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# Output Feedback Stabilization for SISO Nonlinear Systems with an Observer in the Original Coordinates

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***Abstract*— We address the problem of designing a stabilizing output feedback, via the separation principle. Our aim is to propose a more usable technique. The system can be written in any coordinates and is supposed to be locally uniformly observable. Starting form a known state feedback we do one step of backtepping to have access to the input derivative. This is sufficient to design a high gain observer in the original coordinates that we modify to prevent peaking and constrain the estimated state to remain in the observability domain**

## I. Introduction

We address the problem of designing a stabilizing output feedback. We remain in line with the many contributions dealing with general systems satisfying a stabilizability and observability property, following the separation principle and relying on a high gain observer. For example, in [1], the state estimate is reconstructed through a function of the output (and its derivatives) and the input (and its derivatives); the ouput derivatives being obtained from a high-gain observer, and the input derivatives are given by several steps of back-stepping. In [2] the uniform observability assumption of [1] is relaxed. In [3] the authors proposed a different approach but they require minimum phase assumptions. Finally in [4] a solution is proposed with the observer designed in the original coordinates.

In this context, our goal is to propose a design easier to use in applications. But for this we ask for observability and stabilizability properties which are too restrictive from the theoretical view point. (Compare with [8], [2]). A first simplification comes with the fact that, in the general case, we need only one step of backstepping, reducing in this way the need of formal computing. Also, taking advantage of recent results in [5], and inspired by [6], [4], we reduce the order of the observer itself and design it in the original coordinates. But, as observed in [4], in doing so, the task of managing the estimated state is made harder. We solve this problem thanks to a convexity restriction on the observable set, as in [4], but with a different solution.

We follow the arguments of [12, Chapter 12.3] to prove that all these ingredients can be merged appropriately.

## II. Problem Statement

We consider a nonlinear system whose dynamics are:

$$\dot{x} = f(x,u) \quad , \qquad y = h(x,u) \tag{1}$$

with state $x \in X$, measured output $y \in \mathbb{R}$, and control $u \in \mathcal{U}$, where $X$ and $\mathcal{U}$ are connected open sets containing the origin in $\mathbb{R}^n$, and $\mathbb{R}$, respectively. The function $f : X \times \mathcal{U} \to \mathbb{R}^n$ and $h : X \times \mathcal{U} \to \mathbb{R}$ are sufficiently many times differentiable and zero at the origin. We are interested in the design of a stabilizing *output* feedback starting from the knowledge of a stabilizing *state* feedback and knowing the state $x$ is observable via the ouput $y$. The precise context is as follows.

Define recursively the following functions $\varphi_i : \mathbb{R}^n \times \mathbb{R}^{i+1} \to \mathbb{R}$, for $i = \{1, \dots, n-1\}$, as

$$\begin{aligned}\varphi_0(x, v_0) &= h(x, v_0)\\ \varphi_i(x, v_0, \dots, v_i) &= \frac{\partial \varphi_{i-1}}{\partial x} f(x, v_0) + \sum_{k=0}^{i-2} \frac{\partial \varphi_{i-1}}{\partial v_k} v_{k+1} .\end{aligned}$$

Then, with the notation $\check{v} = (v_0 \cdots v_{n-1})^T$, let the functions $\Phi_c$, $\overline{\Phi}_c$ and $\Phi$ be defined as:

$$\begin{aligned}\Phi_c :\ & X \times \mathcal{U} \times \mathbb{R}^{n-1} \to \mathbb{R}^n\\ & (x, \check{v}) \mapsto \Big(\varphi_0(x, v_0), \dots, \varphi_{n-1}(x, v_0, \dots, v_{n-1})\Big) ,\\ \overline{\Phi}_c :\ & X \times \mathcal{U} \times \mathbb{R}^{n-1} \to \mathcal{U} \times \mathbb{R}^{2n-1}\\ & (x, \check{v}) \mapsto \Big(\check{v} , \Phi_c(x, \check{v})\Big) ,\\ \Phi :\ & X \times \mathcal{U} \to \mathbb{R}^n\\ & (x, u) \mapsto \Phi_c(x, u, 0, \dots, 0) .\end{aligned}$$

*Assumption 1 (Stabilizability):* There exists a sufficiently many times differentiable function $\theta : X \to \mathcal{U}$ such that the origin of the system (1), with $u = \theta(x)$, is asymptotically stable with domain of attraction $X_s$.

*Assumption 2 (Observability):* There exists a connected open set $\mathcal{O} \subset X$ containing the origin such that:

O1. the function $\overline{\Phi}_c$ is injective on $\mathcal{O} \times \mathcal{U} \times \mathbb{R}^{n-1}$ ;

O2. the matrix $\frac{\partial \Phi}{\partial x}(x, u)$ is invertible for any $(x, u) \in \mathcal{O} \times \mathcal{U}$;

O3. for any positive real number $r$, the set

$$\{x \in \mathcal{O} \ : \exists u \in \mathcal{U} \ : |u| \le r, \ |\Phi(x, u)| \le r\}$$

is bounded.

*Remark 1:*

- Because we insist on having a design in the given coordinates, the observability assumption, though of the same nature as the one in [1], is more restrictive since we impose at most $n-1$ derivatives in $\Phi_c$. On the other hand, as in [4], because this is more realistic, we do not assume this assumption holds globally.
- Because the state feedback takes values in $\mathcal{U}$, we are forced to have the same input set $\mathcal{U}$ in the observability assumption as in the stabilizability assumption.

