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Adding an integration with prescribed local behavior

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Abstract— **Among the non-linear control techniques, some Lyapunov design methods (Forwarding / Backstepping) take advantage of the structure of the system (Feedforward-form / Feedback-form) to formulate a continuous control law which stabilizes globally and asymptotically the equilibrium. In addition to stabilization, we focus on the local behaviour of the closed loop system, providing conditions under which we can predetermine the behaviour around the origin for Feedforward systems.**

Index Terms— **Stabilization, Lyapunov design, Forwarding, First order approximation, Feedforward form.**

I. Introduction

The synthesis of a stabilizing control law for systems described by nonlinear differential equations has been the subject of great interest for the scientific community during the last three decades. Depending on the structure of the model, some techniques are now available and in use to synthesize control laws ensuring global and asymptotic stabilization of the equilibrium point.

For instance, we can refer to the popular backstepping approach (see [1] and the reference therein), or the forwarding approach (see [2], [3], [4]) and some others based on energy considerations (see [5]) for a survey of the available approaches.

Although the global asymptotic stability of the equilibrium point can be achieved in some specific cases, it remains difficult to address performance issues of a nonlinear system in a closed loop. However, when first order approximation of the non-linear model is considered, some performance aspects can be addressed by using linear optimal control techniques (using LQ controller for instance)

Hence, it is interesting to raise the question of synthesizing a nonlinear control law which guarantees the global asymptotic stability of the origin while ensuring a prescribed local linear behavior. This type of question has been discussed already in the literature when backstepping design is used to synthesize a nonlinear continuous control law (see [6]).

In the present paper, we consider the same problem in the case of a system whose structure allows forwarding design technique (see [2], [3])).

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The paper is organized as follows : In Section II, the problem under consideration is described. Section III is devoted to the statement of the main theorem and to its discussion in the case of systems that are obtained after adding one integrator. The proof of this result is detailed in Section IV. Finally, Section V gives the conclusion.

II. Problem description

To present the problem under consideration, we introduce a general controlled nonlinear system described by the following ordinary differential equation:

$$\dot{\xi} = \Phi(\xi, u) \ , \tag{1}$$

with ξ in $\mathbb{R}^N$ and $\Phi : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$ is a C^1 function such that $\Phi(0,0) = 0$ and u is a scalar control law. For this system, we can introduce the matrices describing its first order approximation which is assumed to be controllable:

$$\frac{\partial \Phi}{\partial \xi}(0,0) \ , \ \frac{\partial \Phi}{\partial u}(0,0) \ .$$

For system (1), the problem we intend to solve can be described as follows:

Stabilization with prescribed local behavior:

Assume the linear state feedback law $u = K_o \xi$ stabilizes the first order approximation of system (1). We are looking for a stabilizing control law $\alpha_o : \mathbb{R}^n \to \mathbb{R}$, differentiable at 0 such that:

1) the origin of the system:

$$\dot{\xi} = \Phi(\xi, \alpha_o(\xi)) \ ,$$

is globally and asymptotically stable.

2) The first order approximation of the control law α_o satisfies:

$$\frac{\partial \alpha_o}{\partial \xi}(0) = K_o \ .$$

In the case where the system (1) is in Feedback form, this problem has been solved in [6]. It should also be noted that some sufficient conditions allowing to solve this problem have been given in [7]. In such a work [7], no structure on the function Φ is given. However, the set of local linear controllers K_o are those which satisfy a specific linear matrix inequality.

In this regard, here, we consider the case of a system in Feedforward form (or chain of integrators, see system (10)).

III. A PRELIMINARY RESULT

A. Result in the case of adding one integration

In this section, we consider the case in which the state ξ of system (1) can be decomposed as $\xi = (y,x)$ with y in $\mathbb{R}$ and x in $\mathbb{R}^n$ and where the dynamics are:

$$\left\{ \begin{array}{l} \dot{y} = h(x) \ , \\ \dot{x} = f(x) + g(x)u \ , \end{array} \right. \tag{2}$$

with $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ are C^p functions, $p \geq 2$ such that $h(0) = 0$ and $f(0) = 0$ and u is the control input in $\mathbb{R}$.

