A Note on Observability Canonical Forms for Nonlinear Systems

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Abstract: For nonlinear systems affine in the input with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}$ and output $y \in \mathbb{R}$, it is a well-known fact that, if the function mapping $(x, u, \ldots, u^{(n-1)})$ into $(u, \ldots, u^{(n-1)}, y, \ldots, y^{(n-1)})$ is an injective immersion, then the system can be locally transformed into an observability normal form with a triangular structure appropriate for a high-gain observer. In this technical note we extend this result to the case of systems not necessarily affine in the input and such that the injectivity condition holds for the function mapping $(x, u, \ldots, u^{(p-1)})$ into $(u, \ldots, u^{(p-1)}, y, \ldots, y^{(p-1)})$ with $p \geq n$. The forced uncertain harmonic oscillator is taken as elementary example to illustrate the theory.

1. INTRODUCTION

The paper deals with a nonlinear single-input single-output nonlinear system of the form

$$\dot{x} \;=\; f(u,x) \quad , \qquad y \;=\; h(u,x) \tag{1}$$

in which x is the state living in an open bounded subset X of $\mathbb{R}^n$, u is the input taking values in an open bounded subset U of $\mathbb{R}$ and $y \in \mathbb{R}$ is the measured output of the system. With ${}^cX$ and ${}^cU$, denoting the closure of X and U respectively, the functions f and h are assumed to be defined on an open set $\mathcal{O}$ containing ${}^cU \times {}^cX$ and on which they are sufficiently many times differentiable.

Our problem is to give conditions under which solutions of this system are related to those of a system in an observability form (see (3) and (4) below).

To state the most known answers about this problem, we define recursively functions $\varphi_i : \mathbb{R}^n \times \mathbb{R}^{i+1} \to \mathbb{R}$ as

$$\begin{aligned} \varphi_0(x, v_0) &= \quad h(x, v_0) \\ \varphi_i(x, v_0, \ldots, v_i) &= \frac{\partial \varphi_{i-1}}{\partial x} f(x, v_0) + \sum_{k=0}^{i-2} \frac{\partial \varphi_{i-1}}{\partial v_k} v_{k+1} \,. \end{aligned}$$

and we let :

$$\overline{\Phi}_{i+1}(x, \check{v}_i) \;=\; \begin{pmatrix} \check{v}_i \\ \cdots\cdots\cdots \\ \Phi_{i+1}(x, \check{v}_i) \\ \\ \end{pmatrix} = \begin{pmatrix} \check{v}_i \\ \cdots\cdots \\ \varphi_0(x, \check{v}_0) \\ \vdots \\ \varphi_i(x, \check{v}_i) \end{pmatrix} \tag{2}$$

with the notation $\check{v}_i = \begin{pmatrix} v_0 & \cdots & v_i \end{pmatrix}^T$.

We know (see [1, p. 13] for instance) that if, for some integer p, $\overline{\Phi}_p$ is an injective immersion from $X \times U \times \mathbb{R}^{p-1}$ to $\mathbb{R}^{2p}$, then, to each solution of (1), we can associate a solution of the following system, called phase-variable representation, a special kind of observability form :

$$\begin{pmatrix} \dot{z}_0 \\ \vdots \\ \dot{z}_{p-2} \\ \dot{z}_{p-1} \end{pmatrix} = \begin{pmatrix} z_1 \\ \vdots \\ z_{p-1} \\ F(\check{u}_p, \check{z}_{p-1}) \end{pmatrix} \tag{3}$$

where $y = z_0$, and with the notations

$$\check{z}_i \;=\; \begin{pmatrix} z_0 & \cdots & z_i \end{pmatrix}^T \quad , \quad \check{u}_{p-1} \;=\; \begin{pmatrix} u, u^{(1)}, \cdots, u^{(p-1)} \end{pmatrix}^T ,$$

where $u^{(i)}$ is the ith time derivative of the input u.