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Since the system is not affine in $u$ and $u$ is an argument of the functions involved in the observability assumption, we choose to consider the input $u$ as part of the state. This is done by considering the extended system:

$$\dot{x}_e \;=\; f_e(x_e, v)$$

which is a compact notation for:

$$\dot{x} \;=\; f(x,u) \;, \qquad \dot{u} \;=\; v \tag{2}$$

with extended state $x_e = (x, u)$ .

With Assumption 1 and relying for instance on the backstepping technique, we know there exists a sufficiently many times differentiable function $\theta_e : X \times \mathcal{U} \to \mathbb{R}$ such that the origin of the system (2), with $v = \theta_e(x,u)$, is asymptotically stable with domain of attraction $\mathcal{S} = X_s \times \mathcal{U}$. Also there exists a function $V_e : \mathcal{S} \to \mathbb{R}$ for the extended system which is $C^1$ positive definite and proper on $\mathcal{S}$ and such that the function $x \mapsto \frac{\partial V_e}{\partial x}(x,u) f(x,u) + \frac{\partial V_e}{\partial u}(x,u)\theta_e(x,u)$ is negative definite on $\mathcal{S}$. Unfortunately, in general we do not know this latter function but instead one which has the right properties only on a (strict) subset of $\mathcal{S}$ that we denote $\mathcal{S}_m$.

On the observability side, to exploit Assumption 2, we need a mechanism to keep the state estimate in $\mathcal{O}$ and possibly to prevent peaking. To be able to design it, we may have to consider a (strict) subset $\mathcal{O}_r$ of $\mathcal{O}$ having a convexity property to be made precise below (see H3).

These various points leads us to pay our attention to the open subset $\mathcal{SO}$ of $X_s \times \mathcal{U}$ we define as follows. Let $v_\infty$ be the real number:

$$v_\infty = \inf_{(x,u) \in \mathcal{S}_m,\, (x,u) \notin \mathcal{O}_r \times \mathcal{U}} V_e(x,u) \tag{3}$$

and $\mathcal{SO}$ be the set:

$$\mathcal{SO} = \left\{ (x,u) \in \mathcal{S}_m \,:\, V_e(x,u) < v_\infty \right\} \, .$$

This is a sublevel set of the function $V_e$ related to stabilization. By construction, it is made forward invariant by the state feedback $\theta_e$. Since it is a subset of $(X_s \cap \mathcal{O}) \times \mathcal{U}$, by imposing the system state remains in $\mathcal{SO}$, we are guaranteed that both stabilizability and observability properties hold. Unfortunately $\mathcal{SO}$ may be much "smaller" than $(X_s \cap \mathcal{O}) \times \mathcal{U}$, but, at this time, we do not know how to design a state feedback making forward invariant this less restrictive set.

*Proposition 1:* If Assumptions 1 and 2 are satisfied, then for any compact set $C_{x,u}$ contained in $\mathcal{SO}$, there exist functions $\gamma$, $h_2$, $\mathrm{sat}_\theta$ and $\theta_{e,mod}$, matrices $K$, $P$ and a real number $\underline{\ell} \geq 1$ such that, for all $\ell \geq \underline{\ell}$, the origin of the system (1) in closed loop with:

$$\begin{aligned} \dot{\hat{x}} &= f(\hat{x},u) + \left(\frac{\partial \Phi}{\partial x}(\hat{x},u)\right)^{-1} \mathcal{L} K [y - h(\hat{x},u)] + \mathcal{M}(\hat{x},u) \\ \dot{u} &= \mathrm{sat}_\theta(\theta_{e,mod}(\hat{x},u)) \end{aligned} \tag{4}$$

where:

$$\mathcal{L} = \mathtt{diag}\,(\ell, \ldots, \ell^n) \tag{5}$$

$$\mathcal{M}(\hat{x},u) = -\gamma(\hat{x},u)\left(\frac{\partial \Phi}{\partial x}(\hat{x},u)\right)^{-1} \mathcal{L} P^{-1} \mathcal{L} \left(\frac{\partial \Phi}{\partial x}(\hat{x},u)\right)^{-1\,T} \times \frac{\partial h_2}{\partial x}(\hat{x},u)^T h_2(\hat{x},u) \, . \tag{6}$$

is asymptotically stable with domain of attraction containing the set $\{(\hat{x}, x, u) \in \mathcal{O} \times C_{x,u} \,:\, h_2(\hat{x},u) < \frac{1}{2}\}$.

In the next section we show a possible design for the functions $\gamma$, $h_2$, $\mathrm{sat}_\theta$ and $\theta_{e,mod}$ and matrices $K$ and $P$.

## III. Output feedback design

We start form the normal form introduced in [5]. It is an extension to the case of systems non affine in the control of the one given in [9]. Specifically, under Assumption 2, the image $z$ in $\mathbb{R}^n$ by $\Phi$ of $(x,u)$ in $\mathcal{O} \times \mathcal{U}$, satisfies:

$$\begin{pmatrix} \dot{z}_1 \\ \vdots \\ \dot{z}_{n-1} \\ \dot{z}_n \end{pmatrix} = \begin{pmatrix} z_2 \\ \vdots \\ z_n \\ a(u, z_1, \ldots, z_n) \end{pmatrix} + \begin{pmatrix} b_1(u, z_1) \\ \vdots \\ b_{n-1}(u, z_1, \ldots, z_{n-1}) \\ b_n(u, z_1, \ldots, z_n) \end{pmatrix} \dot{u}. \tag{7}$$

Our interest in this form comes from the fact, established in [10, Theorem 6.2.2] for instance, that a high gain observer can be used to estimate the state $(z, u)$ of this system using $(z_1, u)$ as measurement and with $v = \dot{u}$ as input.

