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A weak version of the small-gain theorem

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Abstract— **A weak version of the small-gain theorem is derived. Connections with the classical *linear* and *nonlinear* small-gain conditions are established. The necessity of the weak small-gain conditions is discussed.**

I. Introduction

Small-gain theorems have been widely used to establish stability properties of nonlinear interconnected systems. It is possible to provide several versions of the small-gain theorem, depending of the input-output property that is used to quantify the input-output behavior of the interconnected subsystems. Possible selections include the L_2-gain, yielding an L_2 small-gain theorem [11], [10] (which generalizes to the nonlinear setting the linear H_∞ small-gain theorem [4]), and the property of Input-to-State Stability (ISS), which leads to the derivation of nonlinear small-gain theorems, such as the one in [7]. Other versions of the small-gain theorem have been developed in [5], [1], [6], in which interconnections of possibly non-ISS subsystems have been considered. Finally, small-gain theorems for large scale interconnected systems and for systems interconnected by means of communication channels have recently been developed in [3].

The purpose of this paper is to develop a weak version of the small-gain theorem, in the spirit of the Matrosov theorem derived in [2]. As a matter of fact, the paper partly extends, to a class of interconnected systems, the results therein which provide a weak version of Matrosov theorem. Note, however, that the results in [2] are somewhat stronger, since under some stability assumptions it is possible to establish strong convergence claims.

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We consider a nonlinear system described by equations of the form

$$\dot{x} \;=\; f(x) \,, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state of the system, and f is locally Lipschitz continuous. In addition, without loss of generality, we assume that $x = 0$ is an equilibrium of the system.

The ISS small-gain theorem, see [7], allows to establish asymptotic stability of the equilibrium of the system (1) when there exist

- two C^1 functions $V_i : \mathbb{R}^n \to \mathbb{R}_+$ such that $V_1 + V_2$ is positive definite and radially unbounded,
- two class K_∞ functions, $\alpha_i : \mathbb{R}_+ \to \mathbb{R}_+$ and two continuous functions $\beta_i : \mathbb{R}_+ \to \mathbb{R}_+$,

satisfying, along the solutions of system (1), the differential inequalities

$$\begin{array}{rcl} \dot{V}_1 & \leq & -\alpha_1(V_1) + \beta_1(V_2), \\ \dot{V}_2 & \leq & -\alpha_2(V_2) + \beta_2(V_1), \end{array} \tag{2}$$

and the small-gain condition

$$\beta_2 \circ \alpha_1^{-1} \circ \beta_1 \circ \alpha_2^{-1} < Id, \tag{3}$$

where Id is the identity map.

The problem that we address in this paper is to study what happens *relaxing* the inequalities (2). This relaxation can be carried out in various directions. In particular, we are interested in the case in which the argument of the functions α_i and β_j are not the functions V_k, but some other functions $h_i : \mathbb{R}^n \to \mathbb{R}_+$ so that, along the solutions of system (1), we have

$$\begin{array}{rcl} \dot{V}_1 & \leq & -\alpha_1(h_1(x)) + \beta_1(h_2(x)), \\ \dot{V}_2 & \leq & -\alpha_2(h_2(x)) + \beta_2(h_1(x)). \end{array} \tag{4}$$

Remark 1: Under additional assumptions on the functions V_i the inequalities (4) may be exploited to establish boundedness of all solutions of the system (1).

Remark 2: In the considered set up, borrowing from LaSalle invariance principle, and from the classical small-gain theorem, one may be tempted to conjecture that the ω-limit set of the solutions of the system (1) is contained in the largest invariant set such that

$$\begin{array}{rcl} 0 & = & -\alpha_1(h_1(x)) + \beta_1(h_2(x)), \\ 0 & = & -\alpha_2(h_2(x)) + \beta_2(h_1(x)). \end{array} \quad (5)$$

This, unfortunately, is not true in general.

Remark 3: The small-gain condition (3) and the inequalities (2) imply that the equations

$$\begin{array}{rcl} 0 & = & -\alpha_1(V_1(x)) + \beta_1(V_2(x)), \\ 0 & = & -\alpha_2(V_2(x)) + \beta_2(V_1(x)), \end{array}$$

have the unique solution $x = 0$, *i.e.* that thesystem (1) has a unique equilibrium. This is, however, not implied by the inequalities (5).

