then, for all i, there exist continuous functions $\mathfrak{g}_i : \mathbb{R}^i \to \mathbb{R}$ satisfying (5) for all x in $\mathcal{C}$. ■

Note that the values of φ_m and $\mathfrak{g}_i$ are only imposed on the compact set $\mathbf{H}_m(\mathcal{C})$. In particular, their behavior when $|z|$ tends to infinity is free and can be chosen to satisfy some extra constraints given by the observer design (see Assumption (7) in Section 3).

Note also that no assumption on $\mathbf{H}_m$ is needed for Proposition 3 to hold. But it says nothing on the regularity of the functions $\mathfrak{g}_i$, besides continuity. As we saw in Example 1, even the usual assumption of strong differential observability is not sufficient to make it Lipschitz everywhere. Thus, a triangular canonical form still exists but maybe without Lipschitzness of the functions it involves. Fortunately, as stated in the next result, local Lipschitzness of $\mathfrak{g}_i$ may be lost only at the image via H_i of points where the rank of $\frac{\partial \mathbf{H}_i}{\partial x}$ changes ($z_3 = x_3^3 = 0$ in Example 1).

Proposition 4. Assume there exists a neighborhood of x_* in $\mathcal{S}$ on which $\frac{\partial \mathbf{H}_i}{\partial x}$ has a constant rank q. Then, if the system (1) is uniformly observable, there exists a neighborhood $\mathcal{N}_*$ of x_*, a real number L_* and a function $\mathfrak{g}_i$ defined on $\mathbb{R}^i$, such that

$$L_g L_f^{i-1} h(x) = \mathfrak{g}_i(h(x), ..., L_f^{i-1} h(x)) \qquad \forall x \in \mathcal{N}_*$$

and, for all (z_a, z_b) in $\mathbb{R}^m \times \mathbb{R}^m$,

$$|\mathfrak{g}_i(z_{a,1}, ..., z_{a,i}) - \mathfrak{g}_i(z_{b,1}, ..., z_{b,i})| \leq L_* \sum_{j=1}^{i} |z_{a,j} - z_{b,j}| \ . \blacksquare$$

This shows that, with uniform observability, we should get functions $\mathfrak{g}_i$ in (2) which are locally Lipschitz except maybe close to the image of points where the rank of the Jacobian of $\mathbf{H}_i$ changes. For a strongly differentially observable system of order $m = n$, this cannot happen and we thus recover the result of Proposition 1.

From Khalil and Praly (2013), it can be expected that a standard high gain observer would not work for a canonical form (2) with non-Lipschitz $\mathfrak{g}_i$. Hence we need to be able to design an observer for the canonical form (2) with continuous but non-Lipschitz functions $\mathfrak{g}_i$.

3. OBSERVERS FOR A NON-LIPSCHITZ TRIANGULAR FORM

All along this section it is assumed that the system (1) is differentially observable of order m on an open set $\mathcal{S}$ containing the given compact set $\mathcal{C}$. It is also assumed that, the system is uniformly observable on $\mathcal{S}$. According to Propositions 2 and 3, the image by $\mathbf{H}_m$ of the dynamics (1) is contained [1] in the triangular form (2), with m replacing n and with continuous $\mathfrak{g}_i$ and φ_m functions. The only observer we are aware of able to cope with $\mathfrak{g}_i$ no more than continuous is the one presented in Barbot et al. (1996). Its dynamics are described by a differential inclusion (see Appendix) :

$$\dot{\hat{z}} \in F(\hat{z}, y, u)$$

where $(\hat{z}, y, u) \mapsto F(\hat{z}, y, u)$ is a set valued map. It can be shown that any absolutely continuous solution gives in finite time an estimate of z under the only assumption of boundedness of the $\mathfrak{g}_i$'s. But the set valued map F above does not satisfy the usual basic assumptions (upper semi-continuous with closed and convex values) (see Filippov (1988); Smirnov (2001)). It follows that we are not guaranteed of the existence of absolutely continuous solutions nor of possible sequential compactness of such solutions and therefore of possibilities of approximations of F. So we do not pursue with this observer and look at other possible ones.