However, we do not follow the standard route of implementing the high gain observer in the $(z,u)$ coordinates since this involves to find on line a solution $x$ for the equation $z = \Phi(x,u)$, when $(z,u)$ is given. It is very often a too demanding task to realize. Instead we follow the suggestion of [6] and we write the observer in the original coordinates as was done already in [4]. This explains why in (4) we need only to invert the matrix $\frac{\partial \Phi}{\partial x}(\hat{x},u)$ instead of inverting the function $\Phi$.

### A. Observer design

Consider the observer in (4), with $\mathcal{M} = 0$ for the time being, with the matrix $\mathcal{L}$ defined in (5), and with a vector $K$, whose components are $k_i$, such that the matrix:

$$\mathcal{K} \;=\; \begin{pmatrix} -k_1 & 1 & 0 & \ldots & 0 \\ \vdots & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & & \ddots & 1 \\ -k_n & 0 & \cdots & \cdots & 0 \end{pmatrix}$$

has all its eigenvalues with strictly negative real part. In general this observer does not guarantee that $\hat{x}$ remains in $\mathcal{O}$ and therefore that $\frac{\partial \Phi}{\partial x}(\hat{x},u)$ is invertible. It is to round this problem that, as in [4], we introduce the modification $\mathcal{M}$. Here it is not designed via projection, but by considering a dummy measured output. Namely we assume the knowledge of a $C^1$ function $h_2 : \mathcal{O} \times \mathcal{U} \to \mathbb{R}_{\geq 0}$ such that:

H1. For each $u$ in $\mathcal{U}$, the set $\{x \in \mathbb{R}^n \,:\, h_2(x,u) < 1\}$ is a subset of $\mathcal{O}$.

H2. The function $(x,u) \mapsto \frac{h_2(x,u)}{\left|\frac{\partial h_2}{\partial x}(x,u)\right|}$ is continuous on $\mathcal{O} \times \mathcal{U}$.

H3. For any $u$ in $\mathcal{U}$, for any real number $s$ in $[0,1]$, and any $x_1$ and $x_2$ in $\mathcal{O}$ satisfying:

$$h(x_1, u) \leq s \quad , \qquad h(x_2, u) \leq s$$

we have:

$$h_2(x, u) \leq s$$

for all $x$ which satisfies for some $\lambda$ in $[0, 1]$:

$$\Phi(x, u) \ = \ \lambda\Phi(x_1, u) + (1 - \lambda)\Phi(x_2, u) \ .$$

This means nothing but the fact that, for any non negative real number $s$ and any $u$ in $\mathcal{U}$, the image by $\Phi$ of the set $\{(x, u) \in \mathbb{R}^{n+1} : \, h_2(x, u) \leq s\}$ is convex.

H4. The set $\mathcal{O}_r$ defined as:

$$\mathcal{O}_r = \bigcap_{u \in \mathcal{U}} \{x \in \mathcal{O} : \, h_2(x, u) = 0\} \tag{8}$$

has a non empty interior which contains the origin;

H5. For any strictly positive real number $r$, the set:

$$\{x \in \mathcal{O} \ : \exists u \in \mathcal{U} \ : |u| \leq r, \ |h_2(x, u)| \leq \tfrac{1}{2}\}$$

is bounded.

Existence of the function $h_2$ is always guaranteed under the observability assumption as we show in Appendix.

The set $\mathcal{O}_r$ defined in (8) is the one mentioned in Section II. It is such that we have:

$$h_2(x, u) = 0 \qquad \forall (x, u) \in \mathcal{O}_r \times \mathcal{U} \ .$$

This motivates us for introducing a dummy measured output:

$$y_2 = h_2(x, u)$$

and to consider that its measured value is always 0. With this we modify the observer by adding the modification term $\mathcal{M}$ defined in (6), with $\gamma : \mathbb{R}^n \times \mathcal{U} \times \mathbb{R} \to \mathbb{R}_{>0}$ a locally Lipschitz function for which we give a lower bound in (24).

An important feature is that, thanks to this additional term $\mathcal{M}$, no other modification – saturation, ...– is needed.

We may dislike the convexity property mentioned in H3 above. Unfortunately it is in some sense necessary if we choose to keep an Euclidean distance in the image by $\Phi$ as a Lyapunov function for the error system incoporating the modification $\mathcal{M}$. Indeed, in this case the correction term must dominate all the other ones in the expression of $\dot{\hat{x}}$ when $h_2$ becomes too large. Namely we need an infinite gain margin, as defined in Definition 2.8 in [11]. Then as proved in Lemma 2.7 [11] the convexity assumption is necessary.

### B. State feedback design

Because both the input and its first time derivative are used in the observer, our very first step is to design a stabilizing state feedback for the extended system (2).