Remark 4: The differential inequalities in [2] are a special case of the inequalities (4), obtained by setting β_1 to zero. This selection yields a triangular structure of the inequalities which, exploiting properties of asymptotically autonomous vector fields [8], dictates very specific properties for the ω-limit set of the solutions of the underlying system. In particular the ω-limit set is a chain recurrent set. This property is however lost in the current scenario, since there is no *driving inequality*.

The paper is organized as follows. In Section II a preliminary lemma, which generalizes the result in [2] and introduces a new small-gain condition, is stated. Section III discusses the new small-gain condition, establishes connections with the classical, nonlinear, small-gain condition, and clarifies the necessity of the new small-gain property. Section IV provides the main result of the paper, namely a weak version of the small-gain theorem. Finally, Section V contains a simple example and Section VI contains a few concluding remarks and observations.

II. A PRELIMINARY RESULT

This section contains a preliminary result which is instrumental to establish the weak small-gain theorem formulated in Section IV.

Lemma 1: Let $i = 1, 2$. Let $a_i : \mathbb{R}_+ \to [-\bar{a}, \bar{a}]$, be bounded absolutely continuous functions and $b_i : \mathbb{R}_+ \to [0, \bar{b}]$ be bounded, piecewise continuous, functions.

Assume there exist continuous positive definite functions $\alpha_i : \mathbb{R}_+ \to \mathbb{R}_+$, continuous functions $\beta_i : \mathbb{R}_+ \to \mathbb{R}_+$, which are zero at zero, and a real number ε in $]0, 1[$ such that the following hold.

1) The differential inequalities

$$\begin{array}{rcl} \dot{a}_1(t) & \leq & -\alpha_1(b_1(t)) + \beta_1(b_2(t)), \\ \dot{a}_2(t) & \leq & -\alpha_2(b_2(t)) + \beta_2(b_1(t)) \end{array} \quad (6)$$

hold for almost all t in $\mathbb{R}_+$.

2) The small-gain like condition

$$\beta_1(b_2)\beta_2(b_1) \leq (1 - \varepsilon)\alpha_2(b_2)\alpha_1(b_1) \quad (7)$$

holds for all (b_1, b_2) in $[0, \bar{b}]^2$.

Then

$$\liminf_{t \to +\infty} [b_1(t) + b_2(t)] \ = \ 0. \quad (8)$$

III. THE SMALL-GAIN CONDITION (7)

In this section we study the condition (7) and we relate this condition with the classical nonlinear small-gain condition.

To start with, we observe that, if there exist real numbers ψ_1 and ψ_2 such that

$$\psi_1 \ = \ \sup_{b_1 \in]0, \bar{b}]} \frac{\beta_2(b_1)}{\alpha_1(b_1)}, \qquad \psi_2 \ = \ \sup_{b_2 \in]0, \bar{b}]} \frac{\beta_1(b_2)}{\alpha_2(b_2)} \ ,$$

then the condition

$$\psi_1 \, \psi_2 \ \leq \ (1 - \varepsilon) \quad (9)$$

implies condition (7). The converse statement is also true. Namely, if condition (7) holds then the numbers ψ_1 and ψ_2 exist and satisfy condition (9).

This property justifies the terminology ”linear small-gain condition” for condition (7).

We are now ready to relate the condition (7) to the classical nonlinear small-gain condition. To this end, and to simplify the discussion, assume that the functions β_i and α_i are defined on $\mathbb{R}_+$ and that the functions α_i are invertible. Assume also that $\bar{b}$, in (7), is infinity. Then, from the theory of interconnected nonlinear systems we would expect that stability properties be related to the nonlinear small-gain condition (4), namely

$$\beta_2 \circ \alpha_1^{-1} \circ \beta_1 \circ \alpha_2^{-1}(s) \ < \ s \qquad \forall s > 0. \quad (10)$$

Lemma 2: Condition (7) implies, but it is not implied by, condition (10).

While necessity of the small-gain condition (7) is difficult to establish, we now show that violation of the non-strict inequality yields the existence of functions a_i and b_i such that the convergence result of Lemma 1 does not hold.

Lemma 3: Assume there exist strictly positive real numbers b_{1a}, b_{2b} and b_{2c} such that

$$\frac{\beta_1(b_{2b})\beta_2(b_{1a})}{\alpha_2(b_{2b})\alpha_1(b_{1a})} > 1 \qquad \frac{\beta_1(b_{2c})\beta_2(b_{1a})}{\alpha_2(b_{2c})\alpha_1(b_{1a})} < 1. \quad (11)$$

Then there exist functions a_i and b_i such that the convergence result in Lemma 1 does not hold.