Let the first order approximation of system (2) be:

$$\left\{ \begin{array}{l} \dot{y} = H'x \ , \\ \dot{x} = Fx + Gu \ , \end{array} \right. \tag{3}$$

with the matrices H, F and G given as

$$F = \frac{\partial f}{\partial x}(0) \ , \ H' = \frac{\partial h}{\partial x}(0) \ , \ G = g(0) \ . \tag{4}$$

We assume the stabilization problem is solved for the subsystem in x in system (2). More precisely, we make the following assumption on the functions f and g:

Assumption 1: For all vector K_x in $\mathbb{R}^n$ such that the matrix $F + GK_x'$ is Hurwitz, there exists a function $\alpha_x : \mathbb{R}^n \to \mathbb{R}$ of class C^p such that the following two properties are satisfied:

1) the origin of the system:

$$\dot{x} = f(x) + g(x)\alpha_x(x) \ . \tag{5}$$

is globally and asymptotically stable;

2) the first order approximation of this function satisfies:

$$\frac{\partial \alpha_x}{\partial x}(0) = K_x' \ . \tag{6}$$

This assumption means that we can assign any local behavior while ensuring global stabilization for the x-subsystem in equation (2). With Assumption 1, and employing forwarding design method developed in [2] (see also [3], [4]) the stabilization problem of the origin of the complete system (2) can be solved. However to solve the stabilization with a prescribed local behavior, we need to tune properly the parameters of the forwarding design and to modify adequately the controller obtained from [2]. Our result is stated as follows:

Theorem 1 (Adding integration with prescribed behavior): Assume the System (2) satisfies Assumption 1. Given a linear controller $(K_{o,y}, K_{o,x}')$ in $(\mathbb{R} \times \mathbb{R}^n)$ such that the matrix:

$$A = \left[\begin{array}{cc} 0 & H' \\ GK_{o,y} & F + GK_{o,x}' \end{array} \right] \ , \tag{7}$$

is Hurwitz then there exists a C^{p-1} function $\alpha_o : \mathbb{R}^{n+1} \to \mathbb{R}$ such that the following properties are satisfied :

1) the origin of the system

$$\dot{y} = h(x) \ , \ \dot{x} = f(x) + g(x)\alpha_o(x,y) \tag{8}$$

is globally and asymptotically stable;

2) the function α_o satisfies:

$$\frac{\partial \alpha_o}{\partial y}(0,0) = K_{o,y} \ , \ \frac{\partial \alpha_o}{\partial x}(0,0) = K_{o,x}' \ . \tag{9}$$

B. Discussion on Theorem 1

Assumption (1) is stronger than a stabilizability property since it is assumed that all local behaviors can be recovered for the closed loop system. However, employing the result obtained in [6], yields that Assumption 1 is satisfied in the case in which the x-subsystem is in feedback form and when the functions f and g are sufficiently smooth using backstepping design.

Note that the conclusion of Theorem 1 is that Assumption 1 is valid on the entire system with $p := p - 1$. Consequently, with an iterative procedure, higher order systems can be considered. Indeed, let system (1) be with $\xi = (y_1, \ldots, y_{n_y}, x)$ in the form:

$$\left\{ \begin{array}{l} \dot{y}_1 = h_1(y_2, \ldots, y_{n_y}, x) \ , \\ \vdots \\ \dot{y}_{n_y} = h_{n_y}(x) \ , \\ \dot{x} = f(x) + g(x)u \ , \end{array} \right. \tag{10}$$

with x in $\mathbb{R}^n$, y_i in $\mathbb{R}$, $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^n$ and $h_i : \mathbb{R}^{n_y - 1 + n} \to \mathbb{R}$ are C^{i+1} functions, such that $h_i(0, \ldots, 0) = 0$ and $f(0) = 0$ and u is the control input in $\mathbb{R}$. Based on the result obtained from Theorem 1, we can show the following result:

Theorem 2 (Case of higher order systems): Assume the x-subsystem of (10) satisfies Assumption 1 with $p = n_y + 1$. For all vector $(K_{o,y_1}, \ldots, K_{o,y_{n_y}}, K_{o,x}')$ in $(\mathbb{R}^{n_y} \times \mathbb{R}^n)$ which stabilizes globally and asymptotically the first order approximation of system (10), there exists a C^1 function $\alpha_o : \mathbb{R}^{n_y + n} \to \mathbb{R}$ such that the following properties are satisfied :

1) the origin of the system (10) in closed loop with $u = \alpha_o(y_1, \ldots, y_{n_y}, x)$ is globally and asymptotically stable;
2) the function α_o satisfies:

$$\frac{\partial \alpha_o}{\partial y_i}(0, \ldots, 0) = K_{o,y_i} \ , i = 1, \ldots, n_y \ ,$$

$$\frac{\partial \alpha_o}{\partial x}(0, \ldots, 0) = K_{o,x}' \ .$$

Proof : First, employing Theorem 1 it is shown that the (y_{n_y}, x)-subsystem in system (10) satisfies Assumption 1 with $p = n_y$. Recursively, we apply again Theorem 1 and we found that the result holds. □

