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# Solution of a Riccati Equation for the Design of an Observer Contracting a Riemannian Distance

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***Abstract*—We propose a method to design an intrinsic observer guaranteeing that the Riemannian distance between the estimate it generates and the state of the system is decreasing in time, at least locally. The design relies on the existence of a Riemannian metric, the Lie derivative of which along the system vector field is negative in the space tangent to the level sets of the output function. We show that, at least when the system is uniformly strongly infinitesimally observable (i.e., each time-varying linear system resulting from the linearization along a solution to the system satisfies a uniform observability property), there exists such a metric and it can be obtained as a solution to an algebraic-like Riccati equation. For such systems, we propose also an algorithm to numerically approximate the metric by griding the space and integrating ordinary differential equations.**

## I. INTRODUCTION

For systems of the form

$$\dot{x} = f(x) , \quad y = h(x) \tag{1}$$

with state $x \in \mathbb{R}^n$ and output $y \in \mathbb{R}^p$, we consider the problem of designing an observer, the state of which, denoted $\hat{x}$, is the estimated state and such that, for some metric, the Riemannian distance between $x$ and $\hat{x}$ tends to zero.

Different approaches for the solution to this problem have appeared in the literature. The idea of exploiting a possible non-expansivity property of the flow generated by the observer emerged from [1]. Actually, non-expansive flows have been studied in a variety of contexts in earlier work, including [2], [3], [4], [5]. A historical survey of such results appeared in [6].

A possible way to define the distance needed to characterize non-expansivity is by introducing a Riemannian metric. Such an approach, with a metric depending only on $f$ were employed in [7], [8] for the design of observers for systems whose dynamics follow from a principle of least action, such as Euler-Lagrange systems.

In [9], [10], the formalism of Riemannian geometry was employed to derive sufficient conditions for the existence of observers. These conditions involve mainly two properties (see definitions in the Glossary (next section).)

1) The existence of a Riemannian metric (given by a symmetric covariant 2-tensor) (see G-3) with a Lie derivative $\mathcal{L}_f P$ (see G-8) satisfying
$$\mathcal{L}_f P(x) \leq \rho(x)\, dh(x) \otimes dh(x) - q\, P(x) , \tag{2}$$
where $\rho : \mathbb{R}^n \to [0, +\infty)$ is a continuous function, $q$ is a strictly positive real number, $dh$ is the differential form of $h$ (see G-1) and $\otimes$ is a tensor product (see G-2).
2) The output function $h$ is geodesically monotonic for this metric (see G-9).

Under these conditions, an intrinsic expression for a semi-globally convergent observer is (see G-3 and G-4)

$$\dot{\hat{x}} = f(\hat{x}) - k_E(\hat{x})\, \mathrm{grad}_P h(\hat{x}) \frac{\partial \delta}{\partial y_a}(h(\hat{x}), y)^\top \tag{3}$$

where $k_E$ is a function to be chosen with sufficiently large values and the function $(y_a, y_b) \mapsto \delta(y_a, y_b)$ is related to the geodesic monotonicity of the output function (see (4)).

Actually, in [9], it is shown that a weak form of condition 1 above is a necessary condition for the existence of an observer such that the zero-error set $\{(x, \hat{x}) \in \mathbb{R}^{2n} : x = \hat{x}\}$ is asymptotically stable.

In this paper, we focus on the existence condition 1. In [10, Theorem 2.9] (see also [9, Proposition 3.2]) it is established that, if we have a bounded Riemannian metric which is bounded away from zero and satisfies (2), then each linear (time varying) system given by the first order approximation of (1) (assumed to be forward complete) along any of its solutions is uniformly detectable. In [9, Proposition 3.2] it is also claimed that if this uniform linear detectability is strengthened into a uniform reconstructibility property (or say uniform infinitesimal observability [11, Section I.2.1]), then a Riemannian metric satisfying (2) does exist. In this paper we re-establish this last property by showing that the uniform reconstructibility property implies the existence of a solution to

$$\mathcal{L}_f P(x) = dh(x) \otimes dh(x) - P(x) Q(x) P(x)$$

which can be seen as the nonlinear counterpart of the algebraic Riccati equation involved in the asymptotic Kalman filter. In this way, the observer we obtain has a strong connection with the well known Extended Kalman filter (see, e.g., [12]).

## GLOSSARY

G-1 Given a function $h : \mathbb{R}^n \to \mathbb{R}^p$, $dh$ denotes its differential form whose expression in coordinates $x$ is $\frac{\partial h_k}{\partial x_j}(x)$, for $k$ in $\{1, \dots, p\}$ and $j$ in $\{1, \dots, n\}$.

G-2 The tensor product $dh(x) \otimes dh(x)$ is a symmetric covariant 2-tensor whose expression in coordinates $x$

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is given by

$$\sum_{k=1}^{p} \frac{\partial h_k}{\partial x_i}(x) \frac{\partial h_k}{\partial x_j}(x) \ .$$

G-3 A Riemannian metric is a symmetric covariant 2-tensor with positive definite values.

