52nd IEEE Conference on Decision and Control
December 10-13, 2013. Florence, Italy

A small-gain-like theorem for large-scale systems

Giordano Scarciotti, Laurent Praly and Alessandro Astolfi

***Abstract*— The behavior of the solutions of large-scale nonlinear dynamical systems close to their omega-limit sets is studied. Exploiting small-gain like conditions we extend the results in [1], considering the interconnection of p of subsystems, and the results in [2], presenting a block version of the weak nested Matrosov theorem.**

I. INTRODUCTION

The qualitative study of the asymptotic behavior of the solutions of large-scale nonlinear dynamical systems is a challenging problem.

For dealing with asymptotic stability, an elegant tool is the small-gain theorem. Several versions of the theorem have been developed, depending on which property is used to describe the input-output behavior of the various subsystems. Classical versions of the theorem use the L_p-gain, yielding an L_p small-gain (see e.g. [3], [4] and [5]), whereas more recent versions are based upon a Lyapunov formulation (see e.g. [6]) derived from the property of input-to-state stability (ISS) (see e.g. [7] and [8]). Other formulations have been presented (see e.g. [9], [10] and [11]). Therein interconnections between possibly non-ISS subsystems have been considered. The large-scale version of the theorem for interconnected linear systems can be found in [3], whereas a more general nonlinear formulation has recently been developed in [12].

Herein, in the spirit of the results of [2] and [1], we wish to go beyond stability and study the behavior of the solutions approaching their omega-limit sets, in particular when the convergence is not fast enough to guarantee some asymptotic phase/shadowing property. We propose a weak (to be defined) version of a very special kind of small-gain theorem. The paper extends both the results in [1], considering the interconnection of a large number p of subsystems, and the results in [2], presenting a block version of the weak nested Matrosov theorem.

Our analysis studies the properties of functions $h_i : \mathbb{R}^n \to \mathbb{R}_+$ satisfying the (square) system of differential inequalities[1]

$$\begin{aligned}
\dot V_1(t) &\leq -\alpha_1(h_1(t)) + \beta_{12}(h_2(t)) + \cdots + \beta_{1p}(h_p(t)),\\
\dot V_2(t) &\leq -\alpha_2(h_2(t)) + \beta_{21}(h_1(t)) + \cdots + \beta_{2p}(h_p(t)),\\
&\vdots\\
\dot V_p(t) &\leq -\alpha_p(h_p(t)) + \beta_{p1}(h_1(t)) + \cdots + \beta_{p(p-1)}(h_{p-1}(t)),
\end{aligned} \tag{1}$$

where the functions $\alpha_i : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous and positive definite, the functions $\beta_{ij} : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous and the functions $V_i : \mathbb{R}_+ \to \mathbb{R}$ are absolutely continuous and bounded. When compared for instance with the Lyapunov version of the small-gain theorem (see below) the key feature here is, in line with the approach followed in [2] and [1], that the arguments h_i of the function α_i and β_{ij} are not related a priori with the functions V_k in the left-hand side.

As discussed in [1] and illustrated in the counter-example therein, the result that we are going to prove here may not hold when the functions α and β are nonlinear and they satisfy a nonlinear small-gain like condition. A more restrictive linear small-gain like condition may be required. This justify the use of the linear framework in the following.

In the particular case in which, in equation (1), $V_i = h_i$ it is possible to interpret our analysis within the usual small-gain paradigm. Specifically, consider a nonlinear system described by the equation

$$\dot x = f(x), \tag{2}$$

where $x \in \mathbb{R}^n$ is the state of the system and the function f is locally Lipschitz. System (2) is viewed as an interconnection of p subsystems, namely

$$\begin{aligned}
\Sigma_1:\quad \dot x_1 &= f_1(x_1, x_2, \ldots, x_p),\\
\Sigma_2:\quad \dot x_2 &= f_2(x_1, x_2, \ldots, x_p),\\
&\vdots\\
\Sigma_p:\quad \dot x_p &= f_p(x_1, x_2, \ldots, x_p).
\end{aligned} \tag{3}$$

We assume there exists an equilibrium point which we choose as the origin of the coordinates, i.e., $f_i(0) = 0$, and for each i, we have a positive definite C^1 function $x_i \mapsto V_i(x_i)$ such that, when evaluated along the solutions,

G. Scarciotti is with the Dept. of Electrical and Electronic Engineering, Imperial College London, London, SW7 2AZ, UK, E-mail: gs3610@ic.ac.uk.