In the paper [7], the stabilization with prescribed local behavior has been addressed and studied on an inverted pendulum model. In some specific coordinates, this inverted pendulum model can be put in feedforward form and a forwarding control law has been introduced in [2]. It is noticed in [7] that, statistically, for all local behavior obtained from a *LQ* approach, the problem under consideration could be solved. Consequently, Theorem 1 establishes a theoretical justification on the fact that the approach of [7] applies on the feedforward form model of the cart pendulum.

IV. PROOF OF THEOREM 1

The proof of this result is divided in three parts. In the first part, we show that a quadratic Lyapunov function associated with the local stabilizer can be rewritten in the form of a Lyapunov matrix that would have been obtained by following a forwarding design method of [2]. In the second part of the proof, we construct a candidate Lyapunov function V such that the control $u = K_{o,y}y + K_{o,x}'x$ makes the time derivative of this function negative definite in a small neighborhood of the origin. In the third part, from this candidate function we construct a control law makes negative the time derivative of the candidate Lyapunov function and has the prescribed local behavior.

1) Part 1: Forwarding local Lyapunov function: The matrix A given in (7) being by assumption Hurwitz, given a symmetric positive definite matrix S in $\mathbb{R}^{(n+1)\times(n+1)}$, there exists a symmetric positive definite matrix P in $\mathbb{R}^{(n+1)\times(n+1)}$ such that :

$$A'P + PA = -S \ , \ P = \begin{bmatrix} p & Q' \\ Q & R \end{bmatrix} \ . \tag{11}$$

In this part of the proof, we show that the Lyapunov matrix P associated with the matrix A can be rewritten in the form of a Lyapunov matrix that would have been obtained following the Forwarding design method of [2] or [4]. Indeed, note that the Lyapunov function associated to the matrix P can be decomposed as follows.

$$\begin{bmatrix} y & x' \end{bmatrix} P \begin{bmatrix} y \\ x \end{bmatrix} = x'R_x x + p\left(y - M'x\right)^2 \ , \tag{12}$$

where R_x and M are respectively a matrix in $\mathbb{R}^{n\times n}$ and a vector in $\mathbb{R}^n$ defined as:

$$R_x = R - \frac{QQ'}{p} \ , \ M = -\frac{Q}{p} \ . \tag{13}$$

If we compare the decomposition in equation (12) and the structure of the Lyapunov function obtained by forwarding in [4, equation (3)], we see that the matrix P would be a Lyapunov matrix obtained by a forwarding design technique if there exists K_x a vector in $\mathbb{R}^n$ such that the following two requirements are satisfied:

1) $u = K_x'x$ is a control law for the x-subsystem associated to the Lyapunov matrix R_x. In other words, $F + GK_x$ is a Hurwitz matrix and the following inequalities are satisfied:

$$R_x > 0 \ , \ R_x(F + GK_x') + (F + GK_x')'R_x < 0 \ ; \tag{14}$$

2) The following algebraic equation is satisfied:

$$M'(F + GK_x') = H' \ . \tag{15}$$

Consequently, in this part of the proof we show that such a vector K_x does exist. Indeed, we can decompose $A = A_1 + A_2$ with[1]

$$A_1 = \begin{bmatrix} 0 & H' \\ 0_{n,1} & F \end{bmatrix} \ , \ A_2 = \begin{bmatrix} 0 & 0_{1,n} \\ GK_{opt,y} & GK_{opt,x}' \end{bmatrix} \ .$$

[1]the symbol $0_{a,b}$ stands for a $a \times b$ zero matrix.

The matrix T in $\mathbb{R}^{(n+1)\times n}$ defined as:

$$T = \begin{bmatrix} -\dfrac{RG}{Q'G} \\ \\ I_n \end{bmatrix} \ ,$$

satisfies

$$T'PA_2 = 0 \ .$$

Pre- and post- multiplying equality (11) by the matrix T' and T gives:

$$(F + GK_x')'R_x + R_x(F + GK_x') = -T'ST \ .$$

with:

$$K_x = -\frac{Q'F + pH'}{Q'G} \ . \tag{16}$$

Note that T is full rank, consequently $T'ST$ is a symmetric positive definite matrix in $\mathbb{R}^{n\times n}$. Moreover, the matrix P being symmetric positive definite its Schur complement, $R - \frac{QQ'}{p}$, is also positive definite. Consequently, equality (14) is satisfied with K_x defined in (16).

