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Dynamic extension without inversion for observers

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***Abstract*— Dynamic extension is useful in the design of observers for non linear systems. The idea is to immerse the given system dynamic into a *bigger* one having a structure more appropriate for building an observer. The main drawback of such a procedure is the need for inverting on-line an injective immersion. In this work we suggest a solution to overcome this difficulty based on an extension of the *initial* state space. One of the two crucial steps of our approach is the problem of continuous completion of a full column-rank matrix into an invertible one.**

I. INTRODUCTION

A. Context and motivation

For linear systems it is well established that dynamic extension is useless to solve most problems of observation theory. This is not at all the case for nonlinear systems since such an extension allows to immerse the system dynamic into one which can possibly be written in a form more appropriate for observer design. For example, in [10], Lévine and Marino call for dynamic extension to enlarge the class of systems which can be expressed as linear systems up to addition of output nonlinear functions (See [11] for an extension of this result). It is used also, for instance in [6], [12], for uniformly observable systems when the order of differential observability is larger than the system state dimension to obtain an observer form to which the paradigm of high gain observers applies. In [5], Besançon and Ticlea take advantage of the degrees of freedom given by dynamic extension to obtain a form, called state affine up to triangular nonlinearity, to which Kalman-like observers can be applied. In [2], the authors show that, by going to dimension $2(n+1)$ when the system state dimension is n, it is possible to obtain an observer with linear dynamic. All this is just a short list of the many contributions demonstrating the usefulness of dynamic extension.

In all these works dynamic extension is via an injective immersion τ^* from an n-dimensional manifold $\mathcal{M}$, in which the given system state evolves, into an m-dimensional ($m > n$) manifold $\mathcal{M}_e$. As a consequence the state-observer is living in $\mathcal{M}_e$ and the estimated system state is obtained via a left inverse τ of τ^*. Usually it is impossible to get an expression for this left inverse. This makes the observer implementation quite intricate.

Here we study the possibility of having the manifold $\mathcal{M}_e$ as the Cartesian product of $\mathcal{M}$ with an (m-n)-dimensional manifold. The motivation is that, in this case, the left inverse can be simply the Cartesian projection.

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B. Problem statement

1) Starting point: For this introductory presentation of our research, we work within the simplest context. We consider the given system with dynamics :

$$\dot{x} = f(x) \quad , \qquad y = h(x) \ , \tag{1}$$

with x in $\mathbb{R}^n$ and y in $\mathbb{R}$. We assume :

<u>A.II=Injective immersion</u> *: For this system, we are given a C^1 function $\tau^* : \mathbb{R}^n \to \mathbb{R}^m$ whose restriction to the closure $cl(\mathcal{S})$ of an open subset $\mathcal{S}$ of $\mathbb{R}^n$ is an injective immersion.*

Actually this injective immersion should be such that the image $L_f\tau^*$ of the vector field f, is in a form such that an observer can be designed. We write the corresponding observer as :

$$\dot{\hat{\xi}} = \varphi(\hat{\xi}, \hat{x}, y) \quad , \qquad \hat{x} = \tau(\hat{\xi}) \ , \tag{2}$$

where the functions φ and τ are continuous, the latter being nothing but a left inverse of τ^*, i.e., we have :

$$\tau(\tau^*(x)) \ = \ x \qquad \forall x \in \mathcal{S} \ .$$

We assume this observer is designed in such a way that we have :

<u>A.C=Convergence</u> *: We know continuous functions φ and τ such that, any solution $(X(x,t), \hat{\Xi}((\hat{\xi}, x), t))$ of the cascade system :*

$$\dot{x} \ = \ f(x) \quad , \qquad \dot{\hat{\xi}} = \varphi(\hat{\xi}, \tau(\hat{\xi}), h(x))$$

with the initial condition $(x, \hat{\xi})$ in $\mathcal{S} \times \mathbb{R}^m$, which is defined on $[0, +\infty)$ with values in $\mathcal{S} \times \mathbb{R}^m$ satisfies :

$$\lim_{t \to +\infty} \left| \tau^*(X(x,t)) - \hat{\Xi}((\hat{\xi}, x), t) \right| \ = \ 0 \ .$$

2) Example: To illustrate our "starting point" and the results to come, we consider the celebrated harmonic oscillator with unknown frequency. Our interest is not in comparing with the other very many solutions but in having the possibility of doing elementary computations. This system is :

$$\dot{x}_1 \ = \ x_2 \ , \quad \dot{x}_2 \ = \ -x_1 x_3 \ , \quad \dot{x}_3 \ = \ 0 \ , \quad y \ = \ x_1 \ .$$

