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A LaSalle version of Matrosov theorem

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***Abstract*— A weak version of Matrosov theorem, in the spirit of LaSalle invariance principle, is established. The result is clarified by means of two examples. The case of nested-Matrosov theorem is also discussed.**

I. Introduction

The property of stability of equilibria (and motions) of dynamical systems has been studied extensively during the last century. Several fundamental results and characterizations have been derived, see *e.g.* [1], [2], [3], [4] for a *classical* account and [5], [6] for a *modern* discussion.

In the present paper we study a variation of Matrosov theorem with the aim to establish stability properties, or otherwise, of sets rather than of equilibria. In particular, unlike the classical Matrosov theorem which allows to prove asymptotic stability of an equilibrium provided a linear combination of some functions is positive definite, we study the case in which such a linear combination is only positive semi-definite. In this respect, our result stays at Matrosov theorem as LaSalle invariance principle stays at Lyapunov theorem for asymptotic stability. Unlike LaSalle invariance principle, however, the conclusions that can be drawn from our weak version of Matrosov theorem are not necessarily of the LaSalle type, since additional technical assumptions are required to prove convergence results.

For convenience, we recall (the simplest version of) Matrosov theorem, for time-invariant nonlinear systems, as proved in [7], see also [8] for a proof based on the use of a Lyapunov function and [9] for some generalizations.

Theorem 1 (Matrosov): Consider a nonlinear, time-invariant, system described by the equation

$$\dot{x} = f(x), \tag{1}$$

where $x(t) \in \mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz continuous. Assume $x = 0$ is an equilibrium point, *i.e.* $f(0) = 0$. Consider the following conditions.

(M1) There exists a differentiable, positive definite and radially unbounded function $V_0 : \mathbb{R}^n \to \mathbb{R}_+$ such that

$$\dot{V}_0 \leq 0. \tag{2}$$

(M2) There exists two differentiable functions $V_1 : \mathbb{R}^n \to \mathbb{R}$ and $V_2 : \mathbb{R}^n \to \mathbb{R}$, two continuous, positive semi-definite functions $N_1 : \mathbb{R}^n \to \mathbb{R}_+$ and $N_2 : \mathbb{R}^n \to \mathbb{R}_+$ and a continuous function $\psi_2 : \mathbb{R} \to \mathbb{R}$, satisfying $\psi_2(0) = 0$, such that

$$\dot{V}_1 \leq -N_1, \tag{3}$$
$$\dot{V}_2 \leq -N_2 + \psi_2(N_1). \tag{4}$$

(M3) There exists a positive definite function $\omega : \mathbb{R}^n \to \mathbb{R}$ such that

$$N_1(x) + N_2(x) \geq \omega(x). \tag{5}$$

Then the following hold.

- (M1), (M2) and (M3) imply that the equilibrium $x = 0$ of system (1) is globally asymptotically stable.
- (M2) and (M3) imply that all bounded trajectories of system (1) converge to the equilibrium $x = 0$.

Remark 1: The functions V_1 and V_2 are called auxiliary functions. Often, Matrosov theorem is used in the simplified version in which $V_1 = V_0$.

The problem that we address in this paper is to study what happens replacing the condition (5) with the following condition.

$(M3)'$ The function

$$N_1(x) + N_2(x) \tag{6}$$

is positive semi-definite.

One might expect, borrowing from LaSalle invariance principle, that the use of the new, weaker, condition on $N_1 + N_2$ would allow to prove convergence of the trajectories of the system to the largest invariant set contained in $\Omega_N = \{x \in \mathbb{R}^n \mid N_1(x) + N_2(x) = 0\}$. This is unfortunately not true in general, but may be true under additional technical assumptions (which clarify the role of stability in the proof of such convergence result).

The rest of the paper is organized as follows. In Section II we present the main results of the paper, which are then proved using a technical result given in Section III. Section IV contains two examples which highlight the significance of the new stability theorem. Finally, Section V provides some concluding remarks.

II. Main results

The first result of this section is a weak version of Theorem 1, whereas the second result is a weak version of the general result of [8], in which several auxiliary functions are used.