Note also, that M defined in (13) and K_x given in (16) satisfies the algebraic equation (15).

Consequently, for the time, we have shown that the Lyapunov matrix associated to the linear controller of the first order approximation can be decomposed in a forwarding-like manner. From this, we will be able to get a candidate Lyapunov function associated the local controller.

2) Part 2: Construction of the global CLF: The construction of the candidate Lyapunov function is based on a modified forwarding technique inspired from [2]. First, with Assumption 1, and the local stabilizer K_x given in (16), there exists a C^p function $\alpha_x : \mathbb{R}^n \to \mathbb{R}$ such that the origin of the system (5) is globally and asymptotically stable and the local property (6) is satisfied.

Now, we can apply the following Lemma whose proof is given in Appendix A.

Lemma 1: There exists a C^∞ Lyapunov function $V_x : \mathbb{R}^n \to \mathbb{R}_+$, proper and positive definite, such that:

- V_x is a Lyapunov function associated to the closed loop system (5). In other words, we have:

$$W_x(x) := \frac{\partial V_x}{\partial x}(x)\left[f(x) + g(x)\alpha_x(x)\right] < 0 \ , \ \forall \ x \ \neq 0 \ ; \tag{17}$$

- V_x is locally quadratic and its local approximation is R_x defined in (13). We have:[2]

$$\mathscr{H}(V_x)(0) = 2R_x \ . \tag{18}$$

For the non linear system (2), following the forwarding design described in [8] and [2], we can introduce the function $\mathscr{M} : \mathbb{R}^n \to \mathbb{R}$ defined as:

$$\mathscr{M}(x) = \int_\infty^0 h(X_1(x,s))\,ds \ , \tag{19}$$

where $X_1(x,s)$ is the solution initiated from x of the system:

$$\dot{x} = f(x) + g(x)\alpha_x(x) \ .$$

[2]The symbol $\mathscr{H}$ denotes the operator which gives the Hessian of given function in $\mathbb{R}^n$.

The following Lemma can be obtained from [8, Lemma 6.88]. Its proof is written in Section B.

Lemma 2 ([8]): The function $\mathscr{M}$ defined in (19) is C^p and satisfies the following partial differential equation:

$$\frac{\partial \mathscr{M}}{\partial x}(x)\Big[f(x)+g(x)\alpha_x(x)\Big] = h(x) \ ,\forall \ x \ \in \mathbb{R}^n \ . \qquad (20)$$

With V_x obtained from Lemma 1, the function $\mathscr{M}$ given in (19) and according to [2]; we can now introduce the C^p candidate Lyapunov function $V : \mathbb{R}^{n+1} \to \mathbb{R}_+$ as:

$$V(y,x) = V_x(x) + p(y-\mathscr{M}(x))^2 \ . \qquad (21)$$

Note that this function is proper and positive definite.

To complete Part 2 of the proof, it remains to show that $u = K_{o,y}y + K'_{o,x}x$ makes the time derivative of the candidate Lyapunov function negative definite in a small neighbourhood of the origin. To do so, it is sufficient to show that $\mathscr{H}(V)(0,0) = 2P$. Note that

$$\mathscr{H}(V)(0,0) = \qquad (22)$$

$$\begin{bmatrix} 2p & -2p\frac{\partial \mathscr{M}}{\partial x}(0) \\ -2p\left(\frac{\partial \mathscr{M}}{\partial x}(0)\right)' & \mathscr{H}(V_x)(0) + 2p\left(\frac{\partial \mathscr{M}}{\partial x}(0)\right)'\frac{\partial \mathscr{M}}{\partial x}(0) \end{bmatrix}$$

So, let us compute $\frac{\partial \mathscr{M}}{\partial x}(0)$. Since we have:

$$\frac{\partial \mathscr{M}}{\partial x}(x) = \int_{+\infty}^{0} \frac{\partial h}{\partial x}(X_1(x,s))\frac{\partial X_1}{\partial x}(x,s)ds \ .$$

Since $X_1(0,s) = 0$, this implies that:

$$\frac{\partial \mathscr{M}}{\partial x}(0) = H\int_{+\infty}^{0} \frac{\partial X_1}{\partial x}(0,s)ds \ . \qquad (23)$$