To illustrate the result in Lemma 3 consider the differential inequalities

$$\dot{a}_1 \leq -a_1 + b_2^2, \qquad \dot{a}_2 \leq -b_2 + \gamma\sqrt{a_1},$$

with $\gamma > 0$. Note that the linear small-gain condition is violated, while the nonlinear one holds for $\gamma < 1$.

Let k be in $]0, 1[$ and

$$b_2(t) = \sqrt{1 - k\cos(t)}.$$

Then

$$a_1(t) = 1 - \frac{k}{\sqrt{2}} \cos\left(t - \frac{\pi}{4}\right)$$

is a solution of the first inequality and

$$\liminf_{t\to+\infty} [b_1(t) + b_2(t)] > 0.$$

To conclude, it remains to establish that we can find a bounded absolutely continuous function a_2 which satisfies the second differential inequality. To this end note that, for all k in $]0, 1[$,

$$\rho(k) = \frac{\displaystyle\int_0^{2\pi} \sqrt{1 - \frac{k}{\sqrt{2}}\cos\left(t - \frac{\pi}{4}\right)}dt}{\displaystyle\int_0^{2\pi} \sqrt{1 - k\cos(t)}dt} \geq 1.$$

As a result, for all γ in $]1/\rho(k), 1[$,

$$\lim_{t\to+\infty} \int_0^t \left[-b_2(s) + \gamma\sqrt{a_1(s)}\right] ds \; = \; +\infty,$$

which implies that a function a_2 does exist[1].

IV. A WEAK SMALL-GAIN THEOREM

In this section we state the main result of the paper, namely a weak version of the small-gain theorem.

Theorem 1: Consider the nonlinear, time-invariant, system (1). Suppose there exist continuous functions $\beta_i : \mathbb{R}_+ \to \mathbb{R}_+$, which are zero at zero, C^1 functions $V_i : \mathbb{R}^n \to \mathbb{R}$, continuous functions $h_i : \mathbb{R}^n \to \mathbb{R}_+$, continuous positive definite functions $\alpha_i : \mathbb{R}_+ \to \mathbb{R}_+$, such that the conditions (4) hold. Suppose in addition that we have

$$\beta_1(b_2)\,\beta_2(b_1) \leq (1-\varepsilon)\,\alpha_1(b_1)\,\alpha_2(b_2), \qquad (12)$$

for some $\epsilon > 0$ and all non-negative b_1 and b_2.

Then, for any bounded solutions of system (1),

$$\liminf_{t\to+\infty} [h_1(x(t)) + h_2(x(t))] = 0. \qquad (13)$$

Moreover, if the largest invariant set $\mathcal{N}$ contained in the set

$$\{x \in R^n \, : \, h_1(x) = h_2(x) = 0\},$$

is stable, then

$$\lim_{t\to+\infty} h_1(x(t)) + h_2(x(t)) = 0. \qquad (14)$$

Remark 5: As explained in Section III it is not possible, in general, to obtain stronger convergence results, for example asymptotic convergence to zero of $h_1(x(t)) + h_2(x(t))$, nor to relax the linear small-gain condition (12).

Remark 6: The last point in Theorem 1 rephrases a well-known fact, see for instance [9, Lemma I.4].

V. AN ILLUSTRATIVE EXAMPLE

In this section we illustrate some of the ideas and results established by means of a simple example.

Consider the system

$$\begin{aligned} \dot{x}_1 &= (x_{1+}^2 + x_3^2)\,x_2, \\ \dot{x}_2 &= -(x_{1+}^2 + x_3^2)\,x_1 \\ \dot{x}_3 &= -x_3^3 \; + \; x_{1+}^{9/2}. \end{aligned} \qquad (15)$$