As discussed above, a consequence of Assumption 1, is that we know a function $V : X_s \to \mathbb{R}$ which is positive definite and proper on the given set $X_s$ and such that:

$$\dot{V}(x) = \frac{\partial V}{\partial x}(x) f(x, \theta(x)) = -W(x)$$

where $W$ is positive definite on $X_s$. This function $V$ may be given by the design of $\theta$ but not necessarily be proper on $X_s$. If this is the case, we make it proper by modifying it as follows. With $\bar{c}$ a strictly positive real number satisfying:

$$\bar{c} \leq \min_{x \notin X_s} V(x) \ ,$$

we replace the given function $V$ by $\frac{V}{\bar{c} - V}$. Accordingly $W$ becomes $\frac{W}{(\bar{c} - V)^2}$. But then the given domain attraction $X_s$ is replaced by its strict subset $X_{sm} = \{x \in \mathbb{R}^n : \, V(x) < \bar{c}\}$.

In the following we denote by $V_m$, $W_m$ and $\mathcal{S}_m = X_{sm} \times \mathcal{U}$ these functions and set whether they are modified or not.

Now to design a stabilizing state feedback for the extended system (2), we do one step of backstepping. To deal with the constraint that $u$ should remain in $\mathcal{U}$, following [7, Lemma 1], we consider the function $V_e$ defined as:

$$V_e(x_e) = V_m(x) + \int_{\theta(x)}^{u} \frac{s - \theta(x)}{\mathrm{d}(s, \partial \mathcal{U})} \ \mathrm{d}s$$

where $\mathrm{d}(s, \partial \mathcal{U})$ is the distance between $s$ and the boundary of the set $\mathcal{U}$. To fix the idea we consider the case where $\mathcal{U}$ is the interval $= ]-1, +1[$ choosing $\mathrm{d}(s, \partial \mathcal{U}) = 1 - s^2$, knowing that the general case can be dealt with as easily. The function $V_e$ we obtain is $C^1$, positive definite and proper on $\mathcal{S}_m$. By picking

$$v = \theta_e(x, u) = (1 - u^2) \left[-w_1(x_e) - w_2(x_e)\right] - (u - \theta(x))$$

with:

$$\begin{aligned} w_1(x_e) \quad &= \quad \frac{\partial V_m}{\partial x}(x) \left( \frac{f(x, u) - f(x, \theta(x))}{u - \theta(x)} \right) \\ w_2(x_e) \quad &= \quad -\int_0^1 \frac{2(\theta(x) + s(u - \theta(x)))}{1 - (\theta(x) + s(u - \theta(x)))^2} \ \mathrm{d}s \end{aligned}$$

we obtain:

$$\dot{V}_e(x_e) = -W_e(x_e) = -W_m(x) - \frac{(u - \theta(x))^2}{1 - u^2} \tag{10}$$

where the function $W_e$ is positive definite on $\mathcal{S}_m$. Note that $\theta_e$ may be defined only on $\mathcal{S}_m$. In the following, we need this function to be defined on $\mathbb{R}^{n+1}$. So we may need an extension. See after (11). We denote by $\theta_{e,mod}$ this extension.

Now, as discussed in Section II, to be guaranteed that both stabilizability and observability hold, we consider the open set:

$$\mathcal{SO} = \Big\{(x, u) \in \mathcal{S}_m \ : \ V_e(x, u) < v_\infty\Big\}$$

with $v_\infty$ defined in (3). Then, for $v_i$ in $[0, v_\infty)$, we define the compact set:

$$\Omega_{v_i} = \{x_e \in \mathcal{S}_m : \, V_e(x_e) \leq v_i\} \ .$$

and, given the compact set $C_{x,u}$ of the statement, we consider the following three sets contained in $\mathcal{SO}$ and forward invariant for the extended system under $\theta_{e,mod}$:

$$C_{x,u} \ \subsetneq \ \Omega_{v_1} \ \subsetneq \ \Omega_{v_2} \ \subsetneq \ \Omega_{v_3} \ \subsetneq \ \mathcal{SO} \ . \tag{11}$$

Now we are in position to modify $\theta_e$ if needed. As written above this function may be defined on $\mathcal{S}_m$ only. If this is the case, we modify it as:

$$\begin{aligned} \theta_{e,mod}(x_e) \quad &= \quad \rho(V_e(x_e))\, \theta_e(x_e) \qquad \forall x_e \in \Omega_{v_3} \ , \\ &= \quad 0 \qquad\qquad\qquad\quad\ \forall x_e \notin \Omega_{v_3} \ , \end{aligned}$$

where $\rho : \mathbb{R}_{\geq 0} \to [0, 1]$ is any $C^1$ function satisfying:

$$\begin{aligned} \rho(s) \quad &= \quad 1 \quad \text{if} \quad s \leq v_2 \\ &= \quad 0 \quad \text{if} \quad v_3 < s \ . \end{aligned}$$

In this way we obtain a (extended) state feedback law

$\theta_{e,mod} : \mathbb{R}^{n+1} \to \mathbb{R}$ guaranteeing that $u$ remains in $\mathcal{U}$ and stabilizing the origin with domain of attraction containing $\Omega_{v_2}$. This function $\theta_{e,mod}$ is the one used in (4). In the following we denote by $\theta_{e,mod}$ this function whether it is modified or not.