On the other hand, we have:

$$\frac{d}{ds}\frac{\partial X_1}{\partial x}(0,s) = (F+GK'_x)\frac{\partial X_1}{\partial x}(0,s) \ , \frac{\partial X_1}{\partial x}(0,0) = I_n \ .$$

Finally, we obtain:

$$\frac{\partial X_1}{\partial x}(0,s) = \exp\Big((F+GK'_x)s\Big) \ ,$$

and by taking into consideration (23) and (15), we get:

$$\frac{\partial \mathscr{M}}{\partial x}(0) = H'(F+GK'_x)^{-1} = M' = -\frac{Q'}{p} \ .$$

Hence, equality (22) becomes:

$$\mathscr{H}(V)(0,0) = 2\begin{bmatrix} p & Q' \\ Q & R \end{bmatrix} = 2P \ . \qquad (24)$$

With equality (11), we get that the control law $u = K_{o,y}y + K_{o,x}x$ makes the time derivative of V negative definite in a small neighborhood of the origin. In other words, there exists $\varepsilon > 0$ such that along the trajectories of system (2) with the control law $u = K_{o,y}y + K_{o,x}x$, we have:

$$\overset{\cdot}{\widehat{V(y,x)}} < 0 \ , \ \forall (y,x) \ \in \ \left\{(y,x) \in \mathbb{R}^{n+1}, 0 < V(y,x) < \varepsilon\right\} \ . \qquad (25)$$

Hence $M'G \neq 0$.

3) Part 3 Constructing the control law: To finish the proof it remains to construct a control law from the candidate Lyapunov function. By looking at the time derivative of V, we can see that a control law ensuring global and asymptotic stabilization of the origin of the system (2) can be obtained simply as:

$$\alpha_g(y,x) = \alpha_x(x) - \frac{\partial V_x}{\partial x}(x)g(x) + p(y-\mathscr{M}(x))\frac{\partial \mathscr{M}}{\partial x}(x)g(x) \ .$$

Indeed, this gives along the trajectory of the system (2) :

$$\overset{\cdot}{\widehat{V(y,x)}} = -W_x(x) - \left(\frac{\partial V_x}{\partial x}(x)g(x) - p(y-\mathscr{M}(x))\frac{\partial \mathscr{M}}{\partial x}(x)g(x)\right)^2 \ . \qquad (26)$$

which is negative definite due to the fact that $\frac{\partial \mathscr{M}}{\partial x}(0)g(0) = M'G \neq 0$ by controllability of the first order approximation of system (2) and (15).

The control law $u = \alpha_g(y,x)$ is not a solution to our problem since its local behavior is not the prescribed one. However, due to the fact that the local optimal controller $u = K_{o,y}y + K_{o,x}x$ and the function V satisfies inequality (25), it yields that both controllers make negative definite the time derivative of the same Lyapunov function. Therefore, we can follow [9] to unite the two controllers by interpolation in order to obtain the result. As in [9], we consider the following new controller for all (y,x) in $\mathbb{R}^{n+1}$:

$$\alpha_o(x) \ = \ \mathscr{I}(y,x) \ - \ k\,c(y,x)\frac{\partial V}{\partial x}(y,x)^T g(x) \ , \qquad (27)$$

where: $\mathscr{I}(y,x) = (1-\rho(y,x))(K_{o,y}y + K_{o,x}x) + \rho(y,x)\alpha_g(y,x)$ and ρ is any smooth function such that:

$$\rho(y,x) \ = \ \begin{cases} 0 & \text{if } V(y,x) \ \leq \ \frac{1}{2}\varepsilon \ , \\ 1 & \text{if } V(y,x) \ \geq \ \varepsilon \ , \end{cases}$$

and the function c is any smooth function such that:

$$c(y,x)\begin{cases} = 0 & \text{if } V(y,x) \geq \varepsilon \ \text{ or } V(y,x) \ \leq \ \frac{1}{2}\varepsilon \ , \\ > 0 & \text{if } \frac{1}{2}\varepsilon < V(y,x) < \varepsilon \ , \end{cases} \qquad (28)$$

and k is a positive real number sufficiently large to ensure that V is a Lyapunov function of the closed-loop system. The existence of k is obtained employing compactness arguments (see analogous arguments in [10, Lemma 2.13]). For the sake of completeness we rewrite the arguments which can be found in [9].