Note that all solutions are bounded, since

$$\dot{\overbrace{x_1^2 + x_2^2}} \; = \; 0,$$

and the x_3 sub-system is ISS. Therefore, for any solution, there exists a constant c such that $x_1^2 + x_2^2$ and x_3^2 are bounded by c^2. In what follows we assume that these bounds hold. To apply the small-gain theorem let

$$V_1(x) = x_2\;, \quad V_2(x) \; = \; \frac{k}{2}\,x_3^2\;.$$

Then, Young's inequality yields

$$\begin{aligned} \dot{V}_1 &= -(x_{1+}^2 + x_3^2)\,x_{1+} + (x_{1+}^2 + x_3^2)\,|x_{1-}| \\ &\leq -x_{1+}^3 + c\,x_3^2 \end{aligned}$$

$$\begin{aligned} \dot{V}_2 &= -k\,x_3^4 \; + \; k\,x_3\,x_{1+}^{9/2} \\ &\leq -\left(k - \frac{k^4\ell^8}{4}\right)x_3^4 \; + \; \frac{3}{4}\left(\frac{k}{\ell^2}\right)^{3/4} x_{1+}^6, \end{aligned}$$

[1]For example $a_2(t) \; = \; \mathtt{sat}\left(\int_0^t \left[-b_2(s) + \gamma\sqrt{a_1(s)}\right] ds\right)$.

which motivate the choice

$$h_1(x) = x_{1+}^3, \qquad h_2(x) = x_3^2,$$
$$\alpha_1(s) = s, \qquad \beta_1(s) = cs \ ,$$
$$\alpha_2(s) = \left(k - \tfrac{k^4\ell^2}{4}\right) s^2, \qquad \beta_2(s) = \tfrac{3}{4}\left(\tfrac{k}{\sqrt{\ell}}\right)^{3/4} s^2 \ .$$

The linear small-gain condition does not hold for such functions, although the nonlinear one does hold selecting

$$k < \left(\frac{16}{27c^6}\right)^{1/4}, \qquad \ell \ = \ \frac{\left(27c^6)\right)^{1/8}}{k} \ .$$

Note that the conclusion of Theorem 1 would be, in this case,

$$\liminf_{t\to+\infty} \left[x_{1+}(t)^3 + x_3(t)^2\right] \ = \ 0 \ . \qquad (16)$$

As we shall prove this is not the case if $x_3(0)$ is nonzero. To this end, note that, at any equilibrium, $x_{1+} = x_3 = 0$. Moreover, since we have a locally Lipschitz system, any solution not starting from an equilibrium cannot reach an equilibrium in finite time. As a result, along any solution $x_{1+}^2 + x_3^2$ remains strictly positive.

Consider now a solution with $x_3(0) \neq 0$. The above remark motivates the introduction of the function

$$\tau(t) \ = \ \int_0^t (x_{1+}(t)^2 + x_3(t)^2)\, dt$$

where $x_{1+}(t)^2 + x_3(t)$ is obtained from the solution. The function is strictly increasing and, since

$$|x_3(t)| \ \geq \ \exp\left(-\int_0^t x_3(s)^2 ds\right) |x_3(0)| \ ,$$

the integral $\int_0^t x_3(s)^2 ds$ and therefore $\tau(t)$ go to $+\infty$ as t goes to ∞. Therefore there exists a time t_0 such that $\tau(t)$ is larger than 2π for all $t \geq t_0$.

Using τ, we can express the (x_1, x_2)-components of the solution as

$$x_1(t) + ix_2(t) \ = \ \exp(-i\tau(t))\, x_1(0) + ix_2(0)$$

where $i^2 = -1$. It follows that, in any interval $[\tau(s), \tau(s) + 2\pi]$, there exists an interval of length $\frac{\pi}{2}$ in which x_1 and therefore x_{1+} is larger than or equal to $\frac{\sqrt{2[x_1(0)^2+x_2(0)^2]}}{4}$. As a result, for all $t \geq t_0$,

$$\begin{aligned}\int_0^{\tau(t)} & \exp(\tau(s))x_{1+}(\tau(s))^{9/2} ds \\ & \geq \sum_{k=0}^{K(t)} \exp(2k\pi)\, \tfrac{\pi}{2} \left(\tfrac{\sqrt{2[x_1(0)^2+x_2(0)^2]}}{4}\right)^{9/2} , \\ & \geq \tfrac{\pi}{2} \tfrac{\exp(2(K(t)+1)\pi)-1}{\exp(2\pi)-1} \left(\tfrac{\sqrt{2[x_1(0)^2+x_2(0)^2]}}{4}\right)^{9/2} ,\end{aligned}$$

where $K(t)$ is the largest integer k satisfying $\tau(t) \geq 2k\pi$.