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Theorem 2: Consider the nonlinear, time-invariant, system (1) and the following conditions.

(M4) The largest invariant set contained in the set

$$\Omega_{N_1,N_2} = \{x \in \mathbb{R}^n \mid N_1(x) = N_2(x) = 0\} \quad (7)$$

is stable.

(M5) The set

$$\Omega_{N_2} = \{n_2 \in \mathbb{R}_+ \mid \exists x : N_2(x) = n_2, N_1(x) = 0\} \quad (8)$$

is a singleton, *i.e.* N_2 takes a single value on the set $\{x \in \mathbb{R}^n | N_1(x) = 0\}$.

Then the following hold.

1) (M1), (M2) and (M3)$'$ imply that all trajectories of the system (1) are such that

$$\liminf_{t\to\infty} [N_1(x(t)) + N_2(x(t))] = 0. \quad (9)$$

2) (M1), (M2), (M3)$'$ and one of (M4) and (M5) imply that all trajectories of the system (1) are such that

$$\lim_{t\to\infty} [N_1(x(t)) + N_2(x(t))] = 0. \quad (10)$$

3) (M2) and (M3)$'$ imply that all bounded trajectories of the system (1) satisfy condition (9).
4) (M2), (M3)$'$ and one of (M4) and (M5) imply that all bounded trajectories of the system (1) satisfy condition (10).

Theorem 2 can be generalized to the case in which several auxiliary functions are used, thus providing a weak version of the Matrosov Theorem proved in [8], as detailed in the following statement, in which only the case exploiting condition (M4) is considered. The proof of the statement, relying on Lemma 2, and on arguments similar to those in the proof of Theorem 2 is omitted.

Theorem 3: Consider the nonlinear, time-invariant, system (1) and the following conditions.

(M2)$_v$ There exists $k \geq 2$ differentiable functions $V_1 : \mathbb{R}^n \to \mathbb{R}, \cdots, V_k : \mathbb{R}^n \to \mathbb{R}$, k continuous, positive semi-definite functions $N_1 : \mathbb{R}^n \to \mathbb{R}_+, \cdots, N_k : \mathbb{R}^n \to \mathbb{R}_+$ and $k-1$ continuous functions $\psi_2 : \mathbb{R} \to \mathbb{R}, \cdots, \psi_k : \mathbb{R} \to \mathbb{R}$, satisfying $\psi_i(0) = 0$, such that

$$\begin{array}{lcl} \dot{V}_1 & \leq & -N_1, \\ \dot{V}_2 & \leq & -N_2 + \psi_2(N_1), \\ & \vdots & \\ \dot{V}_{k-1} & \leq & -N_{k-1} + \psi_{k-1}(N_1, N_2, \cdots, N_{k-2}), \\ \dot{V}_k & \leq & -N_k + \psi_k(N_1, N_2, \cdots, N_{k-1}). \end{array} \quad (11)$$

(M3)$'_v$ The function

$$N_1(x) + N_2(x) + \cdots + N_k(x) \quad (12)$$

is positive semi-definite.

(M4)$_v$ The largest invariant set contained in the set

$$\Omega_{N_1,\cdots,N_k} = \{x \in \mathbb{R}^n \mid N_1(x) = \cdots = N_k(x) = 0\} \quad (13)$$

is stable.

Then the following hold.

1) (M1), (M2)$_v$ and (M3)$'_v$ imply that all trajectories of the system (1) are such that

$$\liminf_{t\to\infty} [N_1(x(t) + N_2(x(t)) + \cdots + N_k(x(t))] = 0. \quad (14)$$

2) (M1), (M2)$_v$, (M3)$'_v$ and (M4)$_v$ imply that all trajectories of the system (1) are such that

$$\lim_{t\to\infty} [N_1(x(t) + N_2(x(t)) + \cdots + N_k(x(t))] = 0. \quad (15)$$

3) (M2)$_v$ and (M3)$'_v$ imply that all bounded trajectories of the system (1) satisfy condition (14).
4) (M2)$_v$, (M3)$'_v$ and (M4)$_v$ imply that all bounded trajectories of the system (1) satisfy condition (15).