There is no indistinguishability problems if we restrict our attention to the invariant set :

$$\mathcal{S} \ = \ \left\{ x \in \mathbb{R}^3 \ : \ x_1^2 + x_2^2 \in (\frac{1}{r}, r) \ , \ x_3 \in (\frac{1}{r}, r) \right\} \ ,$$

where r is some arbitrary strictly positive real number. We define the following[1] function with values in $\mathbb{R}^4$:

$$\tau^*(x) \;=\; (x_1\,,\, x_2\,,\, -x_1x_3\,,\, -x_2x_3)$$

Its restriction to $\mathcal{S}$ is injective. This is trivial for x_1 and x_2 and follows for x_3 from that the equation :

$$x_1\,[x_{3a}-x_{3b}] \;=\; x_2[x_{3a}-x_{3b}] \;=\; 0$$

implies :

$$[x_1^2+x_2^2][x_{3a}-x_{3b}] \;=\; 0$$

and further that $x_{3a}=x_{3b}$ if $x_1^2+x_2^2$ is not zero. Also its Jacobian is :

$$\frac{\partial\tau^*}{\partial x}(x) \;=\; \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ -x_3 & 0 & -x_1\\ 0 & -x_3 & -x_2 \end{pmatrix} .$$

It has rank 3 on $\mathcal{S}$. Hence τ^* is an injective immersion on $\mathcal{S}$. Its interest is in the fact that it satisfies

$$\dot{\overbrace{\tau^*(x)}} = \begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{pmatrix}\tau^*(x) + \begin{pmatrix}0\\0\\0\\x_1x_3^2\end{pmatrix} \qquad \forall x\in\mathcal{S} .$$

It is in the very basic form to which high gain observers apply. This leads to the observer :

$$\dot{\hat{\xi}} = \begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{pmatrix}\hat{\xi} + \begin{pmatrix}0\\0\\0\\\mathtt{sat}(\hat{x}_1\hat{x}_3^2)\end{pmatrix} + \begin{pmatrix}\ell k_1\\\ell^2k_2\\\ell^3k_3\\\ell^4k_4\end{pmatrix}[y-\hat{\xi}_1] ,$$

$$\hat{x} = \operatorname*{Argmin}_{x\in\mathcal{S}}\left|\hat{\xi} - \begin{pmatrix}x_1\\x_2\\-x_1x_3\\-x_2x_3\end{pmatrix}\right|^2 , \qquad (3)$$

where $\mathtt{sat}$ is a saturation function whose values are in $[-r^3, r^3]$, ℓ is to be chosen large enough and the k_i so that the following polynomial is Hurwitz :

$$\lambda^4+k_1\lambda^3+k_2\lambda^2+k_3\lambda+k_4 .$$

3) Problem statement: The minimization involved in (3) and, in the general case, finding an expression of the left inverse of τ^* may be very difficult. This motivates us for looking for another implementation of the observer (2). For this we augment the given x coordinates with extra ones w in $\mathbb{R}^{m-n}$. Then our problem is to find an "extension" of the injective immersion $\tau^*:\mathcal{S}\to\mathbb{R}^m$ into a diffeomorphism $\tau_e^*:\mathcal{S}\times\mathbb{R}^{m-n}\to\mathbb{R}^m$ allowing us to pull back the observer dynamics, as given above in the $\hat{\xi}$ coordinates, into one in the (x,w) coordinates.

[1]Another such function is

$$\tau^*(x) \;=\; \left(\frac{\lambda_1x_1+x_2}{\lambda_1^2+x_3}\,,\,\frac{\lambda_2x_1+x_2}{\lambda_2^2+x_3}\,,\,\frac{\lambda_3x_1+x_2}{\lambda_3^2+x_3}\,,\,\frac{\lambda_4x_1+x_2}{\lambda_4^2+x_3}\right)^T$$

where the λ_i are distinct complex numbers with strictly negative real part.

II. Sufficient properties and problem reduction

A. Sufficient properties on the extension τ_e^*

Proposition 1: Assume A.C hold. Let $\mathcal{S}_w$ be an open subset of $\mathbb{R}^{m-n}$ and $\tau_e^*:\mathcal{S}\times\mathcal{S}_w\to\mathbb{R}^m$ be a C^1 function such that the following properties hold :

P1 τ_e^* is full rank on $\mathcal{S}\times\mathcal{S}_w$;

P2 There exists a class $\mathcal{K}$ function $\alpha:\mathbb{R}_{\geq0}\to\mathbb{R}_{\geq0}$ such that, for all x_a in $\mathcal{S}$, x_b in $\mathcal{S}$ and w in $\mathcal{S}_w$ we have

$$|w| \;+\; |x_a-x_b| \;\leq\; \alpha\left(|\tau_e^*(x_a,w)-\tau^*(x_b)|\right) .$$

Then the observer given by

$$\dot{\overbrace{\begin{bmatrix}\hat{x}\\\hat{w}\end{bmatrix}}} = \left(\frac{\partial\tau_e^*}{\partial(\hat{x},\hat{w})}(\hat{x},\hat{w})\right)^{-1}\varphi(\tau_e^*(\hat{x},\hat{w}),\hat{x},y) \qquad (4)$$

is such that, for any initial condition $(x,\hat{x},\hat{w})$ such that the corresponding solutions $(X(x,t),\ \hat{X}(x,\hat{x},\hat{w},t),\ \hat{W}(x,\hat{x},\hat{w},t))$ are defined on $[0,+\infty)$ and with values in $\mathcal{S}\times\mathcal{S}\times\mathcal{S}_w$, we have :

$$\lim_{t\to+\infty}\left|\hat{W}(x,\hat{x},\hat{w},t)\right| + \left|X(x,t)-\hat{X}(x,\hat{x},\hat{w},t)\right| = 0 . \quad (5)$$