In the case, studied in [2], where $p = n$, the state dimension, and the vector field f in (1) is affine in u, the observability form (3) can be replaced by :

$$\begin{pmatrix} \dot{z}_0 \\ \vdots \\ \dot{z}_{p-2} \\ \dot{z}_{p-1} \end{pmatrix} = \begin{pmatrix} z_1 \\ \vdots \\ z_{p-1} \\ F(\check{z}_{p-1}) \end{pmatrix} + \begin{pmatrix} \ell_0(\check{z}_0) \\ \cdots \\ \ell_{p-2}(\check{z}_{p-2}) \\ \ell_{p-1}(\check{z}_{p-1}) \end{pmatrix} u \tag{4}$$

where again $y = z_0$, and with a triangular structure for the control vector field given by the ℓ_i's and no time derivative of the input.

Here, we extend this last result in two directions. First we allow p to be strictly larger than n. Second we allow f to be non-affine in u but then at the price of having $\dot{u}$ instead of u in (4).

2. MAIN RESULT

Let $V_i = U \times \mathbb{R}^{i-1}$ and ${}^cV_i = {}^cU \times \mathbb{R}^{i-1}$ be its closure.

Proposition 1. If, for some integer p, the function $\overline{\Phi}_p$ is injective on ${}^cX \times {}^cV_p$ then there exist a C^1 function $T : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^p$, continuous functions $F : \mathbb{R} \times \mathbb{R}^p \to \mathbb{R}$ and $\ell_i : \mathbb{R} \times \mathbb{R}^i \to \mathbb{R}$, $i = 1, \ldots, p$, such that :

– for any C^{p-1} function $u : t \mapsto u(t)$ taking values in U on some open time interval $\mathcal{I}_u$ containing 0;

– for any solution $x(t)$ of system (1) under the effect of the input u, takeing values in X on some open interval $\mathcal{I}_x \subset \mathcal{I}_u$ containing 0,

the function $t \in \mathcal{I}_x \mapsto z(t) = T(u(t), x(t))$ is solution of

$$\begin{pmatrix} \dot{z}_0 \\ \vdots \\ \dot{z}_{p-2} \\ \dot{z}_{p-1} \end{pmatrix} = \begin{pmatrix} z_1 \\ \vdots \\ z_{p-1} \\ F(u, \check{z}_{p-1}) \end{pmatrix} + \begin{pmatrix} \ell_0(u, \check{z}_0) \\ \vdots \\ \ell_{p-2}(u, \check{z}_{p-2}) \\ \ell_{p-1}(u, \check{z}_{p-1}) \end{pmatrix} \dot{u} \qquad (5)$$

Proposition 2. In the case where f is affine in u, i.e.

$$f(x, u) \;=\; a(x) \;+\; b(x)\, u$$

then under the same assumption as in the previous Proposition, we get the existence of a C^1 function $T : \mathbb{R}^n \to \mathbb{R}^p$, continuous functions $F : \mathbb{R}^p \to \mathbb{R}$ and $\ell_i : \mathbb{R}^i \to \mathbb{R}$, $i = 1, \dots, p$, such that $z(t) = T(x(t))$ is solution of (4).

Remark 1.

(1) In the case where $p = n$ and $\overline{\Phi}_p$ is not only an injective function from ${}^cX \times {}^cU \times \mathbb{R}^{n-1}$ to $\mathbb{R}^{2n}$ but also the function $x \in X \mapsto \Phi_n(x, 0)$ is an immersion, then the result of Proposition 2 holds with a function T which is a C^1 diffeomorphism . In this way, we recover the result of [2].

(2) We may require the functions ℓ_i and F in the normal form (5) to be locally Lipschitz. For this it is sufficient that, besides the injectivity of $\overline{\Phi}_p$ on ${}^cX \times {}^cU$, that the function :

$$(u, x) \in {}^cU \times {}^cV \;\mapsto\; \begin{pmatrix} u \\ \Phi_{i+1}(x, (0, \dots, 0)) \end{pmatrix}$$

be an immersion. Indeed this implies that the function Υ introduced in the proof below is Lipschitz on its compact set of definition. In such a case, in the proof below, instead of Tietze extension theorem, we use Kirszbaum extension theorem.

(3) If the normal form is to be used to solve an observer problem, we need to know the input time derivative. This is usually possible in the case of feedback with a backstepping design.

3. PROOFS

Here we prove Proposition 1. The proof of Proposition 2 follows by similar arguments.