3.1 Homogeneous observer

Homogeneous observers are extensions of high gain observers able to cope with some non Lipschitz functions. In our context they take the form :

[1] not equal since we have an immersion and not a diffeomorphism.

$$\begin{array}{c|lllll} & & & j & & \\ \hline & \frac{m-1}{m} & & & & \\ & \frac{m-2}{m} & \frac{m-2}{m-1} & & & \\ & \vdots & \vdots & & & \\ \alpha_i \geq & \frac{2}{m} & \frac{2}{m-1} & \cdots & \frac{2}{3} & \\ & \frac{1}{m} & \frac{1}{m-1} & \cdots & \cdots & \frac{1}{2} \\ & 0 & 0 & \cdots & \cdots & 0 \end{array}$$

Table 1 : Hölder restrictions on the $\mathfrak{g}_i$ when $d_0 = -1$

$$\begin{aligned} \dot{\hat{z}}_1 &= \hat{z}_2 + \mathfrak{g}_1(\hat{z}_1)u - k_1 \lfloor \hat{z}_1 - z_1 \rceil^{\frac{1-d_0(m-2)}{1-d_0(m-1)}} \\ \dot{\hat{z}}_2 &= \hat{z}_3 + \mathfrak{g}_2(\hat{z}_1, \hat{z}_2)u - k_2 \lfloor \hat{z}_1 - z_1 \rceil^{\frac{1-d_0(m-3)}{1-d_0(m-1)}} \\ &\vdots \\ \dot{\hat{z}}_m &= \varphi_m(\hat{z}) + \mathfrak{g}_m(\hat{z})u - k_m \lfloor \hat{z}_1 - z_1 \rceil^{\frac{1+d_0}{1-d_0(m-1)}} \end{aligned} \tag{6}$$

where the k_i's are gains to be tuned and d_0 is a parameter to be chosen in $[-1, 0)$. We refer to Notation (1) for the case $d_0 = -1$.

We have :

Proposition 5. Assume there exist d_0 in $[-1, 0)$ and $c_0 \geq 0$ such that, for all $(\mathbf{z}_i, \mathbf{e}_i)$ in $\mathbb{R}^i \times \mathbb{R}^i$, we have :

$$\begin{aligned} &|\mathfrak{g}_i(z_1 + e_1, ..., z_i + e_i) - \mathfrak{g}_i(z_1, ..., z_i)| \\ &\qquad\qquad \leq c_0 \sum_{j=1}^{i} |e_j|^{\frac{1-d_0(m-i-1)}{1-d_0(m-j)}} \end{aligned} \tag{7}$$

and φ_m verifies the same condition as $\mathfrak{g}_m$. Then, there exist $(k_1, \ldots, k_m)$ such that, for all $(x, \hat{z}, u)$ in $\mathbb{R}^n \times \mathbb{R}^m \times L^\infty_{loc}(\mathbb{R}_+; U)$ such that $X(x,t) \in \mathcal{C}$ for all t in $\mathbb{R}_+$, any absolutely continuous solution $\hat{Z}(\hat{z}, x, t)$ of (6) is defined on $\mathbb{R}_+$ and there exists T_0, depending on $(\hat{z}, x)$, such that :

$$\hat{Z}(\hat{z}, x, t) \ = \ \mathbf{H}_m(X(x,t)) \ , \ \forall t \geq T_0 \ . \qquad \blacksquare$$

For the case $d_0 \in (-1, 0)$, the proof of this proposition follows from the arguments in Andrieu et al. (2008) (see also Qian (2005)), more specifically from the proof of (Andrieu et al., 2009, Corollary 7) and (Andrieu et al., 2008, Theorem 5.1) and from the fact that the function $\mathbf{H}_m$ is injective.

For the case, $d_0 = -1$, we refer the reader to (Levant, 2001 b, Theorem 1). Although the arguments there are not sufficient. Actually it is possible to prove this result using a Lyapunov argument which we omit here due to space limitations.

Assumption (7) imposes that, for each i, $\mathfrak{g}_i$ is Hölder with order α_i larger than the values given in Table 1 below for the case $d_0 = -1$, where i is the index of $\mathfrak{g}_i$ and j is the index of e_j.

3.2 Cascade of homogeneous observers

In the case where Assumption (7) is not satisfied, we can still take advantage of the fact that the homogeneous observer dot not ask for any restriction besides boundedness on the last function $\mathfrak{g}_m$ when $d_0 = -1$.