The key point in the observer (4) is that, instead of left-inverting the function τ^* via τ as in (2), we invert only a matrix. On the other hand, even if we know that $X(x,t)$ remains in $\mathcal{S}$, there is no guarantee that $\hat{X}(x,\hat{x},\hat{w},t),\hat{W}(x,\hat{x},\hat{w},t)$ remains in $\mathcal{S}\times\mathcal{S}_w$ where this matrix is invertible. To round this difficulty we are currently exploring two routes. One consists in modifying/extending the expression of τ_e^* in $\mathbb{R}^m\setminus(\mathcal{S}\times\mathcal{S}_w)$ (see [3]). The other consists in introducing a mechanism preventing the observer state to leave $\mathcal{S}\times\mathcal{S}_w$. Very preliminary results for the latter can be found in [4] for instance.

Proof: First, because of P1, the observer given by (4) is well defined on $\mathcal{S}\times\mathcal{S}\times\mathcal{S}_w$. Second, along the solutions in $\mathcal{S}\times\mathcal{S}\times\mathcal{S}_w$, we have :

$$\begin{aligned}\dot{\overbrace{\tau_e^*(\hat{x},\hat{w})}} &\\ &= \left(\frac{\partial\tau_e^*}{\partial(\hat{x},\hat{w})}(\hat{x},\hat{w})\right)\left(\frac{\partial\tau_e^*}{\partial(\hat{x},\hat{w})}(\hat{x},\hat{w})\right)^{-1}\varphi(\tau_e^*(\hat{x},\hat{w}),\hat{x},y)\\ &= \varphi(\tau_e^*(\hat{x},\hat{w}),\hat{x},y) .\end{aligned}$$

Hence, for any solution $t\mapsto(\hat{X}(x,\hat{x},\hat{w},t),\hat{W}(x,\hat{x},\hat{w},t))$ of (4), $t\mapsto\tau_e^*\left(\hat{X}(\hat{x},\hat{w},t),\hat{W}(\hat{x},\hat{w},t)\right)$ is a solution of (2), for all time interval in which the former is in $\mathcal{S}\times\mathcal{S}_w$.

Finally, consider a solution

$$(X(x,t),\hat{X}(x,\hat{x},\hat{w},t),\hat{W}(x,\hat{x},\hat{w},t))$$

defined on $[0,+\infty)$ and with values in $\mathcal{S}\times\mathcal{S}\times\mathcal{S}_w$. With the assumption A.C, it satisfies :

$$\lim_{t\to+\infty}\left|\tau^*(X(x,t))-\tau_e^*\left(\hat{X}(\hat{x},\hat{w},t),\hat{W}(\hat{x},\hat{w},t)\right)\right| \;=\; 0 .$$

With P2 this gives (5). ■

For our illustration, consider the function $\tau_e^*:\mathbb{R}^4\to\mathbb{R}^4$:

$$\tau_e^*(x,w) \;=\; (x_1\,,\,x_2\,,\,-x_1x_3+x_2w\,,\,-x_2x_3-x_1w)$$

It is C^1 and its Jacobian is :

$$\frac{\partial \tau_e^*}{\partial x}(x,w) \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -x_3 & w & -x_1 & x_2 \\ -w & -x_3 & -x_2 & -x_1 \end{pmatrix} .$$

This establishes that this function is full rank on $\mathcal{S}\times\mathbb{R}$. So P1 holds. Also the inverse function is :

$$\tau_e(\xi) \;=\; \left(\xi_1\,,\,\xi_2\,,\,-\frac{\xi_1\xi_3+\xi_2\xi_4}{\xi_1^2+\xi_2^2}\,,\,\frac{\xi_2\xi_3-\xi_1\xi_4}{\xi_1^2+\xi_2^2}\right)$$

It is C^1 on $\mathbb{R}^4\setminus\left(\{(0,0)\}\times\mathbb{R}^2\right)$ and therefore Lipschitz on the compact image by τ^* of :

$$\left\{(x,w)\in\mathbb{R}^4 :\, x_1^2+x_2^2\in[\frac{1}{r},r]\,,\,x_3\in[\frac{1}{r},r]\,,\,w\in[-r,r]\right\}.$$

Since we have also :

$$\tau_e^*(x,0) \;=\; \tau^*(x)\;,$$

P2 holds. From the above Proposition we get the observer

$$\begin{pmatrix}\dot{\hat{x}}_1\\ \dot{\hat{x}}_2\\ \dot{\hat{x}}_3\\ \dot{\hat{w}}\end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\hat{x}_3 & \hat{w} & -\hat{x}_1 & \hat{x}_2 \\ -\hat{w} & -\hat{x}_3 & -\hat{x}_2 & -\hat{x}_1 \end{pmatrix}^{-1}\times$$
$$\times\left[\begin{pmatrix}\hat{x}_2\\ -\hat{x}_1\hat{x}_3+\hat{x}_2\hat{w}\\ -\hat{x}_2\hat{x}_3-\hat{x}_1\hat{w}\\ \mathtt{sat}(\hat{x}_1\hat{x}_3^2)\end{pmatrix}+\begin{pmatrix}\ell k_1\\ \ell^2 k_2\\ \ell^3 k_3\\ \ell^4 k_4\end{pmatrix}[y-\hat{x}_1]\right] .$$

With Proposition 1 we are left with building the function τ_e^* from the function τ^*. Fortunately, when $\mathcal{S}$ is bounded, we can simplify further the problem.