We start by observing that injectivity of $\overline{\Phi}_p$ and its relation with Φ_p imply the existence of a (unique) function Ψ_p : $\Phi_p({}^cX \times {}^cV_p) \to {}^cX$ satisfying

$$x = \Psi_p(\Phi_p(x, \check{v}_p), \check{v}_p) \quad \forall\, (x, \check{v}_p) \in {}^cX \times {}^cV_p\,, \qquad (6)$$

Let us us add an integrator to system (1), namely

$$\dot{x} \;=\; f(x, u)\,, \quad \dot{u} \;=\; v \quad , \qquad y \;=\; h(x, u)$$

that is regarded as a system with input v, output y and state $\xi = \mathrm{col}(u, x)$. By letting $A(\xi) = \mathrm{col}(f(x, u), 0)$, $B = \mathrm{col}(0, 1)$, $H(\xi) = h(x, u)$, the previous system can be compactly rewritten as

$$\dot{\xi} = A(\xi) + Bv \qquad y = H(\xi)$$

The C^1 function $T : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^p$ claimed in the proposition is

$$T(u, x) = \Phi_p(x, (u, 0, \dots, 0)) = \begin{pmatrix} H(\xi) \\ L_{A(\xi)} H(\xi) \\ \dots \\ L_{A(\xi)}^{p-1} H(\xi) \end{pmatrix} . \qquad (7)$$

The variable $z = T(u, x)$ is governed by the dynamics

$$\begin{aligned} \dot{z}_0 \;&=\; z_1 + g_0(\xi) v \\ &\;\;\vdots \\ \dot{z}_{p-2} \;&=\; z_{p-1} + g_{p-2}(\xi) v \\ \dot{z}_{p-1} \;&=\; \bar{F}(\xi) + g_{p-1}(\xi) v \end{aligned}$$

where $\bar{F}(\xi) = L_{A(\xi)}^p H(\xi)$ and $g_i(\xi) = L_B L_{A(\xi)}^i H(\xi)$, $i = 0, \dots, p-1$.

Consider now the C^1 function $\Gamma : \mathcal{O} \to \mathbb{R}^{p+1}$ defined as

$$\Gamma(u, x) = \begin{pmatrix} u \\ T(u, x) \end{pmatrix}$$

By assumption, its restriction to ${}^cU \times {}^cX$ is injective. This set being compact, Γ is a topological emmbedding and so there exists a continuous function $\Upsilon : \Gamma({}^cU \times {}^cX) \to {}^cU \times {}^cX$ that associates to each $\mathfrak{z} = \mathrm{col}(u, z) \in \Gamma({}^cU \times {}^cX)$ the value

$$\Upsilon(\mathfrak{z}) = \begin{pmatrix} u \\ \Psi_p(z, (u, 0, \dots, 0)) \end{pmatrix} .$$

and which satisfies

$$\Upsilon(\Gamma(\xi)) = \xi \qquad \forall\, \xi \in {}^cU \times {}^cX\,.$$

Now let $\bar{g}_i : \Gamma({}^cU \times {}^cX) \to \mathbb{R}$, $i = 1, \dots, p$, be the continuous function defined as

$$\bar{g}_i(\mathfrak{z}) \;=\; g_i(\Upsilon(\mathfrak{z}))\,.$$

It turns out that, for all $k = 0, \dots, p-1$, and each pair $\mathfrak{z}^a = (u, z^a)$ and $\mathfrak{z}^b = (u, z^b)$ in $\Gamma({}^cU \times {}^cX)$ satisfying $z_i^a = z_i^b$ for all $i = 0, \dots, k$, we have $\bar{g}_k(\mathfrak{z}^a) = \bar{g}_k(\mathfrak{z}^b)$. This fact follows by an elementary adaptation of the arguments in [2] we write here just for the case $k = 0$.