From the remark that such an observer

(1) can be used for the system

$$\begin{aligned}\dot{z}_1 &= z_2 + r_1(t)\\ &\vdots\\ \dot{z}_{k-1} &= z_k + r_{k-1}(t)\\ \dot{z}_k &= \Phi_k(t)\end{aligned}$$

provided the functions r_i are known and the function Φ_k is unknown but bounded, with known bounds.

(2) gives estimates of the z_i's in finite time,

we see that it can be used as a preliminary step to deal with the system

$$\begin{aligned}\dot{z}_1 &= z_2 + r_1(t)\\ &\vdots\\ \dot{z}_{k-1} &= z_k + r_{k-1}(t)\\ \dot{z}_k &= z_{k+1} + \mathfrak{g}_k(z_1,\ldots,z_k)\,u\\ \dot{z}_{k+1} &= \Phi_{k+1}(t) + \mathfrak{g}_{k+1}(z_1,\ldots,z_{k+1})\,u\end{aligned}$$

Indeed, thanks to the above observer we know in finite time the values of $z_1,\ldots,z_k$, so that the function $\mathfrak{g}_k(z_1,\ldots,z_k)u$ becomes a known signal $r_k(t)$.

From this, we can propose the following observer made of a cascade of homogeneous observers :

$$\begin{aligned}
&\dot{\hat{z}}_{11} \in -k_{11}\,\mathrm{S}(\hat{z}_{11}-y)\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\dot{\hat{z}}_{21} = \hat{z}_{22} + \mathrm{sat}_1(\mathfrak{g}_1(\hat{z}_{11}))u - k_{21}\,\lfloor \hat{z}_{21}-y\rceil^{\frac{1}{2}}\\
&\dot{\hat{z}}_{22} \in -k_{22}\,\mathrm{S}(\hat{z}_{21}-y)\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\quad\vdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\dot{\hat{z}}_{m1} = \hat{z}_{m2} + \mathrm{sat}_1(\mathfrak{g}_1(\hat{z}_{11}))u - k_{m1}\,\lfloor \hat{z}_{m1}-y\rceil^{\frac{m-1}{m}}\\
&\quad\vdots\\
&\dot{\hat{z}}_{m(m-1)} = \hat{z}_{mm} + \mathrm{sat}_{m-1}\left(\mathfrak{g}_m(\hat{z}_{m1},\ldots,\hat{z}_{m(m-1)})\right)u\\
&\qquad\qquad -k_{m(m-1)}\,\lfloor \hat{z}_{m1}-y\rceil^{\frac{1}{m}}\\
&\dot{\hat{z}}_{mm} \in -k_{mm}\,\mathrm{S}(\hat{z}_{m1}-y)
\end{aligned}$$

where the k_{ij} are positive real numbers to be tuned and where we have used the notations

$$\mathrm{sat}_i(\mathfrak{g}_i(z_1,...,z_i)) \;=\; \min\{\bar{G}_i, \max\{-\bar{G}_i, \mathfrak{g}_i(z_1,...,z_i)\}\}\ .$$

with the positive real numbers $\bar{G}_i$ given as

$$\left|\mathfrak{g}_i\Big(h(x),\ldots,L_f^{i-1}h(x)\Big)\right| \le \bar{G}_i \qquad \forall x\in\mathcal{C}$$

As a direct consequence of (Levant, 2001 b, Theorem 1) or Proposition 5, we have

Proposition 6. If, in (2), the functions $\mathfrak{g}_i$ are locally bounded, then we can find positive real numbers (k_{ij}) such that, for all (x,u) in $\mathcal{C}\times L^\infty_{loc}(\mathbb{R}_+;U)$ such that $X(x,t)$, solution in $\mathcal{C}$ for all t in $\mathbb{R}_+$ and, for all $\hat{z}$ in $\mathbb{R}^{\frac{m(m+1)}{2}}$, there exist absolutely continuous solutions $\hat{Z}(\hat{z},x,t)$ of the observer above which are defined on $\mathbb{R}_+$ and for which there exists T_0 such that, by denoting $\hat{\mathbf{Z}}_m$ the last block of m components of $\hat{Z}$, we have

$$\hat{Z}_m(\hat{z},x,t) \;=\; \mathbf{H}_m(X(x,t)) \qquad \forall t\ge T_0\ . \qquad \blacksquare$$

A drawback of the cascade of homogeneous observers is that it gives an observer with dimension $\frac{m(m+1)}{2}$ in general. However, it is possible to reduce this dimension since, for each new block, one may increase the dimension by more than one, when the corresponding added functions $\mathfrak{g}_i$ satisfy (an appropriate version of) (7). For instance, in the example below the first block has dimension 3, leading to an observer of dimension 12 and not 15.