Note that the function α_o is such that $\frac{\partial \alpha_o}{\partial y}(0) = K_{o,y}$ and $\frac{\partial \alpha_o}{\partial x}(0) = K_{o,x}$ and consequently Item 2) of Theorem 1 is satisfied.

It remains to show that, by selecting k sufficiently large, we obtain that the time derivative of the Lyapunov function is negative definite. We introduce the continuous function $\mathring{V} : \mathbb{R}^{n+1} \times \mathbb{N} \to \mathbb{R}$ defined as: for all (y,x,k) in $\mathbb{R}^{n+1} \times \mathbb{N}$,

$$\mathring{V}(y,x,k) = \frac{\partial V}{\partial y}(y,x)h(x) + \frac{\partial V}{\partial x}(y,x)f(x) \ . \qquad (29)$$

$$+\frac{\partial V}{\partial x}(y,x)g(x)\mathscr{I}(y,x)-k\,c(y,x)\left|\frac{\partial V}{\partial x}(y,x)g(x)\right|^2 .$$

To evaluate the sign of this function we consider three different cases:

1) When $0 < V(y,x) \le \frac{\varepsilon}{2}$: In this case, $\alpha_o(y,x) = K_{o,y}y + K_{o,x}x$ and with inequality (25) we get, for all k in $\mathbb{N}$:
$$\mathring{V}(y,x,k) < 0 .$$
2) When $V(y,x) \ge \varepsilon$: In that case, $\alpha_o(y,x) = \alpha_g(y,x)$ and with inequality (26), we get: for all k in $\mathbb{N}$:
$$\mathring{V}(y,x,k) < 0 .$$
3) When $\frac{\varepsilon}{2} < V(y,x) < \varepsilon$: It is now shown that if k is selected sufficiently large then we have the negativeness of $\mathring{V}(y,x,k)$. To prove this, assume that this assertion is wrong and suppose that for each k in $\mathbb{N}$, there exists (y_k,x_k) such that $\frac{\varepsilon}{2} \le V(y_k,x_k) \le \varepsilon$ and
$$\mathring{V}(y_k,x_k,k) \ge 0 . \tag{30}$$
First note that for all k, (y_k,x_k) is in the set $\{(y,x) : \frac{1}{2}\varepsilon \le V(y,x) \le \varepsilon\}$ which is compact. Considering: (29) and (30), it yields that:
$$0 \le c(y_k,x_k)\left|\frac{\partial V}{\partial x}(y_k,x_k)g(x_k)\right|^2 \le \frac{c}{k} , \tag{31}$$
with:
$$c = \max_{\{(y,x):\,\frac{1}{2}\varepsilon\le V(y,x)\le\varepsilon\}}\left\{\left|\frac{\partial V}{\partial y}(y,x)h(x)+ \frac{\partial V}{\partial x}(y,x)f(x)+\frac{\partial V}{\partial x}(y,x)g(x)\mathscr{I}(y,x)\right|\right\}.$$
Moreover, $(y_k,x_k)_{k\in\mathbb{N}}$ is a sequence living in a compact set, thus there exists a subsequence $(y_{k_\ell},x_{k_\ell})_{\ell\in\mathbb{N}}$ which converges to a point denoted (y^*,x^*). With (31), it implies that $c(y^*,x^*)\left|\frac{\partial V}{\partial x}(y^*,x^*)g(x^*)\right|^2 = 0$ and consequently:
$$\lim_{\ell\to+\infty}\mathring{V}(y_{k_\ell},x_{k_\ell},k_\ell) = w(y^*,x^*) ;$$
where:
$$w(y^*,x^*) = \frac{\partial V}{\partial y}(y^*,x^*)h(x^*)+\frac{\partial V}{\partial x}(y^*,x^*)f(x^*) +\frac{\partial V}{\partial x}(y^*,x^*)g(x^*)\mathscr{I}(y^*,x^*).$$
From the fact that $c(y^*,x^*)\left|\frac{\partial V}{\partial x}(y^*,x^*)g(x^*)\right|^2 = 0$, there are only two possibilities left:
   - if $\frac{\partial V}{\partial x}(y^*,x^*)g(x^*) = 0$, the function V being a global CLF, by the Artstein condition, then $\frac{\partial V}{\partial y}(y^*,x^*)h(x^*) + \frac{\partial V}{\partial x}(y^*,x^*)f(x^*) < 0$ and thus $w(y^*,x^*) < 0$;
   - if $c(y^*,x^*) = 0$, then by (28), (y^*,x^*) is in the set $\{(y,x) \neq 0 : V(y,x) \le \frac{1}{2}\varepsilon$ or $V(y,x) \ge \varepsilon\}$. Hence, this implies $w(y^*,x^*) < 0$.