Finally, the function $\text{sat}_\theta$ used in (4) is the saturation:

$$\text{sat}_\theta(s) = \min\left\{1, \frac{\Theta_{\max}}{|s|}\right\} s \ . \tag{12}$$

where:

$$\Theta_{\max} = \max_{x_e \in \Omega_{v_2}} |\theta_e(x_e)| \ .$$

## IV. Proof of the main result

### A. Analysis of the extended system

The function $\text{sat}_\theta$ is bounded and satisfies:

$$|\text{sat}_\theta(s_a) - \text{sat}_\theta(s_b)| \ \le \ |s_a - s_b| \qquad \forall (s_a, s_b) \in \mathbb{R}^2 \ .$$

Also the function $W_e$ is positive definite, the functions $\frac{\partial V_e}{\partial x_e}$, $f_e$ and $\theta_{e,mod}$ are continuous, and the sets $\Omega_{v_1}$ and $\Omega_{v_2}$ satisfy (11) and are compact. This implies:

1. There exists a positive real number $\overline{W}$ such that:
$$\frac{\partial V_e}{\partial x_e}(x,u) f_e(x,u,\text{sat}_\theta(\theta_{e,mod}(\hat{x},u))) \ \le \ \overline{W} \qquad \forall (\hat{x},x,u) \ : \ (x,u) \in \mathbb{R}^n \times \Omega_{v_2} \ .$$
2. Given any strictly positive real number $\delta_{x0}$, there exists a $\mathcal{K}^\infty$ function $\alpha$ such that:
$$\frac{\partial V_e}{\partial x_e}(x,u) \left[f_e(x,u,\text{sat}_\theta(\theta_{e,mod}(\hat{x},u))) - f_e(x,u,\theta_e(x,u))\right] \le \alpha(|\hat{x}-x|) \qquad \forall (\hat{x},x,u) \ : \ (x,u) \in \Omega_{v_2}, \, |\hat{x}-x| \le \delta_{x0}.$$
3. With this last inequality, let $\delta_{xw}$ be the strictly positive real number defined as:
$$\delta_{xw} = \alpha^{-1}\left(\min_{(x,u)\in\Omega_{v_2}\setminus\Omega_{v_1}} W_e(x,u)\right) \ .$$
There exists a strictly positive real number $\underline{W}$ such that:
$$\frac{\partial V_e}{\partial x_e}(x,u) f_e(x,u,\text{sat}_\theta(\theta_{e,mod}(\hat{x},u))) \ \le \ -\underline{W} \qquad \forall (\hat{x},x,u) : (x,u) \in \Omega_{v_2}\setminus\Omega_{v_1}, \, |\hat{x}-x| \le \min\{\delta_{x0}, \delta_{xw}\}.$$

Collecting all this, we have established:

$$\frac{\partial V_e}{\partial x_e}(x,u) f_e(x,u,\text{sat}_\theta \theta_{e,mod}(\hat{x},u))$$
$$\le \quad \overline{W} \qquad \forall \, (x,u) \in \Omega_{v_2} \ , \ \forall \, \hat{x} \in \mathbb{R}^n \tag{13}$$
$$\le \ -W_e(x,u) + \alpha(|\hat{x}-x|) \qquad \forall (x,u) \in \Omega_{v_2} \ , \ \forall \hat{x} : |\hat{x}-x| \le \delta_{x0} \tag{14}$$
$$\le \ -\underline{W} \qquad \forall \, (x,u) \in \Omega_{v_2}\setminus\Omega_{v_1} \ , \ \forall \, \hat{x} : |\hat{x}-x| \le \min\{\delta_{x0}, \delta_{xw}\} \ . \tag{15}$$

### B. Analysis of the observer

*1) Preamble:* Thanks to the property H5 of $h_2$, the set:

$$\mathcal{C} = \{(\hat{x},x,u) \in \mathcal{O}\times\Omega_{v_2} \, : \, |h_2(\hat{x},u)| \le \tfrac{1}{2}\} \tag{16}$$

is a compact subset of $\mathcal{O}\times\Omega_{v_2}$. Hence $\frac{\partial \Phi}{\partial x}(\hat{x},u)$ is invertible when $(\hat{x},x,u)$ is in $\mathcal{C}$.

On another hand, with the observability assumption and the implicit function theorem, we know the set $\Phi(\mathcal{O}\times\mathcal{U})$ is open and the function:

$$(z,u) \mapsto x \, : \, z = \Phi(x,u)$$

is $C^1$. Since (7) implies:

$$\frac{\partial \Phi}{\partial u}(x,u) = b(\Phi(x,u),u) \ ,$$

we conclude that the function $b$ is defined on the open set $\Phi(\mathcal{O}\times\mathcal{U})\times\mathcal{U}$ where it is $C^1$. Moreover, according to (7), the $i$th component of $b$ depends on $u$ and the first $i$ components of $\Phi(x,u)$, only. It follows that there exist real numbers $L_{\Phi^{-1}}$ and $L_{bi}$ (depending on $\mathcal{C}$) such that we have:

$$|\hat{x}-x| \le L_{\Phi^{-1}}|\Phi(\hat{x},u) - \Phi(x,u)| \qquad \forall (\hat{x},x,u) \in \mathcal{C} \tag{17}$$

and, for each $i$,

$$|b_i(\hat{z},u) - b_i(z,u)| \le L_{bi}\sum_{j=1}^{i}|\hat{z}_j - z_j| \quad \forall (\hat{x},x,u) \in \mathcal{C} \tag{18}$$

with the notation:

$$\hat{z}_i \ = \ \Phi_i(\hat{x},u) \ , \quad z_i \ = \ \Phi_i(x,u) \ .$$

*2) Analysis:* We define the error vector $\tilde{z}$ as made with component:

$$\tilde{z}_i \ = \ \frac{\hat{z}_i - z_i}{\ell^{i-1}} \ = \ \frac{\Phi_i(\hat{x},u) - \Phi_i(x,u)}{\ell^{i-1}} \tag{19}$$