G-4 Given a Riemannian metric $P$ and a function $h$, the expression in coordinates $x$ of its (Riemannian) gradient $\mathrm{grad}_P h$ is

$$\mathrm{grad}_P h(x) \ = \ P(x)^{-1} \frac{\partial h}{\partial x}(x)^\top \ .$$

G-5 The length of a $C^1$ path $s \mapsto \gamma(s)$ between points $x_a$ and $x_b$ for a Riemannian metric $P$ is defined as

$$L(\gamma)\Big|_{s_a}^{s_b} \ = \ \int_{s_a}^{s_b} \sqrt{\frac{d\gamma}{ds}(s)^\top P(\gamma(s)) \frac{d\gamma}{ds}(s)}\, ds,$$

where

$$\gamma(s_a) \ = \ x_a \quad , \qquad \gamma(s_b) \ = \ x_b \ .$$

G-6 The Riemannian distance between points $x_a$ and $x_b$ that is induced by $P$ is given by

$$d(x_a, x_b) = \min_{\gamma \in C^1, \gamma(s_a)=x_a, \gamma(s_b)=x_b} L(\gamma)|_{s_a}^{s_b} \ .$$

A minimizer (path) giving the distance is called a minimizing geodesic and is denoted by $\gamma^*$.

G-7 A topological space equipped with a Riemannian distance is complete when every geodesic can be maximally extended to $\mathbb{R}$.

G-8 The Lie derivative of the Riemannian metric $P$ along the vector field $f$ is denoted as $\mathcal{L}_f P$. Given a set of coordinates for $x$, for all $v$ in $\mathbb{R}^n$, the quantity $v^\top \mathcal{L}_f P(x) v$ is given by

$$\lim_{t \to 0} \left[ \frac{[(I + t\frac{\partial f}{\partial x}(x))v]^\top P(X(x,t))[(I + t\frac{\partial f}{\partial x}(x))v]}{t} - \frac{v^\top P(x) v}{t} \right]$$

which is equal to

$$\frac{\partial}{\partial x} \left( v^\top P(x)\, v \right) \, f(x) \ + \ 2\, v^\top P(x) \left( \frac{\partial f}{\partial x}(x)\, v \right)$$

where $t \mapsto X(x,t)$ is the solution to (1) and $I$ is the identity matrix.

We would like the reader to distinguish the notation $\mathcal{L}_f P$ for the Lie derivative of a symmetric covariant 2-tensor from $L_f \varphi$, which is used for the more usual Lie derivative of a function $\varphi$, or equivalently, the vector field induced by a function.

G-9 A function $h : \mathbb{R}^n \to \mathbb{R}^p$ is said to be geodesically monotonic with respect to $P$ if there exists a $C^2$ function $\mathbb{R}^p \times \mathbb{R}^p \ni (y_a, y_b) \mapsto \delta(y_a, y_b) \in [0, +\infty)$ satisfying, for all $x$ in $\mathbb{R}^n$

$$\delta(h(x), h(x)) \ = \ 0 \quad , \quad \frac{\partial^2 \delta}{\partial y_a^2}(y_a, y_b)\Bigg|_{y_a = y_b = h(x)} \ > \ 0 \ ,$$

such that, for any pair $(x_a, x_b)$ in $\mathbb{R}^n \times \mathbb{R}^n$ satisfying

$$h(x_a) \ \neq \ h(x_b)$$

and any minimizing geodesic $\gamma^*$ between $x_a = \gamma^*(s_a)$ and $x_b = \gamma^*(s_b)$, with $s_a \leq s_b$, we have

$$\frac{d}{ds} \delta(h(\gamma^*(s)), h(\gamma^*(s_a))) \ > \ 0 \qquad \forall s \in (s_a, s_b] \ . \quad (4)$$

## II. Existence of $P$ for Linearly Reconstructible Systems

To properly state how uniform reconstructibility or uniform infinitesimal observability implies the existence of a Riemannian metric satisfying (2), we assume the existence of a backward invariant open set $\Omega$ for the system (1). This implies that, for each $x$ in $\Omega$, there exists a strictly positive real number $\sigma_x$, possibly infinite, such that the corresponding solution to (1), $t \mapsto X(x,t)$, is defined with values in $\Omega$ over $(-\infty, \sigma_x)$. For each such $x$, the linearization of $f$ and $h$ evaluated along $t \mapsto X(x,t)$ gives the functions $A_x(t) = \frac{\partial f}{\partial x}(X(x,t))$ and $C_x(t) = \frac{\partial h}{\partial x}(X(x,t))$, which are defined on $(-\infty, \sigma_x)$. To these functions, we associate the following family of linear time-varying systems with state $\xi$ in $\mathbb{R}^n$ and output $\eta$ in $\mathbb{R}^p$:

$$\dot{\xi} \ = \ A_x(t)\, \xi \ , \quad \eta \ = \ C_x(t)\, \xi, \qquad (5)$$

which is parameterized by the initial condition $x$ of the chosen solution $t \mapsto X(x,t)$. Below, $\Phi_x$ denotes the state transition matrix for (5). It satisfies

$$\frac{\partial \Phi_x}{\partial s}(t,s) \ = \ A_x(t) \Phi_x(t,s), \qquad \Phi_x(s,s) \ = \ I \ .$$

*Definition 2.1 (uniform reconstructibility):* The family of systems (5) is said to be uniformly reconstructible on a set $\Omega$ if there exist strictly positive real numbers $\tau$ and $\varepsilon$ such that we have, for all $x$ in $\Omega$,

$$\int_{-\tau}^{0} \Phi_x(t,0)^\top C_x(t)^\top C_x(t) \Phi_x(t,0) dt \ \geq \ \varepsilon I \ . \qquad (6)$$

*Proposition 2.2:* Let $Q$ be a symmetric contravariant 2-tensor. Assume there exist

i) an open set $\Omega \subset \mathbb{R}^n$ that is backward invariant for (1) and on which the family of systems (5) is uniformly reconstructible;

ii) coordinates for $x$ such that the derivatives of $f$ and $h$ are bounded on $\Omega$ and we have

$$0 \ < \ \underline{q}\, I \ \leq \ Q(x) \leq \ \overline{q}\, I \qquad \forall x \in \Omega \ . \qquad (7)$$