4. EXAMPLE

Consider the system

$$\dot{x}_1 \;=\; x_2\ ,\ \dot{x}_2 \;=\; -x_1 + x_3^3\ ,\ \dot{x}_3 \;=\; -x_2 + v\ ,\ y \;=\; x_1 \quad (8)$$

with v as input. When $v=0$, it admits $2x_1^2+2x_2^2+x_3^4$ as positive definite and radially unbounded first integral. So we can expect the existence of solutions remaining in the compact set

$$\mathcal{C} \;=\; \left\{x\in\mathbb{R}^3 : 2x_1^2+2x_2^2+x_3^4 \le c\right\}$$

for instance when v is a small periodic time function, except maybe for some particular pair of input v and initial condition (x_1,x_2,x_3) for which resonance occurs. Also due to their periodic behavior, such solutions are likely to have their x_3 component recurrently crossing zero.

The system (8) is uniformly observable since, whatever v is, the knowledge of the function $t\mapsto y(t)=X_1(x,t)$ and therefore of its derivatives gives us right away x_1, x_2 and x_3. So it satisfies the first basic condition needed for an observer to be designed following the techniques described above. This is done in the following although we are aware of the fact that, by exploiting the monotonicity of $x_3\mapsto x_3^3$, if we know a compact set in which the solutions remain, then we can find k_2 and k_3 such that the following very simple reduced order observer is exponentially convergent

$$\begin{aligned}\dot{\hat{\xi}}_2 &= -y+\hat{x}_3^3-k_2\hat{x}_2 \quad , \qquad \hat{x}_2 = \hat{\xi}_2 + k_2 y\\ \dot{\hat{\xi}}_3 &= -(1+k_3)\hat{x}_2 + u \quad , \qquad \hat{x}_3 = \hat{\xi}_3 + k_3 y\end{aligned}$$

4.1 *Strong differential observability*

After letting $u = v - u_0$, where u_0 is at our disposal, computations give successively

$$\mathbf{H}_3(x) = \begin{pmatrix} x_1\\ x_2\\ -x_1+x_3^3\end{pmatrix}$$

$$\mathbf{H}_5(x) = \begin{pmatrix} \mathbf{H}_3(x)\\ -x_2-3(x_2-u_0)x_3^2\\ (1+3x_3^2)(x_1-x_3^3)+6(x_2-u_0)^2x_3\end{pmatrix}\ .$$

$\mathbf{H}_3$ is a bijection on $\mathbb{R}^3$ but not a diffeomorphism even on $\mathcal{C}$ because of a singularity of its Jacobian at $x_3=0$. So the system is differentially observable of order 3 on $\mathcal{C}$ but not strongly. According to Propositions 2 and 3, it admits a triangular canonical form of dimension 3 but with functions φ_3 and $\{\mathfrak{g}_i\}_{i=1,3}$ maybe non Lipschitz.

We have the same result with the order 4 and $\mathbf{H}_4$.

For the order 5, $\mathbf{H}_5$ is injective on $\mathbb{R}^3$ and an immersion at least on $\mathcal{C}$ when we select u_0 large enough. So the system is strongly differentially observable of order 5 on $\mathcal{C}$ and it admits a triangular canonical form of dimension 5 with φ_5 locally Lipschitz but functions $\{\mathfrak{g}_i\}_{i=1,5}$ maybe non Lipschitz at points where the rank of $\mathbf{H}_i$ is smaller.