Similarly, the error in the $x$-coordinates is denoted:

$$\tilde{x} = \hat{x} - x \ .$$

Also, in view of (7), we define:

$$\Delta_a(\hat{x},x,u) = \frac{a(\hat{x},u) - a(x,u)}{\ell^{n-1}} \ ,$$
$$\Delta_{bi}(\hat{x},x,u,v,\ell) = \frac{b_i(\hat{x},u) - b_i(x,u)}{\ell^{i-1}} v \ ,$$
$$\Delta \ = \ (\Delta_1, \dots, \Delta_n)^T = \left(\Delta_{b1}, \dots, \Delta_{b(n-1)}, \Delta_{bn} + \Delta_a\right)^T \ .$$

With ignoring the modification $\mathcal{M}$ for the time being, the system and observer dynamics give:

$$\dot{\tilde{z}} \ = \ \ell\,\mathcal{K}\,\tilde{z} \ + \ \Delta(\hat{x},x,u,v,\ell) \ .$$

Our choice of the $k_i$'s implies the existence of a symmetric positive definite matrix $P$ ad a strictly positive real number $d$ satisfying:

$$P\mathcal{K} \ + \ \mathcal{K}^T P \ \le \ -d\,P \ . \tag{20}$$

With this, we define the positive definite function $U$ as:

$$U(\tilde{z}) \ = \ \tilde{z}^T P\,\tilde{z} \ . \tag{21}$$

We are interested in this function since, with (17), we have the existence of a strictly positive real number $L_U$ such that:

$$|\tilde{x}| \ \le \ L_U \ell^{n-1}\sqrt{U(\tilde{z})} \qquad \forall (\hat{x},x,u) \in \mathcal{C} \ . \tag{22}$$

Also, still with ignoring the modification $\mathcal{M}$, we have:

$$\dot{U}(\tilde{z}) \ \le \ -\ell\,d\,U(\tilde{z}) + 2\tilde{z}^T P \Delta(\hat{x},x,u,v,\ell) \ . \tag{23}$$

The modification $\mathcal{M}$ augments $\dot{U}(\tilde{z})$ with

$$\gamma(\hat{x},u)\tilde{z}^T\mathcal{L}\left(\frac{\partial\Phi}{\partial x}(\hat{x},u)\right)^{-1T}\frac{\partial h_2}{\partial x}(\hat{x},u)^T h_2(\hat{x},u)$$
$$= \gamma(\hat{x},u)\left[\Phi(\hat{x},u)-\Phi(x,u)\right]^T\left(\frac{\partial\Phi}{\partial x}(\hat{x},u)\right)^{-1T}\times$$
$$\times\frac{\partial h_2}{\partial x}(\hat{x},u)^T h_2(\hat{x},u)\ .$$

But, with the convexity property of $h_2$ in H3 , when $h_2(\hat{x},u)$ is in $[0,1]$ and $h_2(x,u)$ is zero, we have

$$0\ \le\ [\Phi(\hat{x},u)-\Phi(x,u)]^T\left(\frac{\partial\Phi}{\partial x}(\hat{x},u)\right)^{-1T}\times$$
$$\times\frac{\partial h_2}{\partial x}(\hat{x},u)^T h_2(\hat{x},u)\ .$$

We conclude that (23) holds even with the modification $\mathcal{M}$.

Moreover, we compute:

$$\begin{aligned}&\dot{h}_2(\hat{x},u)\\ &= \frac{\partial h_2}{\partial u}(\hat{x},u)\dot{u}+\frac{\partial h_2}{\partial x}(\hat{x},u)\dot{\hat{x}}\\ &= T_x(\hat{x},u)+T_u(\hat{x},u)\\ &-\gamma(\hat{x},u)\left|P^{-\frac{1}{2}}\mathcal{L}\left(\frac{\partial\Phi}{\partial x}(\hat{x},u)\right)^{-1T}\frac{\partial h_2}{\partial x}(\hat{x},u)^T\right|^2 h_2(\hat{x},u)\end{aligned}$$

where we have let:

$$T_x(\hat{x},u)=$$
$$\frac{\partial h_2}{\partial x}(\hat{x},u)\left[f(\hat{x},u,\dot{u})+\left(\frac{\partial\Phi}{\partial x}(\hat{x},u)\right)^{-1}K\mathcal{L}[y-h(\hat{x},u)]\right]$$
$$T_u(\hat{x},u)=\frac{\partial h_2}{\partial x}(\hat{x},u)\,\mathrm{sat}_\theta(\theta_{e,mod}(\hat{x},u))$$

This motivates us for choosing $\gamma$ satisfying:

$$\gamma(\hat{x},u)\ge 4\frac{(4h_2(x,u))^2\left[T_x(\hat{x},u)+T_u(\hat{x},u)\right]}{\left|P^{-\frac{1}{2}}\mathcal{L}\left(\frac{\partial\Phi}{\partial x}(\hat{x},u)\right)^{-1T}\frac{\partial h_2}{\partial x}(\hat{x},u)^T\right|^2}\ . \quad (24)$$

Thanks to H2, the function $(x,u)\mapsto\gamma(x,u)$ defined this way is continuous on $\mathcal{O}\times\mathcal{U}$. So we can use $\gamma(\hat{x},u)$ as long as $(\hat{x},u)$ is in $\mathcal{O}\times\mathcal{U}$. It implies that $\dot{h}_2(\hat{x},u)$ is strictly negative when $h_2(\hat{x},u)$ is strictly larger than $\frac{1}{4}$. With uniqueness of solutions, this implies that, for each $s$ in $[\frac{1}{4},1]$ the set $\{(\hat{x},u):\ h_2(\hat{x},u)\le s\}$ is forward invariant.