Since the function w is continuous at (y^*,x^*), $w(y^*,x^*) < 0$, and the sequence $(y_{k_\ell},x_{k_\ell})_{\ell\in\mathbb{N}}$ converges to (y^*,x^*), there exists ℓ_∞ such that, for all $\ell > \ell_\infty$, $\mathring{V}(y_{k_\ell},x_{k_\ell},k_\ell) \le w(y_{k_\ell},x_{k_\ell}) < 0$. This contradicts (30).

Therefore there exists $k > 0$ such that α_o is a stabilizing control law and solves the problem under consideration.

V. Conclusion

We have studied the problem of designing a stabilizing controller which ensures a prescribed local behavior.

We have shown that all stabilizing local behaviors can be reproduced when using the forwarding design technique developed in [2], [3]. This is made possible by modifying the forwarding design adequately. This result gave a theoretical justification on a statistical result given in [7].

Appendix

A. proof of Lemma 1

The proof of this Lemma is based on recent results obtained in [9]. Indeed, the design of the function V_x is obtained from the uniting of a quadratic local control Lyapunov function (denoted V_0) and a global control Lyapunov function (denoted V_∞) obtained employing converse Lyapunov theorem.

First of all, employing the converse Lyapunov theorem of Kurzweil [11], there exists a C^∞ function $V_\infty : \mathbb{R}^n \to \mathbb{R}_+$ such that:
$$\frac{\partial V_\infty}{\partial x}(x)[f(x)+g(x)\alpha_x(x)] < 0 , \ \forall\ x \neq 0 .$$
On the other hand, with (14), the function $V_0(x) = x'R_xx$ is such that:
$$\frac{\partial V_0}{\partial x}(x)[F+GK_x]x = -x'T'STx .$$
Due to the fact that K_x satisfies equation (6) it yields that the matrix $F+GK_x$ is the first order approximation of the x-subsystem in equation (2) with the control law $u = \alpha_x(x)$. Consequently, it implies that there exists a positive real number ε_1 such that:
$$\frac{\partial V_0}{\partial x}(x)[f(x)+g(x)\alpha_x(x)] < 0 , \ \forall\ |x| \le \varepsilon_1 .$$
Employing [9, Theorem 2.1], it yields the existence of a function V_x and a positive real number ε_2 such that:

1) for all x in $\mathbb{R}^n\setminus\{0\}$,
$$\frac{\partial V_x}{\partial x}(x)[f(x)+g(x)\alpha_x(x)] < 0 .$$
2) for all x in $\mathbb{R}^n$ such that $|x| \le \varepsilon_2$, we have:
$$V_x(x) = V_0(x) ,$$
and consequently $\mathscr{H}(V_x)(0) = 2R_x$.

This conclude the proof of Lemma 1

B. proof of Lemma 2

The proof of this Lemma can be found almost entirely in [8]. Indeed, it is shown in [8] that the function $\mathscr{M}$ given in (19) is well defined and moreover that this one is C^1. It is also shown (already in [2]) that this function is a solution of the PDE given in (20). In this note we just give the main ideas and show how employing the fact that the functions f, g and h are C^p we get that the function $\mathscr{M}$ is also C^p.

First of all, following [8], by employing the continuity of the function h and the exponential convergence of $X_1(x,t)$ toward the origin, it yields the existence of a class $\mathscr{K}_\infty$ function γ_1 and a positive real number c_1 such that:

$$|h(X_1(x,t))| \leq \gamma_1(|x|)\exp(-c_1 t) \ , \ \forall t \ .$$

This implies that the function $\mathscr{M}$ defined in (20) is properly defined.

To show that this function is C^1, in [8], the function $\frac{\partial h}{\partial x}(X_1(x,t))\frac{\partial X_1}{\partial x}(x,t)$ is studied. Following the same procedure as previously, we obtain the existence of a class $\mathscr{K}_\infty$ function γ_2 and a positive real number such that

$$\left|\frac{\partial h}{\partial x}(X_1(x,t))\right| \leq \gamma_2(|x|)\exp(-c_2 t) \ , \ \forall t \ .$$