L. Praly is with MINES ParisTech, CAS, Mathématiques et Systèmes, 35, rue St-Honoré, 77305 Fontainebleau CEDEX, France, E-mail: Laurent.Praly@mines-paristech.fr.

A. Astolfi is with the Dept. of Electrical and Electronic Engineering, Imperial College London, London, SW7 2AZ, UK and DICII, University of Roma "Tor Vergata", Via del Politecnico 1, 00133 Rome, Italy, E-mail: a.astolfi@ic.ac.uk. This work is partly supported by the EPSRC Programme Grant "Control For Energy and Sustainability" EP/G066477.

[1]There is no loss of generality in writing $\sum_{j=r}^{q} \beta_{ij}(h_j)$ instead of $\beta_i(h_r, \ldots, h_q)$. Since we can always pick

$$\beta_{ij}(h) = \max_{h_r \leq h, \ldots, h_{j-1} \leq h, h_{j+1} \leq h, \ldots, h_q \leq h} \beta_i(h_r, \ldots, h_{j-1}, h, h_{j+1}, \ldots, h_q).$$

978-1-4673-5716-6/13/$31.00

we have

$$\begin{aligned}\dot{V}_1 &\leq -\alpha_1(V_1) + \beta_{12}(V_2) + \cdots + \beta_{1p}(V_p),\\ \dot{V}_2 &\leq -\alpha_2(V_2) + \beta_{21}(V_1) + \cdots + \beta_{2p}(V_p),\\ &\vdots\\ \dot{V}_p &\leq -\alpha_p(V_p) + \beta_{p1}(V_1) + \cdots + \beta_{p(p-1)}(V_{p-1}).\end{aligned} \tag{4}$$

The above set of inequalities implies that each subsystem Σ_i, with state x_i, is input-to-state stable (ISS) or integral input-to-state stable (iISS) (depending on the properties of α_i's) with respect to the states of the other $p-1$ subsystems. The small-gain condition implies the (function) invertibility of the "matrix" composed of the gain functions $\gamma_{ij} = \beta_{ij} \circ \alpha_j^{\dagger}$ where $\alpha_j^{\dagger}$ is an "inverse" function of α_j. See [3], [12], [6], [13] for further details.

The rest of the paper is organized as follows. In Section II some technical results are given. After preliminary lemmas and definitions, that are instrumental for the following section, a new small-gain like condition for interconnected systems is stated. Section III provides the main result of the paper, i.e., a weak version of the small-gain theorem for large-scale systems. In Section IV some concluding remarks are drawn.

Due to space limitation, we provide only the proof of Lemma 2. The other proofs will appear in a longer version of the paper.

II. TECHNICAL RESULTS

This section contains a series of technical lemmas and definitions that are instrumental to establish the main result of the paper.

The following lemma is a rephrasing of the results proved in [3, Lemma 6.1.9] and in [14, Chapter 6]. In addition, for the class of matrices studied herein the condition on the determinants of the leading principal minors is equivalent to requiring that the spectral radius of the matrix is less than one (see [3, Lemma 6.1.8], and [13] where this condition is used).

Lemma 1: The inverse of any matrix with non-positive off-diagonal elements, positive diagonal elements and having all the leading principal minors with strictly positive determinant, has non-negative entries.

Definition 1: A family of continuous functions $\alpha_i : \mathbb{R}_+ \to \mathbb{R}_+$ and $\beta_{ij} : \mathbb{R}_+ \to \mathbb{R}_+$, where the α_i are positive definite, is said to satisfy the boundedness assumption if, for all (i, j), with $i \neq j$, there exists a non-positive real number γ_{ij} satisfying[2]

$$\sup_{b>0} \frac{\beta_{ij}(b)}{\alpha_j(b)} \leq -\gamma_{ij}.$$

For a family satisfying this assumption, we call test matrix Γ the matrix with γ_{ij} as off-diagonal elements and 1 as diagonal elements. Note that Γ is a matrix with non-positive off-diagonal elements and positive diagonal elements.

[2]This implies $\beta_{ij}(0) = 0$.