Then, there exists a symmetric covariant 2-tensor $P$ defined on $\Omega$, which admits a Lie derivative $\mathcal{L}_f P$ satisfying

$$\mathcal{L}_f P(x) = dh(x) \otimes dh(x) - P(x) Q(x) P(x) \quad \forall x \in \Omega \, , (8)$$

and there exist strictly positive real numbers $\underline{p}$ and $\overline{p}$ such that, in the coordinates given above, we have

$$0 \ < \ \underline{p}\, I \ \leq \ P(x) \leq \ \overline{p}\, I \qquad \forall x \in \Omega \ . \qquad (9)$$

*Proof:* See Section IV-A. The proof of this result given in Section IV-A relies on a fixed point argument, the core of which is the fact the flow generated by the differential Riccati equation is a contraction. This fact, first established for the discrete time case in [13], is proved in [14] for the continuous-time case. ■

*Remark 2.3:* In his introduction of Riccati differential equations for matrices in [15], [16], Radon has shown that such equations can be solved via two coupled linear differential equations. (See also [17].) In our framework, this leads to obtain a solution to equation (8) by solving in $(\alpha, \beta)$ the coupled system

$$\begin{aligned}\sum_{i=1}^{n} \frac{\partial \alpha}{\partial x_i}(x) f_i(x) &= -\frac{\partial f}{\partial x}(x)^\top \alpha(x) \\ &\quad + \frac{\partial h}{\partial x}(x) \frac{\partial h}{\partial x}(x)^\top \beta(x) , \\ \sum_{i=1}^{n} \frac{\partial \beta}{\partial x_i}(x) f_i(x) &= Q(x)\alpha(x) + \frac{\partial f}{\partial x}(x)\beta(x)\end{aligned} \tag{10}$$

with $\beta$ invertible and then picking

$$P(x) = \alpha(x)\, \beta(x)^{-1} .$$

When the metric is obtained by solving (8), the observer we obtain from (3), with $\delta(y_a, y_b) = |y_a - y_b|^2$, resembles an Extended Kalman Filter (see [12] for instance). Indeed, in some coordinates, our observer is

$$\dot{\hat{x}} = f(\hat{x}) - 2\, k_E(\hat{x})\, P(\hat{x})^{-1} \frac{\partial h}{\partial x}(\hat{x})^\top (h(\hat{x}) - y) , \tag{11}$$

$$\begin{aligned}\sum_{i=1}^{n} \frac{\partial P}{\partial x_i}(\hat{x}) f(\hat{x}) = -P(\hat{x}) \frac{\partial f}{\partial x}(\hat{x}) - \frac{\partial f}{\partial x}(\hat{x})^\top P(\hat{x}) \\ + \frac{\partial h}{\partial x}(\hat{x})^\top \frac{\partial h}{\partial x}(\hat{x}) - P(\hat{x}) Q(\hat{x}) P(\hat{x})\end{aligned} \tag{12}$$

and the corresponding extended Kalman filter would be

$$\dot{\hat{x}} = f(\hat{x}) - P^{-1} \frac{\partial h}{\partial x}(\hat{x})^\top (h(\hat{x}) - y) , \tag{13}$$

$$\begin{aligned}\dot{P} = -P \frac{\partial f}{\partial x}(\hat{x}) - \frac{\partial f}{\partial x}(\hat{x})^\top P \\ + \frac{\partial h}{\partial x}(\hat{x})^\top \frac{\partial h}{\partial x}(\hat{x}) - P Q(\hat{x}) P .\end{aligned} \tag{14}$$

The expressions for $\dot{\hat{x}}$ in (11) and (13) are the same except for the presence of $k_E$ in (11). On the other hand, (12) and (14) are significantly different. The former is a partial differential equation which can be solved off-line as an algebraic Riccati equation. If the assumptions in Proposition 2.2 are satisfied, (12) has a solution, guaranteed to be bounded and positive definite. Nevertheless, condition 2 in Section I may not hold, and, as a consequence, the observer may not be semiglobally convergent as in [10, Lemma 3.6], but only locally convergent.

The differential Riccati equation (14) of the extended Kalman filter is an ordinary differential equation with $P$ being part of the observer state. The corresponding observer is known to be locally convergent but under the extra assumption that $P$ is bounded and positive definite. See [18] for instance. Unfortunately, even when the assumptions in Proposition 2.2 are satisfied, we have no guarantee that $P$ has such properties except may be if $\hat{x}$ remains close enough to $x$ (which is what is to be proved).

The quadratic term $P(x)Q(x)P(x)$ in the "algebraic Riccati equation" (8) can be replaced by $\lambda P(x)$. Specifically, we have the following reformulation of [9, Proposition 3.2].

*Proposition 2.4:* Under the conditions of Proposition 2.2, there exists $\underline{\lambda} > 0$ such that, for each $\lambda \geq \underline{\lambda}$, there exists a symmetric covariant 2-tensor $P$ defined on $\Omega$, that admits a Lie derivative $\mathcal{L}_f P$ satisfying

$$\mathcal{L}_f P(x) = dh(x) \otimes dh(x) - \lambda P(x) \qquad \forall x \in \Omega , \tag{15}$$

and there exist strictly positive real numbers $\underline{p}$ and $\overline{p}$ such that, in the coordinates given by the assumption, (9) holds.

*Proof:* See Section IV-B. ■

## III. An Algorithm for the Computation of Approximations of $P$

From Propositions 2.2 and 2.4, to obtain a Riemannian metric satisfying inequality (2), it is sufficient to find a solution to (8) or (15). From the proofs of these propositions, an algorithm providing such a solution is as follows.