Hence, it remains to show that the function $\frac{\partial X_1}{\partial x}(x,t)$ can be bounded also in the same way. In order to do this, we introduce the function $\tilde{f}(x) = f(x) + g(x)\alpha_x(x)$ and we use the following property:

$$\dot{\overbrace{\frac{\partial X_1}{\partial x}(x,t)}} = \frac{\partial \tilde{f}}{\partial x}(X_1(x,t))\frac{\partial X_1}{\partial x}(x,t) \ .$$

By definition, the matrix $\frac{\partial \tilde{f}}{\partial x}(0)$ is Hurwitz. Employing the continuity of the function $x \mapsto \frac{\partial \tilde{f}}{\partial x}(x)$ and exponential convergence of $X_1(x,t$ toward the origin, it can be shown that

$$\left|\frac{\partial X_1}{\partial x}(x,t)\right| \leq \gamma_3(|x|)\exp(-c_3 t) \ , \ \forall \ (x,t) \ . \quad (32)$$

Consequently, the fonction $t \mapsto \frac{\partial h}{\partial x}(X_1(x,t))\frac{\partial X_1}{\partial x}(x,t)$ is integrable for all x and the function $\mathscr{M}$ is C^1. The proof that this function is also C^2 follows the same lines. In order to do this, we study the n functions $\frac{\partial}{\partial x_i}\left(\frac{\partial h}{\partial x}(X_1(x,t))\frac{\partial X_1}{\partial x}(x,t)\right)$ for $i = 1\ldots,n$. It can be noticed that

$$\frac{\partial}{\partial x_i}\left(\frac{\partial h}{\partial x}(X_1(x,t))\frac{\partial X_1}{\partial x}(x,t)\right) = \frac{\partial}{\partial x_i}\left(\frac{\partial h}{\partial x}(X_1(x,t))\right)\frac{\partial X_1}{\partial x}(x,t) + \frac{\partial h}{\partial x}(X_1(x,t))\frac{\partial X_1}{\partial x_i \partial x}(x,t) \ .$$

As previously, using the continuity and exponential convergence of the function X_1, the first term can be bounded by an exponential. Moreover we have:

$$\dot{\overbrace{\frac{\partial X_1}{\partial x_i \partial x}(x,t)}} = \psi(x,t) + \frac{\partial f}{\partial x}(X_1(x,t))\frac{\partial X_1}{\partial x_i \partial x}(x,t) \ , \quad (33)$$

where ψ_i is the function defined by:

$$\psi_i(x,t) = \frac{\partial}{\partial x_i}\left(\frac{\partial \tilde{f}}{\partial x}(X_1(x,t))\right)\frac{\partial X_1}{\partial x}(x,t) \ .$$

With inequality (32) and the exponential convergence of $X(x,t)$, it gives the existence of a positive and continuous function $\gamma_4(\cdot)$ such that:

$$|\psi_i(x,t)| \leq \gamma_4(|x|)\exp(-c_4 t) \ .$$

This upper bound and the use one more time of the fact that $\frac{\partial \tilde{f}}{\partial x}(0)$ is Hurwitz and the continuity of the function $\frac{\partial \tilde{f}}{\partial x}(x)$, we get that the solutions of (33) satisfy:

$$\left|\frac{\partial X_1}{\partial x_i \partial x}(x,t)\right| \leq \gamma_5(|x|)\exp(-c_5 t) \ .$$

And consequently, the functions $\frac{\partial}{\partial x_i}\left(\frac{\partial h}{\partial x}(X_1(x,t))\frac{\partial X_1}{\partial x}(x,t)\right)$ are integrable. This implies that the function $\mathscr{M}$ is C^2. Following the same procedure it can be shown that the function $\mathscr{M}$ is C^p.

To finish the proof of this Lemma, it remains to show that the function $\mathscr{M}$ is a solution of (20). By definition we have:

$$\frac{\partial \mathscr{M}}{\partial x}(x)[f(x) + g(x)\alpha_x(x)] = \lim_{t\to 0}\frac{\mathscr{M}(X_1(x,t)) - \mathscr{M}(x)}{t} \ .$$

Moreover:

$$\begin{aligned} \mathscr{M}(X_1(x,t)) &= \int_{+\infty}^{0} h(X_1(X_1(x,t),s))ds \ , \\ &= \int_{+\infty}^{t} h(X_1(x,s))ds \ . \end{aligned}$$

which gives:

$$\lim_{t\to 0}\frac{\mathscr{M}(X_1(x,t)) - \mathscr{M}(x)}{t} = \lim_{t\to 0}\frac{\int_0^t h(X_1(x,s))ds}{t} = h(x) \ .$$

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