III. Two lemmas

The following Lemmas are intended to be used for studying solutions of ordinary differential equations which are known to exist on $[0, +\infty)$, and taking values in a compact set. This explains why boundedness of various functions can be assumed.

Lemma 1: Let $i \in \{1, 2\}$. Let $a_i : \mathbb{R}_+ \to [1, +\infty)$ be bounded differentiable functions and $b_i : \mathbb{R}_+ \to \mathbb{R}_+$ be bounded continuous functions.

Suppose there exist continuous positive definite functions $\alpha_i : \mathbb{R}_+ \to \mathbb{R}_+$ and a continuous function $\beta_2 : \mathbb{R}_+ \to \mathbb{R}_+$, satisfying $\beta_2(0) = 0$, such that

$$\begin{array}{lcl} \dot{a}_1 & \leq & -\alpha_1(b_1), \\ \dot{a}_2 & \leq & -\alpha_2(b_2) + \beta_2(b_1). \end{array} \quad (16)$$

Then

$$\lim_{t\to+\infty} \int_0^t \alpha_1(b_1(s))ds < +\infty$$

and

$$\liminf_{t\to+\infty} [b_1(t) + b_2(t)] = 0.$$

Remark 2: As will be illustrated by the examples in Section IV, the hypotheses of Lemma 1, which imply

$$\liminf_{t\to+\infty} [b_1(t) + b_2(t)] = 0,$$

do not guarantee, in general, the condition

$$\lim_{t\to+\infty} [b_1(t) + b_2(t)] = 0.$$

The extension of Lemma 1 to k differential inequalities is given in the following statement.

Lemma 2: Let $k \geq 3$, $i \in \{1, \ldots, k\}$ and $j \in \{2, \ldots, k\}$. Suppose there exist bounded differentiable functions $a_i : \mathbb{R}_+ \to [1, +\infty)$, bounded continuous functions $b_i : \mathbb{R}_+ \to \mathbb{R}_+$, continuous positive definite functions $\alpha_i : \mathbb{R}_+ \to \mathbb{R}_+$

and continuous functions $\beta_j : \mathbb{R}_+^{j-1} \to \mathbb{R}_+$, satisfying $\beta_j(0) = 0$, such that

$$\begin{array}{rcl} \dot{a}_1 & \leq & -\alpha_1(b_1), \\ \dot{a}_2 & \leq & -\alpha_2(b_2) + \beta_2(b_1), \\ & \vdots & \\ \dot{a}_{k-1} & \leq & -\alpha_{k-1}(b_{k-1}) + \beta_{k-1}(b_1, b_2, \cdots, b_{k-2}), \\ \dot{a}_k & \leq & -\alpha_k(b_k) + \beta_k(b_1, b_2, \cdots, b_{k-1}). \end{array}$$

Then

$$\lim_{t\to+\infty} \int_0^t \alpha_1(b_1(s))ds < +\infty$$

and

$$\liminf_{t\to+\infty} [b_1(t) + b_2(t) + \cdots + b_k(t)] = 0.$$

IV. Examples

In this section we present two examples which allow to clarify the results of the paper and to highlight that the condition (10) cannot be improved without additional assumption.

Consider the 3-dimensional system

$$\begin{array}{rcl} \dot{x}_1 & = & (x_{1+}^2 + x_3^p)\, x_2, \\ \dot{x}_2 & = & -(x_{1+}^2 + x_3^p)\, x_1, \\ \dot{x}_3 & = & -x_3^q, \end{array} \tag{17}$$

where $x_{1+} = \max\{x_1, 0\}$, p is a positive even integer and q is a positive odd integer. The set of equilibrium points is given by $\{(x_1, x_2, x_3) : x_{1+} = x_3 = 0\}$.