Let $u_* \in U$, $x_*^a \in X$ and $x_*^b \in X$ be such that $\xi_*^a = (u_*, x_*^a)$ and $\xi_*^b = (u_*, x_*^b)$ satisfy $H(\xi_*^a) = H(\xi_*^b)$ or equivalently $z_{*0}^a = z_{*0}^b$, with $z_*^a = \Gamma(\xi_*^a)$ and $z_*^b = \Gamma(\xi_*^b)$. Assume we have

$$\bar{g}_0(u_*, z_*^a) \;\neq\; \bar{g}_0(u_*, z_*^b)$$

i.e.

$$L_B H(\xi_*^a) = g_0(\xi_*^a) \;\neq\; g_0(\xi_*^b) = L_B H(\xi_*^b)\,.$$

By continuity there exist neighborhoods $\mathcal{N}^a$ and $\mathcal{N}^b \subset U \times X$ of ξ_0^a and ξ_0^b such that

$$L_B H(\xi^a) \neq L_B H(\xi^b) \qquad \forall\, (\xi^a, \xi^b) \in \mathcal{N}^a \times \mathcal{N}^b\,.$$

Consider now the system

$$\dot{x}^a \;=\; f(x^a, u)\,,\; \dot{x}^b = f(x^b, u)\,,\; \dot{u} = v_1$$

with output $\tilde{y} = h(x^a, u) - h(x^b, u)$ and input v_1 taken as the feedback

$$v_1 = \frac{L_{A(\xi^a)} H(\xi^a) - L_{A(\xi^b)} H(\xi^b)}{L_B H(\xi^a) - L_B(\xi^b)}\,.$$

It is motivated by the fact that it gives $\dot{\tilde{y}} = 0$. And it is as many times differentiable as needed as long as (ξ^a, ξ^b) is in $\mathcal{N}^a \times \mathcal{N}^b$.

Let $(\xi^a(t), \xi^b(t))$ be its solution with initial value (ξ^a_*, ξ^b_*). There exists a $T > 0$ such that for all $t \in [0,T)$ $(\xi^a(t), \xi^b(t)) \in \mathcal{N}^a \times \mathcal{N}^b$ and, as a consequence, the components $x^a(t)$ and $x^b(t)$ are in ${}^cX$ and $u(t)$ is in ${}^cU$ for all $t \in [0,T)$.

Furthermore, since $t \mapsto \tilde{y}(t)$ is constant on $[0,T)$ and $\tilde{y}(0) = 0$, it is zero on the whole interval. So the same holds for its $p-1$ first derivatives. By definition of the function Φ, we get $\Phi_p(x^a(t), \breve{u}_{p-1}(t)) = \Phi_p(x^b(t), \breve{u}_{p-1}(t))$ and thus

$$\begin{aligned} x^a(t) &= \Psi_p(\Phi_p(x^a(t), v(t)), v(t)) \\ &= \Psi_p(\Phi_p(x^b(t), v(t)), v(t)) \\ &= x^b(t) \qquad \forall t \in [0,T) \ . \end{aligned}$$

This yields in particular $x^a_* = x^b_*$. So we have $\xi^a_* = \xi^b_*$ and thus $g_1(\xi^a_*) = g_1(\xi^b_*)$. This is a contradiction. In this way, we have shown that, for each pair $\mathfrak{z}^a = (u, z^a)$ and $\mathfrak{z}^b = (u, z^b)$ in $\Gamma(U \times X)$ satisfying $z^a_0 = z^b_0$, we have $\bar{g}_0(\mathfrak{z}^a) = \bar{g}_0(\mathfrak{z}^b)$. Since the function g_1 is continuous on ${}^cU \times {}^cX$, the same holds on $\Gamma({}^cU \times {}^cX)$.

Similar arguments, can be used by induction for $k = 1, \ldots, p$, with an appropriate choice of the input derivative $u^{(k+1)}$.

From the above, it follows that the functions $\bar{g}_i(\mathfrak{z})$ presents a triangular structure in the z_i components of z. Namely, since $\Gamma({}^cU \times {}^cX)$ is a subset of $\mathbb{R} \times \mathbb{R}^p$, we can introduce its projection Γ_i on $\mathbb{R} \times \mathbb{R}^i$, i.e.

$$\begin{aligned} \Gamma_i \ = \Big\{ & (u, \check{z}_i) \in \mathbb{R} \times \mathbb{R}_i \ : \\ & \exists (z_{i+1}, \ldots, z_{p-1}) \in \mathbb{R}^{p-i-1} \ : \\ & \quad (u, (\check{z}_i, (z_{i+1}, \ldots, z_{p-1}))) \in \Gamma({}^cU \times {}^cX) \Big\} \ . \end{aligned}$$

This allows us to define the function $\ell_i : \Gamma_i \to \mathbb{R}$ as :

$$\ell_i(u, \check{z}_i) \ = \ \bar{g}_i(\mathfrak{z}) \qquad \forall \mathfrak{z} \in \Gamma({}^cU \times {}^cX) \ .$$

As $\bar{g}_i$, it is continuous. Let also $F : \Gamma({}^cU \times {}^cX) \to \mathbb{R}$ be

$$F(u, \check{z}_{p-1}) \ = \ \bar{F}(\Upsilon(\mathfrak{z})) \ ,$$

it is also continuous. With these functions we do have obtained the form (5).