4.2 *An homogeneous observer of order* 3

The triangular canonical form of dimension 3 mentioned above is

$$\begin{aligned}\dot{z}_1 &= z_2\\ \dot{z}_2 &= z_3\\ \dot{z}_3 &= \varphi_3(z)+\mathfrak{g}_3(z)u\\ y &= z_1.\end{aligned} \qquad (9)$$

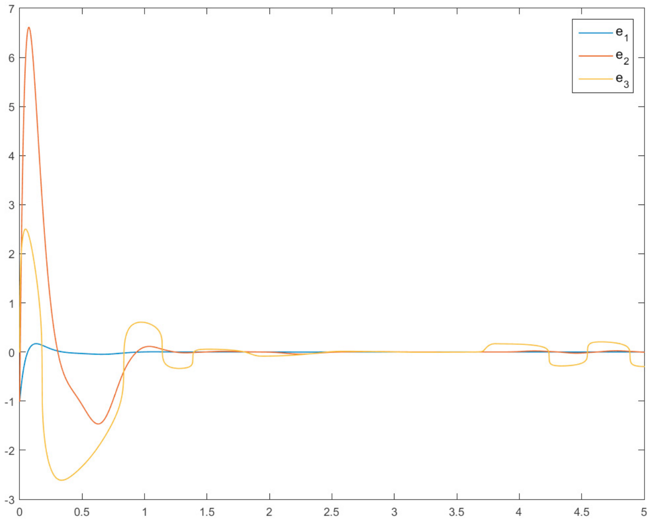

Fig. 1. Errors between the state x and its estimate given by a high gain observer.

where

$$\varphi_3(z) = -z_2 - 3(z_2 - u_0)(z_3 + z_1)^{\frac{2}{3}} ,$$
$$\mathfrak{g}_3(z) = 3(z_3 + z_1)^{\frac{2}{3}} .$$

These functions are not Lipschitz at the points on the hyperplane $z_3 = -z_1$ (image by $\mathbf{H}_3$ of points where $x_3 = 0$) known to be visited possibly recurrently along solutions.

As expected a high gain observer has problems dealing with this system. Figure 1 shows the results of a simulation done with such an observer initialized at the origin, for a system solution issued from $x = (1, 1, 0)$ with $v = 5\sin(t/10)$. Each time the system solution crosses the singularity hyperplane $x_3 = 0$, the error jumps likely because the observer does not manage to compensate for the infinite Lipschitz constant of φ_3 and $\mathfrak{g}_3$ at this point.

On another hand, because $\mathfrak{g}_3$ and φ_3 are Hölder of order $\frac{2}{3}$, we can consider an homogeneous observer as discussed in Proposition 5. It is

$$\dot{\hat{z}}_1 = \hat{z}_2 - k_1 \lfloor \hat{z}_1 - z_1 \rceil^{\frac{1-d_0}{1-2d_0}}$$
$$\dot{\hat{z}}_2 = \hat{z}_3 - k_2 \lfloor \hat{z}_1 - z_1 \rceil^{\frac{1}{1-2d_0}}$$
$$\dot{\hat{z}}_3 = \psi(\hat{z}_1, \hat{z}_2, \hat{z}_3, u) - k_3 \lfloor \hat{z}_1 - z_1 \rceil^{\frac{1+d_0}{1-2d_0}}$$

with d_0 to be chosen in $(-1, 0)$ and where $\psi = \varphi_3 + \mathfrak{g}_3 u$ is given in (9). It converges if the k_i are sufficiently large and the function ψ verify inequality (7), i.e. :

$$|\psi(z_1 + e_1, z_2 + e_2, z_3 + e_3, u) - \psi(z_1, z_2, z_3, u)| \leq L\,[|e_1|^{a_1} + |e_2|^{a_2} + |e_3|^{a_3}] ,$$

with

$$a_1 \geq \frac{1 + d_0}{1 - 2d_0} \quad , \qquad a_2 \geq \frac{1 + d_0}{1 - d_0} \quad , \qquad a_3 \geq 1 + d_0 .$$

In our case, we have $a_1 = \frac{2}{3} = a_3$ and $a_2 = 1$, so that the condition (7) is satisfied with $d_0 = -\frac{1}{3}$. The results of a simulation with this d_0 and $k_1 = 8$, $k_2 = 35$, $k_3 = 75$ are presented in Figure 2.