Proposition 2: Assume A.II hold and $\mathcal{S}$ is bounded. If there exists a continuous function $\psi : \mathtt{cl}(\mathcal{S})\to\mathbb{R}^{m*(m-n)}$ whose values are $m*(m-n)$ matrices satisfying :

$$\mathtt{det}\left(\frac{\partial\tau^*}{\partial x}(x)\quad\psi(x)\right)\neq 0\qquad\forall x\in\mathtt{cl}(\mathcal{S})$$

then there exists a strictly positive real number $\varepsilon_\diamond$ such that, with $\mathcal{S}_w$ defined[2] as :

$$\mathcal{S}_w = \mathcal{B}_1(0)\;,$$

for any ε in $(0,\varepsilon_\diamond]$, a function τ_e^* satisfying conditions P1 and P2 of Proposition 1 is :

$$\tau_e^*(x,w) \;=\; \tau^*(x) \;+\; \varepsilon\,\psi(x)\,w\;. \tag{6}$$

With this proposition, our problem reduces to the one of completing the full column-rank matrix $\frac{\partial\tau^*}{\partial x}(x)$ into an invertible matrix via $\psi(x)$ where ψ is a continuous function.

Proof:

To check if property P1 holds, we compute the Jacobian of τ_e^* in (6) :

$$\frac{\partial\tau_e^*}{\partial(x,w)} \;=\; \left(\frac{\partial\tau^*}{\partial x}(x)+\varepsilon\frac{\partial}{\partial x}[\psi(x)w]\quad \varepsilon\psi(x)\right)$$

[2]For ε in $\mathbb{R}_{>0}$ and z_0 in $\mathbb{R}^p$, $\mathcal{B}_\varepsilon(z_0)$ denotes the open ball centered at z_0 and with radius ε.

Its determinant is in the form

$$\mathtt{det}\left(\frac{\partial\tau_e^*}{\partial(x,w)}\right) \;=\; \varepsilon^{m-n}\times$$
$$\times\left[\mathtt{det}\!\left(\tfrac{\partial\tau^*}{\partial x}(x)\quad\psi(x)\right)+\varepsilon\,P(\varepsilon,x,w)\right]$$

where $P(\varepsilon,x,w)$ is a polynomial of degree $n-1$ in ε whose coefficients are continuous functions of x and w. Since $\mathtt{cl}(\mathcal{S})$ is compact as a bounded and closed set, there exists a strictly positive real number $\overline{p}$ satisfying :

$$|P(\varepsilon,x,w)| \;\leq\; \overline{p}\qquad\forall(\varepsilon,x,w)\in[0,1]\times\mathtt{cl}(\mathcal{S})\times\mathcal{B}_1(0)\;.$$

Also, by assumption, there exists a strictly positive real number $\underline{d}$ such that we have :

$$\left|\mathtt{det}\!\left(\tfrac{\partial\tau^*}{\partial x}(x)\;\;\psi(x)\right)\right| \;\geq\; \underline{d}\;,\;\forall x\in\mathtt{cl}(\mathcal{S})\;.$$

This yields :

$$\left|\mathtt{det}\left(\frac{\partial\tau_e^*}{\partial(x,w)}\right)\right| \;\geq\; \varepsilon^{m-n}\left[\underline{d}-\varepsilon\overline{p}\right]\;.$$

Hence property P1 holds when :

$$\mathcal{S}_w \;=\; \mathcal{B}_1(0)\quad,\qquad\varepsilon_\diamond\;\leq\;\varepsilon_1\;=\;\frac{\underline{d}}{\overline{p}}\;.$$

It remains to check that P2 holds. The function τ_e^* being full rank in $\mathtt{cl}(\mathcal{S})\times\{0\}$, the Implicit function Theorem implies that, for all x in $\mathcal{S}$ there exist $\delta(x)>0$ such that, with $\varepsilon\leq\delta(x)$, the function τ_e^* is injective in the subset $\mathcal{B}_{\delta(x)}(x)\times\mathcal{B}_1(0)$ of $\mathbb{R}^m$. Then, since $\{\mathcal{B}_{\frac{\delta(x)}{3}}(x), x\in\mathtt{cl}(\mathcal{S})\}$ is a covering by open subsets of the compact set $\mathtt{cl}(\mathcal{S})$, there exists a finite set of points $\{(x_\ell)\}$ in $\mathtt{cl}(\mathcal{S})$ satisfying :

$$\mathtt{cl}(\mathcal{S})\subset\bigcup_\ell\mathcal{B}_{\frac{\delta(x_\ell)}{3}}(x_\ell)\;.$$