Given $x$ in $\Omega$ at which $P$ is to be evaluated, pick $\overline{T} > 0$ large enough, and perform the following steps:

**Step 1)** Compute the solution $[-\overline{T}, 0] \ni t \mapsto X(x,t)$ to (1) backward in time from the initial condition $x$ at time $t = 0$, up to a negative time $t = -\overline{T}$;

**Step 2)** Compute the solution $[-\overline{T}, 0] \ni t \mapsto \Pi(t)$ to

$$\begin{aligned}\dot{\pi} &= -\pi \frac{\partial f}{\partial x}(X(x,t)) - \frac{\partial f}{\partial x}(X(x,t))^\top \pi \\ &\quad + \frac{\partial h}{\partial x}(X(x,t))^\top \frac{\partial h}{\partial x}(X(x,t)) - \pi Q(X(x,t))\pi\end{aligned}$$

or, with $\lambda$ large enough, to

$$\begin{aligned}\dot{\pi} &= -\pi \frac{\partial f}{\partial x}(X(x,t)) - \frac{\partial f}{\partial x}(X(x,t))^\top \pi \\ &\quad + \frac{\partial h}{\partial x}(X(x,t))^\top \frac{\partial h}{\partial x}(X(x,t)) - \lambda \pi\end{aligned}$$

with initial condition $\pi(-\overline{T}) = \underline{p}\, I_n$ and using the function $[-\overline{T}, 0] \ni t \mapsto X(x,t)$ obtained in Step 1;

**Step 3)** Define the value of $P$ at $x$ as the value $\Pi(0)$.

By griding the state space of $x$ and approximating $P$ at each such $x$, the method suggested above can be considered as a design tool, at least for low dimensional systems. Note that the computations in Step 1 and Step 2 only require the use of a scheme for integration of ordinary differential equations.

To illustrate this algorithm, we consider a harmonic oscillator with unknown frequency. Its dynamics are

$$\dot{x} = f(x) := \begin{pmatrix} x_2 \\ -x_3 x_1 \\ 0 \end{pmatrix} , \quad y = h(x) := x_1 \tag{16}$$

with $(x_1, x_2, x_3) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}_{>0}$. By formal computations, it is possible to obtain the following expression of a metric $P$ satisfying (15):

$$P(x) = \begin{pmatrix} \dfrac{\lambda^2 + 2x_3}{\lambda(\lambda^2 + 4x_3)} , & \star , & \star \\[2ex] -\dfrac{1}{(\lambda^2 + 4x_3)} , & \dfrac{2}{\lambda(\lambda^2 + 4x_3)} , & \star \\[2ex] \dfrac{-\lambda^3 x_1 + (\lambda^2 - 4x_3)x_2}{\lambda^2(\lambda^2 + 4x_3)^2} , & \dfrac{(3\lambda^2 + 4x_3)x_1 - 4\lambda x_2}{\lambda^2(\lambda^2 + 4x_3)^2} , & a \end{pmatrix}$$

where the various $\star$ should be replaced by their symmetric values and

$$a = \frac{6\lambda^4 + 12\lambda^2 x_3 + 16x_3^2}{\lambda^3(\lambda^2+4x_3)^3} x_1^2 - \frac{4(5\lambda^2+4x_3)}{\lambda^2(\lambda^2+4x_3)^3} x_1 x_2 + \frac{4(5\lambda^2+4x_3)}{\lambda^3(\lambda^2+4x_3)^3} x_2^2$$

Now, we employ the algorithm described above to obtain an approximation of this analytic expression. For this purpose, we use the second differential equation in Step 2. For different values of $\lambda$ and over a grid of $m_{x_1} * m_{x_2} * m_{x_3}$ values of $x$ in $[0,1]\times[1,1.5]\times[0.5,1.5]$ with $m_{x_1} = m_{x_2} = m_{x_3} = 5$, Figure 1 shows at each computation step the max norm over the window $[-\bar{T}, 0]$ of the error matrix between the computed matrix and its analytic expression. The plot shows that this error is very close to zero before $t = 0$ for $\lambda$ larger than or equal to 8. The value of $\bar{T}$ shown in the various plots is chosen depending on $\lambda$ to satisfy

$$\exp(-\lambda\bar{T}) \leq 10^{-4} .$$

The initial condition used is $\hat{\pi}(-\bar{T}) = 5I$.

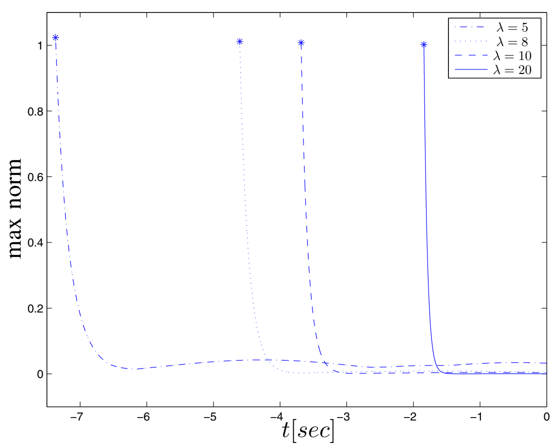


Fig. 1. The $\infty$-norm of the error between the computed matrix and its analytic expression. The initial error is denoted by $*$.

## IV. Sketch of Proofs of the Results

### A. Sketch of proof of Proposition 2.2

We present here the steps we have found for proving Proposition 2.2. They rely on 4 lemmas which are given without proofs, due to space limitations.