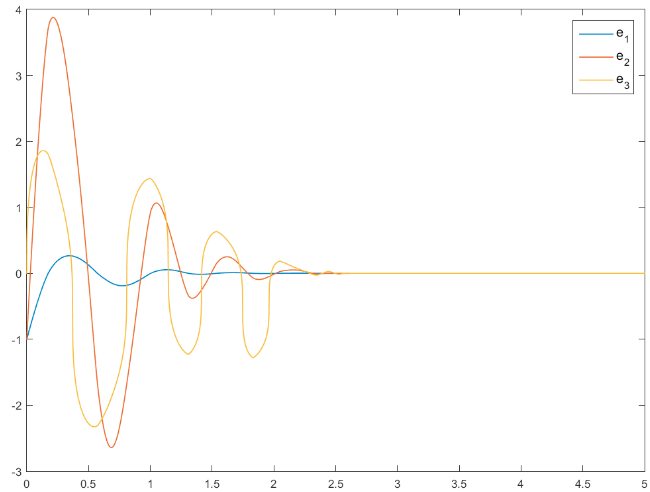

Fig. 2. Errors between the state x and its estimate given by a homogeneous observer.

Note that in this case, the third-order exact differentiator proposed in Levant (2001 b), obtained by picking $d_0 = -1$, would also work.

4.3 A cascade of homogeneous observers

The function $\mathbf{H}_3$ is a bijection but its inverse $\mathbf{H}_3^{-1}$ is not locally Lipschitz around the points such that $z_3 = -z_1$ (i.e. $H_3(\{x_3 = 0\})$). This may lead to too high sensitivity to disturbances on the estimate of x_3. Therefore, it may be preferable to implement the observer in dimension 5. Indeed, $\mathbf{H}_5$ is an injective immersion and has a locally Lipschitz inverse. The triangular canonical form of dimension 5 given by $\mathbf{H}_5$ is

$$\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= z_3 \\
\dot{z}_3 &= z_4 + \mathfrak{g}_3(z_1, z_2, z_3)u \\
\dot{z}_4 &= z_5 + \mathfrak{g}_4(z_1, z_2, z_3, z_4)u \\
\dot{z}_5 &= \varphi_5(z) + \mathfrak{g}_5(z)u \\
y &= z_1 .
\end{aligned}$$

where the functions $\mathfrak{g}_3$, $\mathfrak{g}_4$, $\mathfrak{g}_5$ and φ_5, not expressed here due to space limitations are Hölder of order 2/3, 1/3, 1, and 1 respectively. Hence Assumption (7) is satisfied with $d_0 = -1$. So an homogeneous observer can be used. However, we choose here to illustrate the cascaded observer proposed in Section 3.2. It takes the following form :

$$\dot{\hat{z}}_{11} = \hat{z}_{12} - k_{11} \lfloor \hat{z}_{11} - y \rceil^{\frac{2}{3}}$$
$$\dot{\hat{z}}_{12} = \hat{z}_{13} - k_{12} \lfloor \hat{z}_{11} - y \rceil^{\frac{1}{3}}$$
$$\dot{\hat{z}}_{13} \in -k_{13}\, \mathrm{S}(\hat{z}_{11} - y)$$

. . . .

$$\dot{\hat{z}}_{21} = \hat{z}_{22} - k_{21} \lfloor \hat{z}_{21} - y \rceil^{\frac{3}{4}}$$
$$\dot{\hat{z}}_{22} = \hat{z}_{23} - k_{22} \lfloor \hat{z}_{21} - y \rceil^{\frac{1}{2}}$$
$$\dot{\hat{z}}_{23} = \hat{z}_{24} + \mathrm{sat}_3(\mathfrak{g}_3(\hat{z}_{11}, \hat{z}_{12}, \hat{z}_{13}))u - k_{23} \lfloor \hat{z}_{21} - y \rceil^{\frac{1}{4}}$$
$$\dot{\hat{z}}_{24} \in -k_{24}\, \mathrm{S}(\hat{z}_{21} - y)$$

. . . .

$$\dot{\hat{z}}_{31} = \hat{z}_{32} - k_{31} \lfloor \hat{z}_{31} - y \rceil^{\frac{4}{5}}$$
$$\dot{\hat{z}}_{32} = \hat{z}_{33} - k_{32} \lfloor \hat{z}_{31} - y \rceil^{\frac{3}{5}}$$
$$\dot{\hat{z}}_{33} = \hat{z}_{34} + \mathrm{sat}_3(\mathfrak{g}_3(\hat{z}_{11}, \hat{z}_{12}, \hat{z}_{13}))u - k_{33} \lfloor \hat{z}_{31} - y \rceil^{\frac{2}{5}}$$
$$\dot{\hat{z}}_{34} = \hat{z}_{35} + \mathrm{sat}_4(\mathfrak{g}_4(\hat{z}_{11}, \hat{z}_{12}, \hat{z}_{13}))u - k_{34} \lfloor \hat{z}_{31} - y \rceil^{\frac{1}{5}}$$
$$\dot{\hat{z}}_{35} \in -k_{34}\, \mathrm{S}(\hat{z}_{31} - y)$$