Let $\delta_{\min}$ be the positive real number defined as

$$\delta_{\min}=\min_\ell\delta(x_\ell)\;.$$

The function τ^* being injective and the set $\mathtt{cl}(\mathcal{S})$ being compact, according to [1, Proposition 3] or [9, Claim 3] (see Appendix A for a self-contained proof), there exists a positive real number L such that we have :

$$|x_a-x_b|\leq L\,|\tau^*(x_a)-\tau^*(x_b)|\;. \tag{7}$$

Let us show that P2 holds when we select $\varepsilon_\diamond$ as :

$$\varepsilon_\diamond=\min\left\{\varepsilon_1,\frac{\delta_{\min}}{3L\max_{x\in\mathtt{cl}(\mathcal{S})}|\psi(x)|}\right\}\;.$$

For this we consider (x_a,x_b) in $\mathtt{cl}(\mathcal{S})^2$ and w in $\mathtt{cl}(\mathcal{B}_1(0))$ satisfying :

$$\tau_e^*(x_a,w) \;=\; \tau_e^*(x_b,0) \;=\; \tau^*(x_b)\;. \tag{8}$$

With the definition of τ_e^*, this implies, for any ε in $(0,\varepsilon_\diamond]$,

$$|\tau^*(x_a)-\tau^*(x_b)| \;=\; \varepsilon\,|\psi(x_a)w|$$
$$\leq\;\varepsilon_\diamond\max_{x\in\mathtt{cl}(\mathcal{S})}|\psi(x)|\;\leq\;\frac{\delta_{\min}}{3L}\;.$$

With (7), this yields :

$$|x_a - x_b| \le \frac{\delta_{\min}}{3} \ .$$

But, there exists ℓ such that x_a is in $\mathcal{B}_{\frac{\delta(x_\ell)}{3}}(x_\ell)$. So the last inequality implies that x_b is also in $\mathcal{B}_{\delta(x_\ell)}(x_\ell)$. Hence (x_a, w) and $(x_b, 0)$ are in $\mathcal{B}_{\delta(x_\ell)}(x_\ell) \times \mathcal{B}_1(0)$ on which τ_e^* is injective. It follows that (8) implies :

$$x_a = x_b \quad , \qquad w = 0 \ .$$

With Lemma 1 in Appendix A, we can conclude that there exists a real number L_e such that, for all ε in $(0, \varepsilon_\diamond]$, (x_a, x_b) in $\mathtt{cl}(\mathcal{S})^2$ and w in $\mathcal{B}_1(0)$, we have :

$$|w| \ + \ |x_a - x_b| \ \le \ L_e \, |\tau_e^*(x_a, w) - \tau^*(x_b)| \ .$$

This is P2. ■

III. Continuous completion of a full column-rank matrix into an invertible matrix

As far as we know the first significant contribution to the problem of continuous completion of a full column-rank matrix into an invertible matrix has been done by Wazewski in [13]. His work has been completed later by Eckmann in [7, p 121-134]. (See also [8]). Here we summarize Eckmann results.

A. Universal formula

We start by looking for a universal formula which would not depend on the specific data. In this case the problem is as follows.

Let $\mathcal{G}$ be the subset of $\mathbb{R}^{mn}$ of "good" $m{*}n$ real variables φ_{ij} in the sense that the matrix

$$\varphi = \begin{pmatrix} \varphi_{11} & \cdots & \varphi_{1n} \\ \vdots & \vdots & \vdots \\ \varphi_{m1} & \cdots & \varphi_{mn} \end{pmatrix}$$

has rank n. We look for $m * (m-n)$ continuous functions $\psi_{kl} : \mathcal{G} \to \mathbb{R}$ such that the following matrix is invertible for all $(\varphi_{11}, \ldots, \varphi_{mn})$ in $\mathcal{G}$:

$$M(\varphi_{11}, \ldots, \varphi_{mn}) \ = \ \begin{pmatrix} \varphi & \psi(\varphi_{11}, \ldots, \varphi_{mn}) \end{pmatrix}$$

where :

$$\psi \ = \ \begin{pmatrix} \psi_{11} & \ldots & \psi_{1(m-n)} \\ \vdots & & \vdots \\ \psi_{m1} & \ldots & \psi_{m(m-n)} \end{pmatrix} \ .$$

Wazewski has noticed in [13] that this problem is equivalent to look for a full column rank $m{*}(m-n)$ matrix ψ solution of :

$$\varphi^T \psi \ = \ 0 \ .$$

Indeed when this equations holds, the result follows from the identity :

$$\begin{pmatrix} \varphi^T \\ \psi^T \end{pmatrix} \begin{pmatrix} \varphi & \psi \end{pmatrix} \ = \ \begin{pmatrix} \varphi^T \varphi & 0 \\ 0 & \psi^T \psi \end{pmatrix} \ .$$

Conversely, if $\begin{pmatrix} \varphi & \psi \end{pmatrix}$ is non singular, the same holds for $\begin{pmatrix} \varphi & [I - \varphi(\varphi^T\varphi)^{-1}\varphi^T]\psi \end{pmatrix}$ and we have the identity :

$$\varphi^T \left([I - \varphi(\varphi^T\varphi)^{-1}\varphi^T]\psi \right) \ = \ 0 \ .$$