Lemma 2: Let $i \in \{1, \ldots, p\}$ and $j \in \{1, \ldots, p\}$. Let $a_i : \mathbb{R}_+ \to [-\overline{a}, \overline{a}]$ be bounded absolutely continuous functions and $b_i : \mathbb{R}_+ \to [0, \overline{b}]$ be bounded, piecewise continuous, functions. Assume there exist continuous positive definite functions $\alpha_i : \mathbb{R}_+ \to \mathbb{R}_+$ and continuous functions $\beta_{ij} : \mathbb{R}_+ \to \mathbb{R}_+$ such that the boundedness assumption is satisfied and the following hold.

1) The differential inequalities

$$\begin{aligned}\dot{a}_1 &\leq -\alpha_1(b_1) + \beta_{12}(b_2) + \cdots + \beta_{1p}(b_p),\\ \dot{a}_2 &\leq -\alpha_2(b_2) + \beta_{21}(b_1) + \cdots + \beta_{2p}(b_p),\\ &\vdots\\ \dot{a}_p &\leq -\alpha_p(b_p) + \beta_{p1}(b_1) + \cdots + \beta_{p(p-1)}(b_{p-1}),\end{aligned} \tag{5}$$

hold for almost all t in $\mathbb{R}_+$.

2) The test matrix Γ has all the leading principal minors with strictly positive determinant.

Then

$$\liminf_{t\to\infty} \sum_{i=1}^{p} b_i(t) = 0. \tag{6}$$

Proof: The claim holds if there exists a positive real number $\overline{\alpha}$ such that we have

$$\int_0^t \sum_{i=1}^{p} \alpha_i(b_i(s))ds \leq p\overline{\alpha}, \qquad \forall t \geq 0. \tag{7}$$

To prove this last claim suppose, by contradiction, that there exist $\underline{b}$ and $T > 0$ such that

$$\sum_{i=1}^{p} b_i(t) \geq \underline{b}, \quad \forall t \in [T, +\infty).$$

In fact, there exists $\underline{\alpha}$ such that

$$\sum_{i=1}^{p} \alpha_i(b_i(t)) \geq \underline{\alpha}, \quad \forall t \in [T, +\infty).$$

The inequality (7) in $[T, t]$ gives

$$\underline{\alpha}(t - T) \leq \int_T^t \sum_{i=1}^{p} \alpha_i(b_i(s))ds \leq p\overline{\alpha}, \quad \forall t \in [T, +\infty),$$

hence

$$\underline{\alpha}t \leq \underline{\alpha}T + p\overline{\alpha}, \quad \forall t \in [T, +\infty),$$

which is impossible for t sufficiently large.

It remains to establish (7). Note that, by definition of the matrix Γ, system (5) can be written, in compact form, as

$$[\dot{\boldsymbol{a}}]_i \leq [-\Gamma\boldsymbol{\alpha}]_i, \qquad \forall\, i = 1, \ldots, p. \tag{8}$$

where $\boldsymbol{a} = [a_1, \ldots, a_p]^T$, $\boldsymbol{\alpha} = [\alpha_1, \ldots, \alpha_p]^T$, $[\boldsymbol{v}]_i$ denotes the i-th component of the vector $\boldsymbol{v}$. Since Γ has all the leading principal minors with strictly positive determinant, by Lemma 1, Γ^{-1} has all non-negative entries, hence the relation

$$[\Gamma^{-1}\dot{\boldsymbol{a}}]_i \leq [-\boldsymbol{\alpha}]_i, \tag{9}$$

holds. This can be seen noting that each of the inequalities in (9) is obtained as a weighted sum, with non-negative weights, of the inequalities in (8). Integrating from 0 to t each of these relations yields

$$\begin{aligned}\int_0^t [\boldsymbol{\alpha}(\boldsymbol{b}(s))]_i ds &\leq -\int_0^t [\Gamma^{-1}\dot{\boldsymbol{a}}(s)]_i ds,\\ &\leq [\Gamma^{-1}(\boldsymbol{a}(t)-\boldsymbol{a}(0))]_i,\end{aligned}$$

where $\boldsymbol{b} = [b_1, \ldots, b_p]^T$. Since the functions a_i are bounded, there exists a positive real number $\overline{\alpha}$ such that we have for all i,

$$\int_0^t [\boldsymbol{\alpha}(\boldsymbol{b}(s))]_i ds \leq \overline{\alpha}.$$