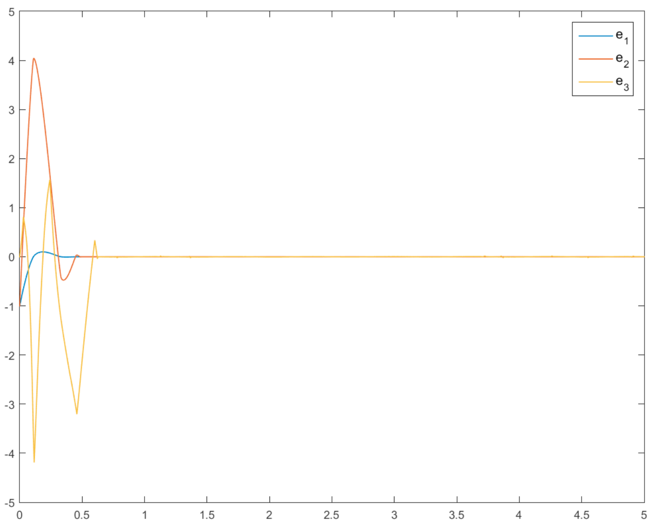

Fig. 3. Errors between the state x and its estimate given by a cascaded homogeneous observer.

The results of a simulation with the gains

$k_{11} = 14,\ k_{12} = 56,\quad k_{13} = 110$
$k_{21} = 28,\ k_{22} = 277,\ k_{23} = 790,\quad k_{24} = 1100$
$k_{31} = 50,\ k_{32} = 947,\ k_{33} = 6232,\ k_{34} = 11842,\ k_{35} = 11000$

and a discretization time step of 10^{-6} are given in Figure 3 (see Levant (2003) for information on the gains settings)

5. CONCLUSION

It is known that observers can be designed for systems which can be transformed into a triangular canonical form, the functions of which are locally Lipschitz. We have established that, when omitting the requirement of local Lipschitzness, such systems include those which are uniformly observable and differentially observable with an order maybe larger than the system state dimension. Hence, in this case, the usual assumption of strong differential observability of order equal to the system state dimension is too restrictive. But then we have to deal with an observer design for a non-Lipschitz triangular canonical form. Homogeneous observers give an answer to this problem for some classes of Hölder functions. We have proposed to combine them in a cascade way to go beyond this Hölderness, imposing only continuity.

All this is only a preliminary work. Many topics remain to be addressed : sensitivity to uncertainties in the dynamics, sensitivity to measurement noise, gain tuning procedure.

APPENDIX : BARBOT ET AL'S OBSERVER

The set valued map proposed in Barbot et al. (1996) to obtain an observer for a triangular canonical form where the functions are only locally bounded is defined as follows. Given $(\hat{z}, y, u)$, $(v_1, \dots, v_m)$ is in $F(\hat{z}, y, u)$ if there exists $(\tilde{z}_2, ..., \tilde{z}_m)$ in $\mathbb{R}^{m-1}$ such that:

$$\begin{aligned}
v_1 &= \tilde{z}_2 + \mathfrak{g}_1(y)\, u \\
\tilde{z}_2 &\in \mathrm{sat}(\hat{z}_2) - k_1\, S(y - \hat{z}_1) \\
&\vdots \\
v_i &= \tilde{z}_{i+1} + \mathfrak{g}_i(y, \tilde{z}_2, \dots, \tilde{z}_i)\, u \\
\tilde{z}_{i+1} &\in \mathrm{sat}(\hat{z}_{i+1}) - k_i\, S(\hat{z}_i - \tilde{z}_i) \\
&\vdots \\
v_m &\in \varphi_m(y, \tilde{z}_2, \dots, \tilde{z}_m) \\
&\qquad + \mathfrak{g}_m(y, \tilde{z}_2, \dots, \tilde{z}_m)\, u - k_m\, S(\hat{z}_m - \tilde{z}_m)
\end{aligned}$$

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