Consider now the identity

$$-x_3^3 \ + \ x_{1+}^{9/2} \ = \ -(x_{1+}^2 + x_3^2)\, x_3 \ + \ (x_{1+}^2 x_3 + x_{1+}^{9/2})$$

and the bound

$$\frac{x_{1+}^2 x_3 + x_{1+}^{9/2}}{x_{1+}^2 + x_3^2} \ \geq \ \frac{x_{1+}^{9/2}}{c^2} \ ,$$

yielding

$$\begin{aligned}\exp(\tau(t))\, x_3(t) \ - \ x_3(0) & \\ & \geq \frac{1}{c^2} \int_0^{\tau(t)} \exp(\tau(s)) x_{1+}(\tau(s))^{9/2} ds \\ & \geq \tfrac{1}{c^2} \tfrac{\pi}{2} \tfrac{\exp(2(K(t)+1)\pi)-1}{\exp(2\pi)-1} \left(\tfrac{\sqrt{2[x_1(0)^2+x_2(0)^2]}}{4}\right)^{9/2}\end{aligned}$$

Finally, exploiting the conditions

$$\lim_{t\to+\infty} \tau(t) \ = \ +\infty \ ,$$
$$1 \ \leq \ \exp([2(K(t)+1)\pi] - \tau(t)) \ \leq \ \exp(2\pi) \ ,$$

we conclude

$$\begin{aligned}\liminf_{t\to+\infty} x_3(t) \ & \geq \ \tfrac{1}{c^2} \tfrac{\pi}{2} \tfrac{1}{\exp(2\pi)-1} \left[\tfrac{\sqrt{2[x_1(0)^2+x_2(0)^2]}}{4}\right]^{9/2} \\ & > \ 0,\end{aligned}$$

which shows that condition (16) does not hold.

Figure 1 displays the state histories of system (15) for the initial condition $(x_1(0), x_2(0), x_3(0)) = (1, 0, 1)$. Note that the x_3 component of the state is bounded away from zero.

VI. Conclusion

A weak version of the small-gain theorem has been established. This result relies upon the properties of a set of differential inequalities together with a linear small-gain condition. The paper provides a non-trivial generalization of the results in [2] in which cascaded systems have been studied.

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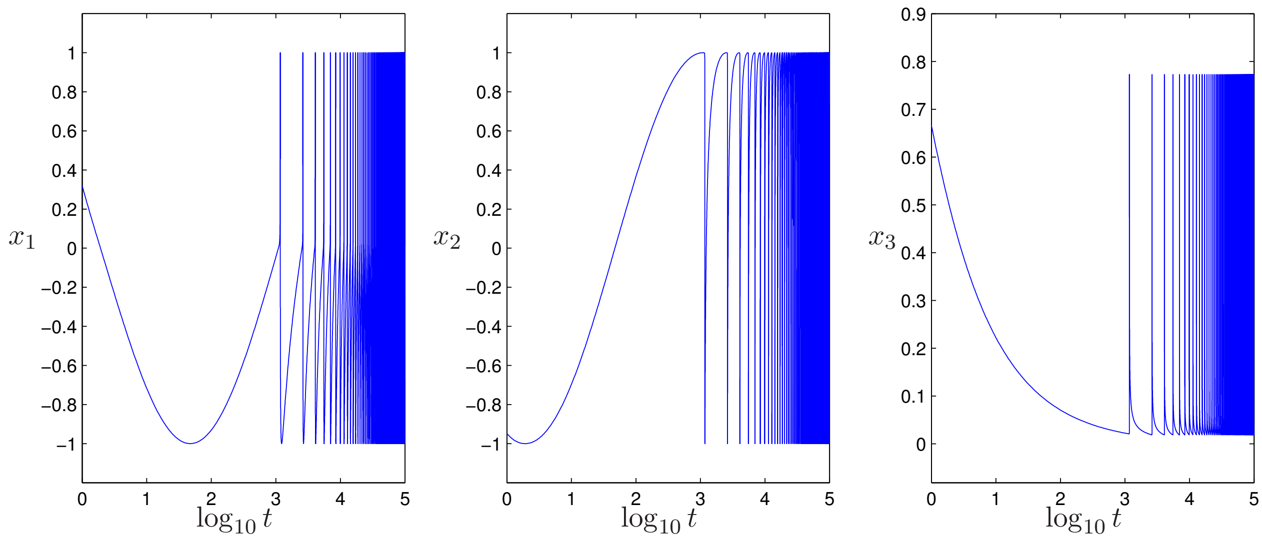

Fig. 1. State histories of the system (15).

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