Finally, the claim follows adding all these inequalities. ∎

Remark 1: For $p = 2$ Lemma 2 is consistent with the results in [1]. In addition, the condition on the determinants of the leading principal minors of the matrix Γ reduces to the small-gain like condition (determinant of Γ)

$$\gamma_{21}\gamma_{12} < 1.$$

For $p = 3$ the condition on the determinants of the leading principal minors of the matrix Γ reduces to the small-gain like conditions

$$\begin{gathered}\gamma_{21}\gamma_{12} < 1 \text{ (or } \gamma_{31}\gamma_{13} < 1, \text{ or } \gamma_{23}\gamma_{32} < 1),\\ \gamma_{21}\gamma_{12} + \gamma_{31}\gamma_{13} + \gamma_{23}\gamma_{32} + \gamma_{32}\gamma_{21}\gamma_{13} + \gamma_{31}\gamma_{12}\gamma_{23} < 1.\end{gathered} \tag{10}$$

Note that the last inequality (determinant of Γ) implies the former three.

For $p \geq 4$, as noted in [3], the condition on the determinant of Γ is no longer sufficient; to establish the claim the determinants of the leading principal minors of Γ have to be strictly positive.

We conclude this section considering the triangular block case that can be seen as a generalization of the weak nested Matrosov theorem of [2]. To this end, we let $r_l = \sum_{k=1}^{l} s_k$, with $r_0 = 0$, be the size of the column vectors

$$\begin{aligned}\boldsymbol{a}_l &= \begin{bmatrix} a_{(r_{l-1}+1)} & a_{(r_{l-1}+2)} & \ldots & a_{r_l}\end{bmatrix}^T,\\ \boldsymbol{b}_l &= \begin{bmatrix} b_{(r_{l-1}+1)} & b_{(r_{l-1}+2)} & \ldots & b_{r_l}\end{bmatrix}^T,\\ \boldsymbol{V}_l &= \begin{bmatrix} V_{(r_{l-1}+1)} & V_{(r_{l-1}+2)} & \ldots & V_{r_l}\end{bmatrix}^T,\\ \boldsymbol{h}_l &= \begin{bmatrix} h_{(r_{l-1}+1)} & h_{(r_{l-1}+2)} & \ldots & h_{r_l}\end{bmatrix}^T,\end{aligned}$$

and introduce also the notation

$$\boldsymbol{\alpha}_l(\boldsymbol{b}_l) = \begin{bmatrix} -\alpha_{(r_{l-1}+1)}(b_{(r_{l-1}+1)}) + \beta_{(r_{l-1}+1)(r_{l-1}+2)}(b_{(r_{l-1}+2)}) \\ \qquad + \ldots + \beta_{(r_{l-1}+1)r_l}(b_{r_l}) \\ \beta_{(r_{l-1}+2)(r_{l-1}+1)}(b_{(r_{l-1}+1)}) - \alpha_{(r_{l-1}+2)}(b_{(r_{l-1}+2)}) \\ \qquad + \ldots + \beta_{(r_{l-1}+2)r_l}(b_{r_l}) \\ \vdots \\ \beta_{r_l(r_{l-1}+1)}(b_{r_{l-1}+1}) + \ldots + \beta_{r_l(r_l-1)}(b_{r_l-1}) - \alpha_{r_l}(b_{r_l}) \end{bmatrix}$$

$$\boldsymbol{\beta}_{lm}(\boldsymbol{b}_m) = \sum_{j=(r_{m-1}+1)}^{r_m} \begin{bmatrix} \beta_{(r_{l-1}+1)j}(b_j) \\ \beta_{(r_{l-1}+2)j}(b_j) \\ \vdots \\ \beta_{(r_l)j}(b_j)\end{bmatrix}.$$

By combining the proof arguments of [2] with those of Lemma 2, we can prove the following statement.

Proposition 1: Let $i \in \{1, \ldots, p\}$ and $j \in \{1, \ldots, p\}$. Let $a_i : \mathbb{R}_+ \to [-\overline{a}, \overline{a}]$ be bounded absolutely continuous functions and $b_i : \mathbb{R}_+ \to [0, \overline{b}]$ be bounded, piecewise continuous, functions. Let $\alpha_i : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous positive definite functions and $\beta_{ij} : \mathbb{R}_+ \to \mathbb{R}_+$ continuous functions with $\beta_{ij}(0) = 0$. Let $\boldsymbol{a}_l$, $\boldsymbol{b}_l$, $\boldsymbol{\alpha}_l$ and $\boldsymbol{\beta}_{lm}$ be vectors of size s_l, with components obtained from the a_i's, b_i's, α_i's and β_{ij}'s, such that the boundedness assumption is satisfied for the vectors $\boldsymbol{\alpha}_l$ and that the following hold.