Unfortunately this problem has a solution only for the pairs (n, m) shown in the following table :

$m =$	even	≥ 2	7	8
$n =$	1	$m-1$	2	3

- for the case $n = 1$ and m even, a solution is obtained by signed permutations. For instance when :

$$\varphi \ = \ \begin{pmatrix} \varphi_{11} \\ \varphi_{21} \\ \varphi_{31} \\ \varphi_{41} \end{pmatrix}$$

  a universal formula is :

$$\psi \ = \ \begin{pmatrix} -\varphi_{21} & -\varphi_{31} & -\varphi_{41} \\ \varphi_{11} & -\varphi_{41} & -\varphi_{31} \\ -\varphi_{41} & \varphi_{11} & \varphi_{21} \\ \varphi_{31} & \varphi_{21} & \varphi_{11} \end{pmatrix} \ .$$

  For our illustration, we observe that, because of the triangular structure, it is sufficient to complete the vector

$$(\varphi_{33} \, , \, \varphi_{43}) \ = \ (-x_1 \, , \, -x_2) \ .$$

  So, the above universal formula gives :

$$(\psi_{31} \, , \, \psi_{41}) \ = \ (x_2 \, , \, -x_1)$$

  which we extend with :

$$\psi_{11} \ = \ \psi_{21} \ = \ 0 \ .$$

- for the case $n = m - 1$, a solution is obtained by picking ψ_{i1} as $(-1)^{m+i}$ times the corresponding $(n{*}n)$ minor of the matrix φ. For our illustration, this universal formula gives :

$$(\psi_{11} \, , \, \psi_{21} \, , \, \psi_{31} \, , \, \psi_{41}) = (x_2 x_3 \, , \, -x_1 x_3 \, , \, -x_2 \, , \, -x_1) \ .$$

B. Towards an "ad hoc" formula

To round the problem of non existence of a universal formula in the general case, we can take advantage of the fact that, in our context, the $m * n$ variables φ_{ij} are not independent but continuous functions of the n components of x. So, our actual problem is :
Given $m * n$ functions φ_{ij} which are continuous in the n-dimensional variable x in $\mathtt{cl}(\mathcal{S})$, look for $m * (m-n)$ functions ψ_{kl} which are continuous on $\mathtt{cl}(\mathcal{S})$ and such that the following matrix is invertible for all x in $\mathtt{cl}(\mathcal{S})$:

$$P(x) \ = \ \begin{pmatrix} \varphi(x) & \psi(x) \end{pmatrix} \ . \qquad (9)$$

We have :

Theorem 1 ([13, Théorèmes 1 et 3] and [7, page 127]): On any subset of $\mathtt{cl}(\mathcal{S})$ which, equipped with the subspace topology of $\mathbb{R}^n$, is a contractible space, there exist continuous functions ψ_{kl} making invertible the matrix P in (9).

Unfortunately this result is mainly an existence result. We have not been able yet to extract a "formula" from its proof. Nevertheless it says when it is or it is not worth looking for a solution.

C. Another example

We consider the following non linear oscillator :

$$\dot{x}_1 = x_2 \ , \quad \dot{x}_2 = -x_1^3 x_3 \ , \quad \dot{x}_3 \ = \ 0 \ , \quad y \ = \ x_1$$

For any x_3, $x_1 = x_2 = 0$ is an equilibrium point and, at this point, x_3 cannot be estimated. This leads us to pick the set $\mathcal{S}$ as :

$$\mathcal{S} \ = \ \mathbb{R}^3 \setminus (\{(0,0) \times \mathbb{R}) \ .$$

It is forward invariant since x_3 and :

$$\mathcal{I}(x_1, x_2, x_3) \ = \ \frac{1}{4} x_1^4 x_3 + \frac{1}{2} x_2^2$$

are constant along the solutions.

For the function τ^*, we propose the following, coming from the high-gain observer methodology,

$$\tau^*(x) \ = \ \begin{pmatrix} x_1 \\ x_2 \\ -x_1^3 x_3 \\ -3x_1^2 x_2 x_3 \\ -6x_1 x_2^2 x_3 + 3x_1^5 x_3^2 \\ -16x_2^3 x_3 + 27x_1^4 x_2 x_3^2 \end{pmatrix}$$

It is injective on $\mathcal{S}$. Indeed the first and second components of $\tau^*(x)$ give x_1 and x_2 uniquely. Then, when x_1 is not zero, we get x_3 uniquely from the third component of $\tau^*(x)$. Finally, in $\mathcal{S}$, when x_1 is zero, x_2 cannot be zero. In this case, we get x_3 uniquely from the sixth component of $\tau^*(x)$. With the same reasoning, we can show that the following Jacobian of τ^* has rank 3 on $\mathcal{S}$:

$$\frac{\partial \tau^*}{\partial x}(x) \ = $$

$$\left(\begin{array}{c|c|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3x_1^2 x_3 & 0 & -x_1^3 \\ -6x_1 x_2 x_3 & -3x_1^2 x_3 & -3x_1^2 x_2 \\ -6x_2^2 x_3 + 15x_1^4 x_3^2 & -12x_1 x_2 x_3 & -6x_1 x_2^2 + 10x_1^5 x_3 \\ 108x_1^3 x_2 x_3^2 & -48x_2 x_3 + 27x_1^4 x_3^2 & -16x_2^3 + 54x_1^4 x_2 x_3 \end{array}\right)$$