Now, with (18), we know there exists a real number $L$ such that, for all $(\hat{x},x,u,v)\in\mathcal{C}\times[-\Theta_{\max},\Theta_{\max}]$, and $\ell\ge 1$, we have:

$$|\Delta_i(\hat{x},x,u,v,\ell)|\ \le\ L\sum_{j=1}^{i}\frac{|\tilde{z}_j|}{\ell^{i-j}}\ .$$

This implies:

$$\tilde{z}^T P\Delta(\tilde{z},\hat{z},u,v,\ell)\le L\,n\,|P\tilde{z}|\,|\tilde{z}|\ \le\ L\,n\,\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}U(\tilde{z})\ ,$$

where $\lambda_{\min}(P)$ is the smallest eigenvalue of $P$ and $\lambda_{\max}(P)$ the largest one. Thus, from (23), we get:

$$\dot{U}(\tilde{z})\ \le\ -\left[\ell\,d-2L\,n\,\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\right]U(\tilde{z}) \quad (25)$$
$$\forall(\hat{x},x,u,v)\ :\ (\hat{x},x,u,v)\in\mathcal{C}\times[-\Theta_{\max},\Theta_{\max}]\ .$$

*C. Stability*

Let $\ell$ be fixed satisfying:

$$\ell\ \ge\ \ell_0\ =\ \max\left\{1,\frac{2L\,n}{d}\,\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\right\}\ .$$

Let $\Gamma$ be the compact set:

$$\Gamma=\{(\hat{x},x,u)\in\mathcal{C}\ :\ V_e(x,u)\le\ v_1\ ,\ |\tilde{x}|\ \le\ \delta_{x0}\}.$$

Since it is a subset of $\mathcal{C}$, the solutions of the closed loop system are well defined as long as they are in its interior $\mathring{\Gamma}$. Moreover the inequalities (14) and (25) are satisfied at all the points in this set. Also, since $\Phi$ is a $C^1$ and $\ell$ larger than 1, there exists a real $L_\Phi$ such that:

$$|\tilde{z}|\ \le\ L_\Phi\,|\tilde{x}| \qquad \forall(\hat{x},x,u)\in\Gamma\ . \quad (26)$$

Finally let $\mathcal{N}_\ell$ be an open neighborhood of the origin contained in $\mathring{\Gamma}$ where $(\hat{x},x,u)$ satisfies:

$$V_e(x,u)\ +\ \alpha\left(L_U\ell^{n-1}\sqrt{\lambda_{\max}(P)}L_\Phi|\hat{x}-x|\right)\ <\ \frac{v_1}{2}\ ,$$
$$|\hat{x}-x|\ <\ \frac{\delta_{x0}}{2L_U\ell^{n-1}\sqrt{\lambda_{\max}(P)}L_\Phi}\ .$$

Now, consider a solution of the closed loop system (2), (4), starting from any point $(\hat{x},x,u)\in\mathcal{N}_\ell$. Let $[0,T[$ be its right maximal interval of definition when it takes its value in the open set $\mathring{\Gamma}$. To simplify the notation we add $(t)$ to denote those variables which are evaluated along this solution. With (25) and (26) we have:

$$U(\tilde{z}(t))\le U(\tilde{z}(0))\le\lambda_{\max}(P)L_\Phi^2|\tilde{x}(0)|^2 \qquad \forall t\in[0,T[\ ,$$

which implies, with (22),

$$|\tilde{x}(t)|\le L_U\ell^{n-1}\sqrt{\lambda_{\max}(P)}L_\Phi|\tilde{x}(0)|<\frac{\delta_{x0}}{2} \qquad \forall\ t\in[0,T[\,. \quad (27)$$

This inequality and (14), where $W_e$ is non negative, give:

$$\begin{aligned}&V_e(x(t),u(t))\\ &\le V_e(x(0),u(0))+\alpha\Big(L_U\ell^{n-1}\sqrt{\lambda_{\max}(P)}L_\Phi|\tilde{x}(0)|\Big)\ , \quad (28)\\ &<\frac{v_1}{2} \qquad \forall\ t\in[0,T[\ .\end{aligned}$$

Thus, if the initial condition $(\hat{x}(0),x(0),u(0))$ is in $\mathcal{N}_\ell$, then the solution remains inside a strict subset of $\mathring{\Gamma}$. Hence $T$ is infinite and from (27) and (28) we can conclude that the origin is stable.

*D. Attractiveness*

Consider again a solution of the closed loop system with initial condition $(\hat{x},x,u)$ such that $(x,u)$ is in $C_{x,u}$ and $\hat{x}$ satisfies $h_2(\hat{x},u)<\frac{1}{2}$. This initial condition is in the interior $\mathring{\mathcal{C}}$ of the compact set $\mathcal{C}$ defined in (16). Let $[0,T[$ be its right maximal interval of definition when it takes its values in the open set $\mathring{\mathcal{C}}$ contained in $\mathbb{R}^n\times\Omega_{v_2}$. Hence, with the definition of $\mathrm{sat}_\theta$, inequalities (25) and (13) are satisfied at any point visited by the solution. So we have, for all $t$ in $[0,T[$,

$$U(\tilde{z}(t))\le\exp\left(-t\left[\ell d-2Ln\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\right]\right)U(\tilde{z}(0))\ , \quad (29)$$

$$V_e(x(t),u(t)) \leq V_e(x(0),u(0)) + \overline{W}\,t \leq v_1 + \overline{W}\,t .$$