1) The differential inequalities

$$\begin{aligned}&\dot{\boldsymbol{a}}_1 \leq \boldsymbol{\alpha}_1(\boldsymbol{b}_1),\\ &\dot{\boldsymbol{a}}_2 \leq \boldsymbol{\beta}_{21}(\boldsymbol{b}_1) + \boldsymbol{\alpha}_2(\boldsymbol{b}_2),\\ &\quad\vdots\\ &\dot{\boldsymbol{a}}_h \leq \boldsymbol{\beta}_{h1}(\boldsymbol{b}_1) + \boldsymbol{\beta}_{h2}(\boldsymbol{b}_2) + \cdots + \boldsymbol{\alpha}_h(\boldsymbol{b}_h),\end{aligned} \tag{11}$$

with $p = r_h$, hold for almost all t in $\mathbb{R}_+$.

2) The test matrix Γ_l associated to each vector $\boldsymbol{\alpha}_l$ has all the leading principal minors with strictly positive determinant.

Then

$$\liminf_{t\to\infty} \sum_{i=1}^{p} b_i(t) = 0. \tag{12}$$

Note that the boundedness assumption is not required for the vectors $\boldsymbol{\beta}_{lm}$.

Example 1: To illustrate the result in Proposition 1 consider the four differential inequalities

$$\begin{aligned}\dot{a}_1(t) &\leq -\alpha_1(b_1(t)) + \beta_{12}(b_2(t)),\\ \dot{a}_2(t) &\leq -\alpha_2(b_2(t)) + \beta_{21}(b_1(t)),\\ \dot{a}_3(t) &\leq -\alpha_3(b_3(t)) + \beta_{31}(b_1(t)) + \beta_{32}(b_2(t)) + \beta_{34}(b_4(t)),\\ \dot{a}_4(t) &\leq -\alpha_4(b_4(t)) + \beta_{41}(b_1(t)) + \beta_{42}(b_2(t)) + \beta_{43}(b_3(t)).\end{aligned}$$

Condition 2) of Proposition 1 are in particular

$$\begin{gathered}\beta_{31}(0) = \beta_{32}(0) = \beta_{41}(0) = \beta_{42}(0) = 0,\\ \sup_{b\in[0,\overline{b}]} \frac{\beta_{21}(b)}{\alpha_1(b)} = \gamma_{21} < +\infty, \ \sup_{b\in[0,\overline{b}]} \frac{\beta_{12}(b)}{\alpha_2(b)} = \gamma_{12} < +\infty,\\ \sup_{b\in[0,\overline{b}]} \frac{\beta_{43}(b)}{\alpha_3(b)} = \gamma_{43} < +\infty, \ \sup_{b\in[0,\overline{b}]} \frac{\beta_{34}(b)}{\alpha_4(b)} = \gamma_{34} < +\infty,\end{gathered}$$

and

$$\gamma_{21}\gamma_{12} < 1, \quad \gamma_{43}\gamma_{34} < 1.$$

Remark 2: Proposition 1 establishes that in the block triangular case it is sufficient to check the small-gain condition for each diagonal block separately.

III. A WEAK SMALL-GAIN THEOREM FOR LARGE-SCALE SYSTEMS

In this section the main result of the paper is stated, i.e. a weak version of the small-gain theorem for large-scale systems.

Theorem 1: Consider system (2), p $\mathcal{C}^1$ functions V_i : $\mathbb{R}^n \to \mathbb{R}$ and p continuous functions $h_i : \mathbb{R}^n \to \mathbb{R}_+$. Assume there exist continuous positive definite functions $\alpha_i : \mathbb{R}_+ \to \mathbb{R}_+$, continuous functions $\beta_{ij} : \mathbb{R}_+ \to \mathbb{R}_+$, which are zero at zero, such that the boundedness assumption is satisfied for the vectors $\boldsymbol{\alpha}_l$, and that the differential inequalities