We start by showing that equation (8) is invariant under a change of coordinates. Let $x$ and $\tilde{x}$ denote two sets coordinates for a point, related as $\tilde{x} = \varphi(x)$, where $\varphi$ is a diffeomorphism. With the definition of the Lie derivative of $P$, the expression of (8) in the $x$-coordinates is

$$P(x)\frac{\partial f}{\partial x}(x) + \frac{\partial f}{\partial x}(x)^\top P(x) + \sum_{i=1}^{n} \frac{\partial P}{\partial x_i}(x) f_i(x) = \frac{\partial h}{\partial x}(x)^\top \frac{\partial h}{\partial x}(x) - P(x)Q(x)P(x) . \quad (17)$$

To write this in the $\tilde{x}$ coordinates, we apply the rule of change of coordinates for, respectively, a vector field, a symmetric covariant 2-tensor, a symmetric contravariant 2-tensor, and a function,

$$\begin{aligned} \tilde{f}(\tilde{x}) &= \frac{\partial \varphi}{\partial x}(x) f(x) , \\ \tilde{P}(\tilde{x}) &= \left[\left[\frac{\partial \varphi}{\partial x}(x)\right]^{-1}\right]^\top P(x) \left[\frac{\partial \varphi}{\partial x}(x)\right]^{-1} , \\ \tilde{Q}(\tilde{x}) &= \frac{\partial \varphi}{\partial x}(x)\, Q(x) \left[\frac{\partial \varphi}{\partial x}(x)\right]^\top , \qquad (18) \\ \tilde{h}(\tilde{x}) &= h(x) . \qquad (19) \end{aligned}$$

Introducing these relations into (17), we obtain

$$\tilde{P}(\tilde{x})\frac{\partial \tilde{f}}{\partial \tilde{x}}(\tilde{x}) + \frac{\partial \tilde{f}}{\partial \tilde{x}}(\tilde{x})^\top P(\tilde{x}) + \sum_{i=1}^{n} \frac{\partial P}{\partial \tilde{x}_i}(x)\tilde{f}_i(\tilde{x}) = \frac{\partial \tilde{h}}{\partial \tilde{x}}(\tilde{x})^\top \frac{\partial \tilde{h}}{\partial \tilde{x}}(\tilde{x}) - \tilde{P}(\tilde{x})\tilde{Q}(\tilde{x})\tilde{P}(\tilde{x}) .$$

The fact that this equation is the same as (17) confirms that we do have the said invariance under a change of coordinates.

With this at hand, let $\mathfrak{P}_{>0}$ be the $n$-dimensional cone of symmetric positive definite matrices and consider the cascade system, written in arbitrary coordinates for the time being, say $(x, \pi)$

$$\begin{aligned} \dot{x} &= f(x) , \\ \dot{\pi} &= F(x,\pi) = -\pi\frac{\partial f}{\partial x}(x) - \frac{\partial f}{\partial x}(x)^\top \pi + \frac{\partial h}{\partial x}(x)^\top \frac{\partial h}{\partial x}(x) - \pi\, Q(x)\, \pi . \end{aligned} \quad (20)$$

We denote by $t \mapsto (X(x,t), \Pi(x,\pi,t))$ its solution issued from $(x,\pi)$ in $\Omega \times \mathfrak{P}_{>0}$. In these equations $\pi$ is to play the role of the metric, i.e. a symmetric covariant 2-tensor. This leads us to study what is the effect on these equations of the following specific class of change of coordinates:

$$\begin{pmatrix} \tilde{x} \\ \tilde{\pi} \end{pmatrix} = D_\varphi(x,\pi) = \begin{pmatrix} \varphi(x) \\ \left[\left[\frac{\partial \varphi}{\partial x}(x)\right]^{-1}\right]^\top \pi \left[\frac{\partial \varphi}{\partial x}(x)\right]^{-1} \end{pmatrix} \quad (21)$$

Note that this makes $D_\varphi$ a diffeomorphism. The image of the vector field $(f, F)$ by $D_\varphi$ is :

$$\tilde{f}(\tilde{x}) = \frac{\partial \varphi}{\partial x}(x) f(x)$$

$$\begin{aligned} \tilde{F}(\tilde{x},\tilde{\pi}) = &- \left[\left[\frac{\partial \varphi}{\partial x}(x)\right]^{-1}\left(\sum_{i=1}^{n} \frac{\partial \frac{\partial \varphi}{\partial x}}{\partial x_i}(x) f_i(x)\right)\left[\frac{\partial \varphi}{\partial x}(x)\right]^{-1}\right]^\top \pi \left[\frac{\partial \varphi}{\partial x}(x)\right]^{-1} \\ &- \left[\left[\frac{\partial \varphi}{\partial x}(x)\right]^{-1}\right]^\top \pi \left[\frac{\partial \varphi}{\partial x}(x)\right]^{-1} \left(\sum_{i=1}^{n} \frac{\partial \frac{\partial \varphi}{\partial x}}{\partial x_i} f_i(x)\right) \left[\frac{\partial \varphi}{\partial x}(x)\right]^{-1} \\ &+ \left[\left[\frac{\partial \varphi}{\partial x}(x)\right]^{-1}\right]^\top F(x,\pi) \left[\frac{\partial \varphi}{\partial x}(x)\right]^{-1} . \end{aligned}$$

It satisfies

$$\tilde{F}(\tilde{x},\tilde{\pi}) = -\tilde{\pi}\frac{\partial \tilde{f}}{\partial \tilde{x}}(\tilde{x}) - \frac{\partial \tilde{f}}{\partial \tilde{x}}(\tilde{x})^\top \tilde{\pi} + \frac{\partial \tilde{h}}{\partial \tilde{x}}(\tilde{x})^\top \frac{\partial \tilde{h}}{\partial \tilde{x}}(\tilde{x}) -$$

$\tilde{\pi}\,\tilde{Q}(\tilde{x})\,\tilde{\pi}$

where $\tilde{Q}$ and $\tilde{h}$ are given in (18) and (19). We conclude that the system (20) too is invariant under a change of coordinates of type (21).