Since $V_e(x(t),u(t))$ is smaller than $v_2$, this implies:

$$T \geq \frac{v_2 - v_1}{\overline{W}}$$

$$V_e\left(x\left(\frac{v_2-v_1}{2\overline{W}}\right), u\left(\frac{v_2-v_1}{2\overline{W}}\right)\right) \leq \frac{v_2+v_1}{2} < v_2 . \tag{30}$$

Then let $U_{\max}$ be the real number defined as:

$$U_{\max} = \sup_{(\hat{x},x,u,\ell)\in\mathcal{C}\times[1,+\infty[} U(\tilde{z}) .$$

There exists $\ell_1$ satisfying:

$$\ell^{2(n-1)} \exp\left(-\frac{v_2-v_1}{2\overline{W}}\ell d + \frac{(v_2-v_1)Ln}{\overline{W}}\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\right) U_{\max} \leq \left[\frac{\min\{\delta_{x0},\delta_{xw}\}}{L_U}\right]^2 \qquad \forall\, \ell \geq \ell_1 .$$

Let $\ell$ be fixed satisfying:

$$\ell \geq \underline{\ell} = \max\{\ell_0, \ell_1\} .$$

From inequalities (29) and (22), we obtain:

$$U(\tilde{z}(t)) \leq \frac{\left[\frac{\min\{\delta_{x0},\delta_{xw}\}}{L_U}\right]^2}{\ell^{2(n-1)}} \qquad \forall\, t \in [\tfrac{v_2-v_1}{2\overline{W}}, T[$$

$$|\tilde{x}(t)| \leq \min\{\delta_{x0},\delta_{xw}\} \qquad \forall\, t \in [\tfrac{v_2-v_1}{2\overline{W}}, T[ . \tag{31}$$

Then, with (15) and (30), we obtain:

$$\max\{V_e(x(t),u(t)), v_1\} \leq \max\left\{V_e\left(x\left(\frac{v_2-v_1}{2\overline{W}}\right), u\left(\frac{v_2-v_1}{2\overline{W}}\right)\right), v_1\right\} < v_2 \tag{32}$$

for all $t$ in $[\frac{v_2-v_1}{2\overline{W}}, T[$. On another hand, we know the set $\{(\hat{x},u) : h_2(\hat{x},u) \leq \max\{h_2(\hat{x}(0),u(0),\frac{1}{4}\}\}$ is forward invariant. We have established that the solution cannot reach the boundary of $\mathring{\widehat{\mathcal{C}}}$ on $[0,T[$. This implies that $T$ is infinite and that the solution remains in $\mathcal{C}$ for all $t$ in $\mathbb{R}_{\geq 0}$. So inequalities (31) and (32) and therefore inequalities (14) and (25) hold for all $t$ larger than $\frac{v_2-v_1}{2\overline{W}}$. From LaSalle invariance principle, we can conclude:

$$\lim_{t\to+\infty} V_e(x(t),u(t)) + U(\tilde{z}(t)) = 0 .$$

and thus that the solution of the closed loop system converges to the origin provided its initial condition $(\hat{x}(0),x(0),u(0))$ is such that $(x(0),u(0))$ is in $C_{x,u}$ and $\hat{x}(0)$ satisfies $h_2(\hat{x}(0),u(0)) < \frac{1}{2}$.

## V. Conclusions

The results presented in this paper are in line with the many contributions on stabilization by output feedback designed from a separation principle with a high gain observer. Our objective is to propose a design more usable in applications. It applies to systems written in a generic form. It assumes the knowledge of a stabilizing state feedback and a (local) uniform complete observability property holds. It takes advantage of the fact that, though possibly non affine in the control, the system can be written in an observability feedback form where one input derivative only is needed. As a consequence, a dynamic extension with only the control as extra state component is needed for the state feedback. On the observer side, we propose a high gain observer in the original coordinates and we use an extra dummy measured output to round the problems of peaking and local observability.

## Appendix

### *Existence of $h_2$ satisfying H1 to H5*

Since the origin is in $\mathcal{O}$, we have:

$$\left|\mathtt{det}\left(\frac{\partial\Phi}{\partial x}(0,u)\right)\right| \neq 0 \qquad \forall u \in \mathcal{U}$$

This determinant has constant sign in $u$ since $\mathcal{U}$ is connected. Let $Q : \mathcal{U} \to \mathbb{R}^{n*n}$ be a $C^1$ function with positive definite symmetric matrices as values. Given an input $u$ in $\mathcal{U}$, let $\mathcal{R}(u)$ be the subset of real numbers $r$ such that the set $\{z \in \mathbb{R}^n : z^T Q(u) z \leq r\}$ is contained in $\Phi(\mathcal{O}\times\{u\})$. We define the function:

$$\Psi(u) = \sup_{r\in\mathcal{R}(u)} r$$

It takes strictly positive values on $\mathcal{U}$. Let $\Psi_s$ be a $C^1$ function lower bounding it but with still strictly positive values on $\mathcal{U}$. Then we select a real number $\mu$ in $(0,1)$ and let:

$$h_2(x,u) = \max\left\{\frac{\Phi(x,u)^T Q(u)\Phi(x,u)}{\Psi_s(u)} - \mu\,,\, 0\right\}^2 .$$

With the property O3 of $\Phi$, we can verify that Properties H1 to H5 are satisfied.

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