However, up to now, the functions ℓ_i and F are defined only on Γ_i and $\Gamma({}^cU \times {}^cX)$, respectively where they are continuous. To extend their definition to $\mathbb{R} \times \mathbb{R}^p$, we use the fact that Γ_i and $\Gamma({}^cU \times {}^cX)$ are compact subsets of $\mathbb{R} \times \mathbb{R}^i$ and $\mathbb{R} \times \mathbb{R}^p$ respectively as images by continuous functions of compact sets. By applying Tietze extension Theorem, we know that the definitions ℓ_i ad F can be extended to the corresponding full spaces.

4. EXAMPLE

We consider the uncertain harmonic oscillator described in the state space by

$$\dot{x}_1 = x_2 \, , \quad \dot{x}_2 = x_3 x_1 + u \, , \quad \dot{x}_3 = 0 \tag{8}$$

with $y = x_1$ and living in the open bounded subset of $\mathbb{R}^3$

$$\begin{aligned} X \ = \ \{ (x_1, x_2, x_3) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} : c_1 < x_1^2 + x_2^2 < c_2 \, , \\ -c_3 < x_3 < -c_4 \} \end{aligned}$$

with $c_1 < c_2$ and $c_3 > c_4$ positive numbers, forced by the input $u \in U$, U an open bounded subset of $\mathbb{R}$. It turns out that the injectivity condition of Proposition 2 is not fulfilled with $p = n = 3$ but it is fulfilled with $p = 4$. As a matter of fact, by defining $\varphi_0(x) = x_1$,

$\varphi_1(x) = x_2 \ , \ \varphi_2(x, v_0) = x_1 x_3 + v_0 \ , \ \varphi_3(x, v_1) = x_2 x_3 + v_1$ and

$$\Phi_4(x, v_0, v_1) = \mathrm{col}\ (\varphi_0, \varphi_1, \varphi_2, \varphi_3) \ ,$$

it turns out that the function $\Psi_4 : \Phi_4({}^cX \times {}^cV) \to {}^cX$ defined as

$$\Psi_4 = \begin{pmatrix} \varphi_0 \\ \varphi_1 \\ \dfrac{(\varphi_2 - v_0)\varphi_0 + (\varphi_3 - v_1)\varphi_1}{\varphi_0^2 + \varphi_1^2} \end{pmatrix}$$

is such that

$$x = \Psi_4(\Phi_4(x, v), v)$$

for any $(x, v) \in {}^cX \times {}^cV$, with $v = (v_0, v_1)$, and ${}^cV = {}^cU \times \mathbb{R}$. The smooth function $T : \mathbb{R}^3 \to \mathbb{R}^4$ claimed in Proposition 2 is

$$T(x) = \mathrm{col}(x_1 \ , \ x_2 \ , \ x_1 x_3 \ , \ x_2 x_3)$$

with the normal form (4) expressed as:

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{pmatrix} = \begin{pmatrix} z_2 \\ z_3 \\ z_4 \\ z_1 z_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ \dfrac{z_1 z_3 + z_2 z_4}{\max\{c_1, z_1^2 + z_2^2\}} \end{pmatrix} u \tag{9}$$

which is defined on $\mathbb{R}^4$ (and not only on $\Phi_4({}^cX \times \{0\})$).

5. CONCLUSIONS

In this note we have studied the existence of observability normal forms (5) for nonlinear systems of the form (1). The main result is detailed in Proposition 1 and relies upon the existence of a left inverse of the function $\overline{\Phi}_p(\cdot)$ defined in (2) for $p \geq n$. The result generalizes known results for systems that are affine in the input and fulfilling the assumption of the paper with $p = n$.

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