We are interested in this system (20) because, if it has an invariant manifold in the following form

$$\left\{(x,\pi)\in\Omega\times\mathfrak{P}_{>0}\;:\;\pi=P(x)\right\}$$

with some function $P$, then, given any $x$ in $\Omega$, the solution $t\mapsto(X(x,t),\Pi(x,P(x),t))$ to (20), defined say on $]T_-,T_+[$, satisfies

$$\Pi(x,P(x),t)\;=\;P(X(x,t))\qquad\forall t\in]T_-,T_+[\ . \qquad (22)$$

Then by computing the following expression, with $v$ in $\mathbb{S}^{n-1}$,

$$E(x,v)\;=\;\lim_{t\to 0} v^\top\left[\frac{[(I+t\frac{\partial f}{\partial x}(x))]^\top P(X(x,t))[(I+t\frac{\partial f}{\partial x}(x))]}{t}-\frac{P(x)}{t}\right]v\ ,$$

and noting that we have

$$\lim_{t\to 0}\frac{\Pi(x,P(x),t)\;-\;[P(x)+t\,F(x,P(x))]}{t}\;=\;0\ ,$$

we obtain

$$\begin{aligned}E(x,v)\;&=\;\lim_{t\to 0} v^\top\left[\frac{[(I+t\frac{\partial f}{\partial x}(x))]^\top\,[P(x)+t\,F(x,P(x))]\,[(I+t\frac{\partial f}{\partial x}(x))]}{t}-\frac{P(x)}{t}\right]v\ ,\\ &=\;2v^\top P(x)\frac{\partial f}{\partial x}(x)\,v\;+\;v^\top F(x,P(x))\,v\ ,\\ &=\;v^\top\left(\frac{\partial h}{\partial x}(x)^\top\frac{\partial h}{\partial x}(x)-P(x)\,Q\,P(x)\right)v\ .\end{aligned}$$

This establishes the expression of (8) in the coordinates in which (20) is written. But we know that (20) is independent of the coordinates. Hence, we have obtained (8).

With the above arguments, the proof of our result is complete if we establish the existence of $P$ satisfying (22). Since we are allowed to work with any coordinates, from now on, we work with the coordinates given in the assumption.

First, we recognize that the second equation in (20) is a differential Riccati equation, for which the following result is well known; see, for instance, [11, pages 109-113].

*Lemma 4.1:* Under the conditions of Proposition 2.2 there exist strictly positive real numbers $\underline{p}$ and $\overline{p}$ such that, for each $(x,\pi)$ in $\Omega\times\mathfrak{P}_{>0}$ satisfying

$$0\;<\;\pi\;\leq\;\overline{p}\,I\ ,$$

the solution $t\mapsto(X(x,t),\Pi(x,\pi,t))$ exists with values in $\Omega\times\mathfrak{P}_{>0}$ on[1] $[0,\sigma_x)$ and satisfies, for all $t$ in $[0,\sigma_x)$,

$$\lambda_{\min}(\pi)\,\exp(-(\overline{p}\,\overline{q}+2a)\,\tau)\;\leq\;\Pi(x,\pi,t)\ , \qquad (23)$$

where

$$a\;=\;\sup_{x\in\Omega}\left|\frac{\partial f}{\partial x}(x)\right|\ ,$$

and, when $\sigma_x$ is strictly larger than $\tau$ (see Definition 2.1),

$$\underline{p}I\;\leq\;\Pi(x,\pi,t)\;\leq\;\overline{p}I\qquad\forall t\in[\tau,\sigma_x)\ . \qquad (24)$$

Now, we equip $\mathfrak{P}_{>0}$ with the metric $\mathcal{D}$ induced from the Riemannian scalar product between two tangent vectors $Y_a$ and $Y_b$ at $S$ in $\mathfrak{P}_{>0}$ defined as follows:

$$g_S(Y_a,Y_b)=\mathrm{trace}(S^{-1}Y_aS^{-1}Y_b). \qquad (25)$$

This makes $\mathfrak{P}_{>0}$ a complete Riemannian manifold [19, Proposition 6.2.2]. Moreover, the distance between $S_a$ and $S_b$ in $\mathfrak{P}_{>0}$ is $\mathcal{D}(S_a,S_b)=\sqrt{\sum_{i=1}^n\log(\lambda_i)^2}$ , where $\lambda_i$ are the eigenvalues of $S_aS_b^{-1}$. Let $\mathfrak{P}_{\underline{p},\overline{p}}$ be the following subset

$$\mathfrak{P}_{\underline{p},\overline{p}}=\left\{\pi\in\mathfrak{P}_{>0}\;:\;\underline{p}I\leq\pi\leq\overline{p}I\right\}$$

with $0<\underline{p}<\overline{p}$. We have the following result.