As for our illustration, we have a triangular structure which allows us to complete only the vector $\left(-x_1^3 \, , \, -3x_1^2 x_2 \, , \, -6x_1 x_2^2 + 10x_1^5 x_3 \, , \, -16x_2^3 + 54x_1^4 x_2 x_3\right)$. So we can proceed as if we were in the case $n = 1$, $m = 4$ for which we have a universal formula. In this way, we obtain the following complement ψ of $\frac{\partial \tau^*}{\partial x}$:

$$\psi(x) \ = $$

$$\left(\begin{array}{c|c|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3x_1^2 x_2 & 6x_1 x_2^2 - 10x_1^5 x_3 & 16x_2^3 - 54x_1^4 x_2 x_3 \\ -\,x_1^3 & 16x_2^3 - 54x_1^4 x_2 x_3 & 6x_1 x_2^2 - 10x_1^5 x_3 \\ 16x_2^3 - 54x_1^4 x_2 x_3 & -x_1^3 & -3x_1^2 x_2 \\ -\,6x_1 x_2^2 + 10x_1^5 x_3 & -3x_1^2 x_2 & -x_1^3 \end{array}\right)$$

Finally, again because of the triangular structure, the extension τ_e^* given by (no ε)

$$\tau_e^*(x, w) \ = \ \tau^*(x) \ + \ \psi(x) \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

is a diffeomorphism on $\mathcal{S} \times \mathbb{R}^3$.

D. A recipe

As illustrated by our examples, to exhibit the extension ψ of φ, it is useful to take advantage of the possibility that the matrix φ has a fixed $p * p$ - minor which is non zero for all x in $\mathtt{cl}(\mathcal{S})$. Indeed, in this case, by permutations of rows and columns, we can assume without loss of generality that the $p * p$ submatrix φ_{ul}, in the upper left corner of φ, is non singular for all x in $\mathtt{cl}(\mathcal{S})$. This leads us to decompose φ as :

$$\varphi \ = \ \begin{pmatrix} \varphi_{ul} & \varphi_{ur} \\ \varphi_{bl} & \varphi_{br} \end{pmatrix} \ .$$

Then $\varphi(x)$ having rank n for all x in $\mathtt{cl}(\mathcal{S})$, the $(m-p) * (n-p)$ matrix $\varphi_{br} - \varphi_{bl} \varphi_{ul}^{-1} \varphi_{ur}$ has rank $n - p$. If it has an $(m-p) * (m-n)$ complement ψ_b giving a non singular $(m-p) * (m-p)$ matrix, then a complement for our original matrix φ is simply :

$$\psi \ = \ \begin{pmatrix} 0 \\ \psi_b \end{pmatrix} \ .$$

IV. Conclusions

In this note we have given a possible solution to overcome the difficult immersion inversion typically involved in observers based on dynamic extension. Our proposition is to extend directly the given state space in order to render the injective immersion part of a diffeomorphism. To follow this approach, we have proposed a two step procedure. The first one consist in the continuous completion of a full column-rank matrix into an invertible one and the second one is the modification of the observer algorithm in order to keep the observer trajectory in a specific set. We have given results concerning the first steps whereas we are still working on the second one.

References

[1] V. Andrieu, Exponential convergence of nonlinear Luenberger observers, IEEE CDC, Atlanta (2010).

[2] V. Andrieu, L. Praly, On the existence of Kazantzis-Kravaris / Luenberger Observers. SIAM J. Control and Optimization, Vol. 45, Issue 2, Pages 432-456, 2006.

[3] V. Andrieu, L. Praly On diffeomorphism extension for observer design. Electronic archive "HAL". `http://hal.archives-ouvertes.fr/hal-00957648`

[4] D. Astolfi, L. Praly, Output feedback stabilization with an observer in the original coordinates for nonlinear systems Proceedings of the 52nd IEEE Conference on Decision and Control, December 2013

[5] G. Besançon and A. Ticlea, An immersion-based observer design for rank-observable nonlinear systems IEEE Transactions on Automatic Control, vol. 52, no. 1, january 2007
J-

[6] P. Gauthier and I. Kupka. Deterministic observation theory and applications. Cambridge University Press, 2001.

[7] B Eckmann Mathematical survey lectures 1943-2004. Springer 2006.