$$\begin{aligned}
\dot{\boldsymbol{V}}_1 &\le \boldsymbol{\alpha}_1(\boldsymbol{h}_1),\\
\dot{\boldsymbol{V}}_2 &\le \boldsymbol{\beta}_{21}(\boldsymbol{h}_1) + \boldsymbol{\alpha}_2(\boldsymbol{h}_2),\\
&\vdots\\
\dot{\boldsymbol{V}}_h &\le \boldsymbol{\beta}_{h1}(\boldsymbol{h}_1) + \boldsymbol{\beta}_{h2}(\boldsymbol{h}_2) + \cdots + \boldsymbol{\alpha}_h(\boldsymbol{h}_h),
\end{aligned} \tag{13}$$

with $p = r_h$, hold for almost all t in $\mathbb{R}_+$.
Finally, suppose that the test matrix Γ_l corresponding to each vector $\boldsymbol{\alpha}_l$ has all the leading principal minors with strictly positive determinant.
Then, for any bounded solution $x(t)$ of (2), we have

$$\liminf_{t\to\infty} \sum_{i=1}^{p} h_i(x(t)) = 0. \tag{14}$$

Moreover, if the largest invariant set $\mathcal{N}$ contained in the set

$$\{x \in \mathbb{R}^n : \, h_1(x) = h_2(x) = \cdots = h_p(x) = 0\},$$

is stable, then

$$\lim_{t\to\infty} \sum_{i=1}^{p} h_i(x(t)) = 0. \tag{15}$$

Proof: The property (14) comes directly from Proposition 1 with $h_i(x(t))$ playing the role of $b_i(t)$. Property (15) is a rephrasing of a well-known fact, see [8, Lemma I.4]. ■

IV. CONCLUSIONS

The properties of the solutions of large-scale nonlinear dynamical systems have been studied. It has been shown that the stability of the resulting omega-limit set plays a crucial role to assess asymptotic properties. A small-gain like condition has been developed, thus extending the results in [1] and [2] to the case of p interconnected subsystems and to the case of general block-triangular systems.

REFERENCES

[1] A. Astolfi and L. Praly, "A weak version of the small-gain theorem," *51st IEEE Conference on Decision and Control (CDC)*, pp. 4586–4590, 2012.

[2] ——, "A LaSalle version of Matrosov theorem," *50th IEEE Conference on Decision and Control and European Control Conference (CDC-ECC)*, pp. 320–324, 2011.

[3] M. Vidyasagar, *Input-Output Analysis of Large-Scale Interconnected Systems*. Springer-Verlag, 1981.

[4] A. Van der Schaft, *$\mathcal{L}_2$-Gain and Passivity Techniques in Nonlinear Control*. Springer-Verlag, 1996.

[5] A. Isidori, *Nonlinear Control Systems II*. Springer-Verlag, 1999.

[6] Z. Jiang, I. Mareels, and Y. Wang, "A Lyapunov formulation of the nonlinear small-gain theorem for interconnected ISS systems," *Automatica*, vol. 32, no. 8, pp. 1211–1215, 1996.

[7] E. Sontag, "Smooth stabilization implies coprime factorization," *IEEE Transactions on Automatic Control*, vol. 34, no. 4, pp. 435–443, 1989.

[8] E. Sontag and Y. Wang, "New characterizations of input-to-state stability," *IEEE Transactions on Automatic Control*, vol. 41, no. 9, pp. 1283–1294, 1996.

[9] H. Ito, "Stability criteria for interconnected iISS systems and ISS systems using scaling of supply rates," *Proceedings of the 2004 American Control Conference*, vol. 2, pp. 1055–1060, 2004.

[10] ——, "State-dependent scaling problems and stability of interconnected iISS and ISS systems," *IEEE Transactions on Automatic Control*, vol. 51, no. 10, pp. 1626–1643, 2006.

[11] D. Angeli and A. Astolfi, "A tight small gain theorem for not necessarily ISS systems," *Proceedings of the 44th IEEE Conference on Decision and Control, and the European Control Conference*, pp. 5427–5431, 2005.

[12] S. Dashkovskiy, B. Rüffer, and F. Wirth, "Small gain theorems for large scale systems and construction of ISS Lyapunov functions," *SIAM J. Control and Optimization*, vol. 48, no. 6, pp. 4089–4118, 2010.

[13] ——, "An ISS small-gain theorem for general networks," *Mathematics of Control, Signals, and Systems*, vol. 19, no. 2, pp. 93–122, 2007.

[14] A. Berman and R. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, ser. Classics in Applied Mathematics. SIAM, 1994, vol. 9.