*Lemma 4.2:* Under the conditions of Proposition 2.2 and with $\underline{p}$ and $\overline{p}$ obtained from Lemma 4.1, for each $(x,\pi_a,\pi_b)$ in $\Omega\times\mathfrak{P}_{\underline{p},\overline{p}}\times\mathfrak{P}_{\underline{p},\overline{p}}$, the solutions $t\mapsto(X(x,t),\Pi(x,\pi_a,t))$ and $t\mapsto(X(x,t),\Pi(x,\pi_b,t))$ are defined on $[0,\sigma_x)$ and satisfy, for all $t$ in $[0,\sigma_x)$,

$$\mathcal{D}(\Pi(x,\pi_a,t),\Pi(x,\pi_b,t))\;\leq\;\exp(-\underline{q}\,\underline{\underline{p}}\,t)\,\mathcal{D}(\pi_a,\pi_b)\ , \qquad (26)$$

where

$$\underline{\underline{p}}\;=\;\underline{p}\,\exp(-[\overline{p}\,\overline{q}+2a]\tau)\ . \qquad (27)$$

This result is an improvement of [14, Lemma 1], in particular, guaranteeing the contraction is uniform with respect to initial conditions in $\mathfrak{P}_{\underline{p},\overline{p}}$.

Let $\mathcal{P}$ be the space of continuously differentiable functions $P:\Omega\to\mathfrak{P}_{\underline{p},\overline{p}}$. When equipped with the distance

$$d(P_a,P_b)=\sup_{x\in\mathcal{C}}\mathcal{D}(P_a(x),P_b(x))$$

this space is complete – this follows from the completeness of $\mathfrak{P}_{>0}$.

With $\tau$ satisfying (6), to any function $P$ in $\mathcal{P}$, we associate a function, denoted $\mathcal{O}(P)$, defined on $\Omega$ and whose values are

$$\mathcal{O}(P)(x)=\Pi(X(x,-\tau),P(X(x,-\tau)),\tau)\qquad\forall x\in\Omega\ . \ (28)$$

This defines an operator $\mathcal{O}$ on $\mathcal{P}$. Since the functions $x\mapsto X(x,-\tau)$ and $(x,\pi)\mapsto\Pi(x,\pi,t)$ are continuous, $\mathcal{O}(P)$ is continuous when $P$ is continuous. Also, $\Omega$ being backward invariant, for any $x\in\Omega$, we have that $X(x,-\tau)\in\Omega$ and $\sigma_{X(x,-\tau)}=\sigma_x+\tau$. Hence, we can use (24) and (26) with $x$ replaced by $X(x,-\tau)$. More precisely, we have

$$\underline{p}I\leq\mathcal{O}(P)(x)\leq\overline{p}I\qquad\forall(x,P)\in\Omega\times\mathcal{P}\ .$$

This implies that the values of $\mathcal{O}$ are in $\mathcal{P}$.

Also, given $P_a$ and $P_b$ in $\mathcal{P}$, we have that, for each $x\in\Omega$,

$$\begin{aligned}&\mathcal{D}(\mathcal{O}(P_a)(x),\mathcal{O}(P_b)(x))\\ &\quad\leq\;\exp(-\underline{q}\,\underline{\underline{p}}\;\tau)\mathcal{D}(P_a(X(x,-\tau)),P_b(X(x,-\tau)))\ ,\\ &\quad\leq\;\exp(-\underline{q}\,\underline{\underline{p}}\;\tau)\,d(P_a,P_b)\ .\end{aligned}$$

Taking the supremum over $x\in\Omega$ gives

$$d(\mathcal{O}(P_a),\mathcal{O}(P_b))\;\leq\;\exp(-\underline{q}\,\underline{\underline{p}}\;\tau)\,d(P_a,P_b)\ .$$

[1]As mentioned at the beginning of Section II, $[0,\sigma_x)$ is the right maximal interval of definition of the solution $t\mapsto X(x,t)$ with values in $\Omega$.

This shows that $\mathcal{O}$ is a contraction over the complete space $\mathcal{P}$. Then, the Banach Fixed Point Theorem implies the following.

*Lemma 4.3:* The operator $\mathcal{O}$ defined in (28) has a unique fixed point $P^* \in \mathcal{P}$ satisfying

$$P^*(x) \;=\; \Pi(X(x,-\tau), P^*(X(x,-\tau)), \tau) \;. \qquad (29)$$

With this result, our proof is completed by showing that this fixed point $P^*$ satisfies (22).

*Lemma 4.4:* $P^*$ in Lemma 4.3 satisfies

$$P^*(X(x,t)) = \Pi(x, P^*(x), t) \qquad \forall t < \sigma_x \;,\; \forall x \in \Omega \;. \; (30)$$

*Remark 4.5:* In view of this proof we have, for all positive integers $k$ and all $\pi$ in $\mathfrak{P}_{\underline{p},\overline{p}}$,

$$\mathcal{D}\left(P^*(x)\,,\,\Pi(X(x,-k\tau),\pi,k\tau)\right) \;\leq\; \exp(-k[q\underline{\underline{p}}\kappa])\mathcal{D}\left(P^*(X(x,-k\tau))\,,\,\pi\right).$$

Hence, the larger $k$ is the better $\Pi(X(x,-k\tau),\pi,k\tau)$ approximates $P^*(x)$.

## B. Sketch of proof of Proposition 2.4

We can show formally, by substitution, that the following expression of $P$

$$P(x) = \lim_{T\to-\infty} \int_T^0 \exp(\lambda t)\Phi_x(t,0)^\top C_x(t)^\top C_x(t)\Phi_x(t,0)\,dt$$

satisfies (15). So it remains to prove that the right hand side has the required properties.