[8] C. Ehresmann, Sur les applications continues d'un espace dans un espace fibré ou dans un revêtement. Bulletin de la S.M.F., tome 72 (1944), p. 27-54
[9] A. Isidori, L. Praly and L. Marconi About the existence of locally Lipschitz output feedback stabilizers for nonlinear systems. SIAM J. Control Optim., Vol.48, no 5, pp. 3389–3402, 2010.
[10] J. Lévine and R. Marino, Nonlinear system immersion, observers, and finite-dimensional filters, Syst. Control Lett., vol. 7, pp. 133–142, 1986
[11] D. Noh, N. Jo and J. Seo, Nonlinear observer design by dynamic observer error linearization, IEEE Transactions on Automatic Control, vol. 49, no. 10, october 2004
[12] A. Rapaport and A. Maloum, Design of exponential observers for nonlinear systems by embedding. Int. j. Robust Nonlinear Control, 2004, 14, 273-288.
[13] T. Wazewski, Sur les matrices dont les éléments sont des fonctions continues. Composito Mathematica, tome 2 (1935), p. 63-68

APPENDIX

A. Technical lemma

Lemma 1: Let be given a C^1 function $\varphi : \mathbb{R}^n \times \mathbb{R}^{m-n} \to \mathbb{R}^m$, which is full rank on some compact subset $C \times \mathtt{cl}(\mathcal{B}_1(0))$ of $\mathbb{R}^m$ and satisfies, when x_a and x_b are in C and w is in $\mathtt{cl}(\mathcal{B}_1(0))$,

$$\varphi(x_a, w) \ = \ \varphi(x_b, 0) \quad \Rightarrow \quad \{x_a = x_b \ , \ w = 0\} \ . \qquad (10)$$

There exists a real number L such that we have :

$$|w| \ + \ |x_a - x_a| \ \leq \ L|\varphi(x_a, w) - \varphi(x_b, 0)| \qquad (11)$$
$$\forall (x_a, x_b, w) \in C^2 \times \mathtt{cl}(\mathcal{B}_1(0)) \ .$$

The proof of this Lemma is inspired from [1, Proposition 3] and [9, Claim 3].

Proof:
For the sake of establishing the claim by contradiction, we assume there is no real number L satisfying (11). This implies that, for any integer n, we can find x_{an} and x_{bn} in C^2 and w_n in $\mathtt{cl}(\mathcal{B}_1(0))$ satisfying :

$$n \, |\varphi(x_{an}, w_n) - \varphi(x_{bn}, 0)| \ \leq \ |w_n| \ + \ |x_{an} - x_{bn}| \ . \ (12)$$

In this way we obtain three sequences in compact sets. So there is an infinite subset of the set of integers such that the corresponding sub-sequences converge to $x_{a\diamond}$, $x_{b\diamond}$ and $w_\diamond$. Since φ is continuous and satisfies (10), the boundedness of the right hand side in (12) leads to :

$$x_{a\diamond} \ = \ x_{b\diamond} \ = \ x_\diamond \quad , \qquad w_\diamond \ = \ 0 \ .$$

Since φ is C^1 and $C \times \mathtt{cl}(\mathcal{B}_1(0))$ is compact, there exists a strictly positive real number $\overline{J}$ that we have :

$$\left| \frac{\partial \varphi}{\partial x, w}(x, w)^{-1} \right| \ \leq \ \overline{J} \qquad \forall (x, w) \in C \times \mathtt{cl}(\mathcal{B}_1(0)) \ .$$

Then we introduce the function $\Delta : C^2 \times \mathtt{cl}(\mathcal{B}_1(0)) \to \mathbb{R}^m$ as :

$$\Delta(x_a, x_b, w) \\ = \ \varphi(x_a, w) - \varphi(x_b, 0) - \frac{\partial \varphi}{\partial x, w}(x_\diamond, 0) \begin{pmatrix} x_a - x_b \\ w \end{pmatrix} \ .$$

The function φ being C^1, we can find a strictly positive real number δ such that we have :

$$\frac{|\Delta(x_a, x_b, w)|}{|x_a - x_b| + |w|} \leq \frac{1}{2\overline{J}}$$
$$\forall (x_a, x_b, w) : \ |x_a - x_\diamond| + |x_b - x_\diamond| + |w| \ \leq \ \delta \ .$$

Then, from :

$$\begin{pmatrix} x_a - x_b \\ w \end{pmatrix} \\ = \ \frac{\partial \varphi}{\partial x, w}(x_\diamond, 0)^{-1} \left[\varphi(x_a, w) - \varphi(x_b, 0) - \Delta(x_a, x_b, w) \right] \ ,$$

we obtain :

$$|x_a - x_b| \ + \ |w| \\ \leq \ \overline{J} \, |\varphi(x_a, w) - \varphi(x_b, 0)| \ + \ \frac{1}{2} \left[|x_a - x_b| + |w| \right] \\ \forall (x_a, x_b, w) : \ |x_a - x_\diamond| + |x_b - x_\diamond| + |w| \ \leq \ \delta$$

and therefore :

$$|x_a - x_b| + |w| \ \leq \ 2\overline{J} \, |\varphi(x_a, w) - \varphi(x_b, 0)|$$
$$\forall (x_a, x_b, w) : \ |x_a - x_\diamond| + |x_b - x_\diamond| + |w| \ \leq \ \delta \ .$$

Finally, since there exists an integer n satisfying :

$$|x_{an} - x_\diamond| \ + \ |x_{bn} - x_\diamond| \ + \ |w_n| \ \leq \ \delta \ ,$$

we have :

$$|x_{an} - x_{bn}| + |w_n| \ \leq \ 2\overline{J} \, |\varphi(x_{an}, w_n) - \varphi(x_{bn}, 0)| \ .$$

This contradicts (12). ∎