By invariance of $\Omega$ and boundedness of $\frac{\partial f}{\partial x}$ and $\frac{\partial h}{\partial x}$, there exist positive scalars $a$ and $c$ such that

$$|\Phi_x(t,s)| \leq \exp(a|t-s|) \quad , \qquad |C_x(t)| \leq c \qquad (31)$$

for each $(x,t,s) \in \Omega\times\mathbb{R}\times\mathbb{R}$. So, by picking $\lambda$ strictly larger than $2a$, we have that, for each $x\in\Omega$, the norm of the argument of the integral defining $P(x)$ is bounded by

$$c^2\,\exp((\lambda-2a)t) \qquad (32)$$

for each $t\leq 0$. By continuity of $x\mapsto(\Phi_x(t,0),C_x(t))$, using (6), we have that there exist $\varepsilon,\tau>0$ such that, for every $x$ in $\mathbb{R}^n$, $P$ satisfies

$$\begin{aligned} P(x) \;&\geq\; \int_{-\tau}^0 \exp(\lambda t)\Phi_x(t,0)^\top C_x(t)^\top C_x(t)\Phi_x(t,0)dt \\ &\geq\; \varepsilon\exp(-\lambda\tau)I. \end{aligned}$$

Moreover, using (32), gives

$$P(x) \;\leq\; c^2\int_{-\infty}^0 \exp(\lambda t)\,\exp(-2at)dt\,I \;=\; \frac{c^2}{\lambda-2a}I,$$

leading to the bounds with $\underline{p}=\varepsilon\exp(-\lambda\tau)$ and $\overline{p}=\frac{c^2}{\lambda-2a}$.

# V. Conclusion

According to [10] a first step in the design of an observer making the Riemannian distance between estimated state and system state decrease along solutions is to design a Riemannian metric the Lie derivative of which along the system vector field is negative in the space tangent to the level sets of the output function. We have shown here that the design of such a metric is possible when the system is strongly infinitesimally observable (i.e., each time-varying linear system resulting from the linearization along a solution to the system satisfies a uniform observability property). In such a case, it is sufficient to solve an "algebraic" (actually a partial differential equation) Riccati equation. This leads to an observer that resembles an Extended Kalman Filter.

With the same strong infinitesimal observability property, we can also proceed with a linear equation instead of the quadratic Riccati equation. In this case, the metric that we obtain is nothing but an exponentially weighted observability Grammian.

The two designs above need the solution of a partial differential equation. But thanks to the method of characteristics, it can be obtained off-line by solving ordinary differential equations on a sufficiently large time interval and over a grid of initial conditions in the system state space.

Unfortunately, as shown in [10], to obtain observers for which convergence holds globally or at least regionally and not only locally, the output function may need to satisfy the extra property of being geodesically monotonic. This topic is not addressed here.

# References


[1] W. Lohmiller and J.-J. Slotine. On contraction analysis for nonlinear systems. *Automatica*, 34(6):671–682, 1998.

[2] D. C. Lewis. Metric properties of differential equations. *American Journal of Mathematics*, 71:294–312, 1949.

[3] P. Hartmann. *Ordinary differential equations*. Wiley, 1964.

[4] B. P. Demidovich. Dissipativity of a nonlinear system of differential equations. *Ser. Mat; Mekh. Part I.6 (1961); Part II.1, 3-8(1962) (in Russian).*, pages 19–27.

[5] S. Z. Németh. *Geometric aspects of Minty-Browder monotonicity*. PhD thesis, E ötvös Loránd University. Budapest., 1998.

[6] J. Jouffroy. Some ancestors of contraction analysis. *Proc. IEEE Conf. Dec. Control*, pages 5450–5455, 2005.

[7] N. Aghannan and P. Rouchon. An intrinsic observer for a class of Lagrangian systems. *IEEE Trans. Automatic Control*, 48(6):936–945, 2003.

[8] S. Bonnabel. A simple intrinsic reduced-observer for geodesic flow. *IEEE Transactions on Automatic Control*, 55(9):2186–2191, 2010.

[9] R. G. Sanfelice and L. Praly. Nonlinear observer design with an appropriate Riemannian metric. In *Proc. 48th IEEE Conference on Decision and Control and 28th Chinese Control Conference*, pages 6514–6519, 2009.

[10] R. G. Sanfelice and L. Praly. Convergence of nonlinear observers on $\mathbb{R}^n$ with a Riemannian metric (Part I). *IEEE Transactions on Automatic Control*, 57(7):1709–1722, July 2012.

[11] J-P. Gauthier and I. Kupka. *Deterministic Observation Theory and Applications*. Cambridge University Press, 2001.

[12] A. H. Jazwinski. *Stochastic Processes and Filtering Theory*. Academic Press, 1970.

[13] P. Bougerol. Kalman filtering with random coefficients and contractions. *SIAM J. Control anfd Optimization*, 31(4):942–959, July 1993.

[14] S. Bonnabel and R. Sepulchre. The geometry of low-rank Kalman filters. In *Matrix Information Geometry*, pages 53–68. Springer, 2013.

[15] J. Radon. ber die oszillationstheoreme der kunjugierten punkte beim probleme von lagrange. *Mnchener Sitzungsberichte*, 57:243–257, 1927. (German).

[16] J. Radon. Zum problem von lagrange. *Hamburger Math. Einzelschr.*, 6(273–299), 1928. (German).

[17] W. T. Reid. *Riccati Differential Equation*. Adademic Press, New York, 1972.

[18] S. Bonnabel and J.-J. Slotine. A contraction theory-based approach to the stability of the deterministic Extended Kalman Filter. *IEEE Transactions on Automatic Control*, 60(2):565–569, February 2015.

[19] R. Bhatia. *Positive Definite Matrices*. Princeton University Press, 2007.