A. *The case* $p = 2$, $q = 3$

Let

$$V_0(x_1, x_2, x_3) = V_1(x_1, x_2, x_3) \; = \; \frac{x_1^2 + x_2^2 + x_3^2}{2}$$

and

$$V_2(x_1, x_2, x_3) \; = \; x_2.$$

Then

$$\begin{array}{rcl} \dot{V}_0 \;=\; \dot{V}_1 & = & -x_3^4 \\ \dot{V}_2 & = & -(x_{1+}^2 + x_3^2)\, x_1 \\ & = & -\left(x_{1+}^2 + x_3^2\right) x_{1+} \; + \; \sqrt{-\dot{V}_1}|x_{1-}|, \end{array} \tag{18}$$

where $x_{1-} = \min\{x_1, 0\}$. As a result, conditions (M1), (M2) and (M3)$'$ hold hence

$$\lim_{t\to+\infty} \int_0^t x_3^4(\tau)d\tau \; < +\infty\,, \quad \liminf_{t\to+\infty} \left(x_3^2(t) + x_{1+}(t)^3\right) = 0. \tag{19}$$

We now study if it is possible to obtain stronger asymptotic properties. For, observe that the first equality in (18) implies that the origin is globally stable and that all the solutions converge to the largest invariant set contained in the intersection of a level set of V_0 with the set $\{(x_1, x_2, x_3) : x_3 = 0\}$. In such a set, we have

$$\dot{V}_2 \; = \; -x_{1+}^3 \; \leq \; 0,$$

and one could be tempted to claim that the solutions converge to the largest invariant set contained also in the set $\{(x_1, x_2, x_3) : x_3 = x_{1+} = 0\}$. This is not the case, since, in general, a generic solution does not inherit the properties of the solutions in the invariant set. To clarify this statement re-write the system using polar coordinates (θ, ρ) in the (x_1, x_2)-plane, *i.e.*

$$\begin{array}{rcl} \dot{\rho} & = & 0, \\ \dot{\theta} & = & -(\rho^2 \cos(\theta)_+^2 + x_3^2), \\ \dot{x}_3 & = & -x_3^3, \end{array}$$

and note that $\rho(t) = \rho(0)$,

$$\theta(t) \; \leq \; \theta(0) \; - \; \int_0^t x_3^2(s)ds$$

and

$$x_3(t) = \exp\left(-\int_0^t x_3^2(s)ds\right) x_3(0).$$

As a result

$$\theta(t) \; \leq \; [\theta(0) - \log(x_3(0))] \; + \; \log(x_3(t))$$

and, since $\lim_{t\to\infty} x_3(t) = 0$, $\theta(t)$ tends to $-\infty$ modulo 2π, *i.e.* $\theta(t)$ does not converge. In addition, since $\rho(t)$ is constant, the vector $(x_1(t), x_2(t))$ has a constant modulus and does not stop turning around the origin, which implies that

$$\lim_{t\to+\infty} x_3(t) = 0, \qquad \liminf_{t\to+\infty} x_{1+}(t) = 0,$$

but also that

$$\limsup_{t\to+\infty} x_{1+}(t) = |x_1(0)|.$$

This last equation shows that the asymptotic property expressed by the second of equations (19) cannot be improved.

Figure 1 shows the phase portrait of the trajectories of the system with initial condition $x(0) = \left[\begin{array}{ccc} 1 & 0 & 1 \end{array}\right]'$, whereas Figure 2 shows the time histories of the states x_1, x_2 and x_3. Note that the time axis is in log-scale. Figure 2 highlights that all trajectories with initial condition off the (x_1, x_2)-plane have an oscillatory behavior with a period that tends to infinity. Note that trajectories with initial conditions such that $x_3(0) = 0$ converge to the set

$$\{(x_1, x_2) \mid x_1^2 + x_2^2 = x_1(0)^2 + x_2(0)^2, x_1 \leq 0\},$$

i.e. to a semi-circle centered at the origin, the size of which depends upon the initial conditions. This set is not stable, hence condition (M4) does not hold.

Remark 3: The ω-limit set of the trajectories of the system starting off the (x_1, x_2)-plane is, as detailed in [10], a chain recurrent set, which strictly contains the ω-limit set of the trajectories of the system starting in the (x_1, x_2)-plane, consistently with the results in [10] on asymptotically autonomous semiflows.

Remark 4: As a consequence of the discussion in this section, the (x_1, x_2)-subsystem of system (17), with $p = 2$ and $q = 3$, and x_3 regarded as an input, does not possess the converging-input converging-state property, see [11]. This

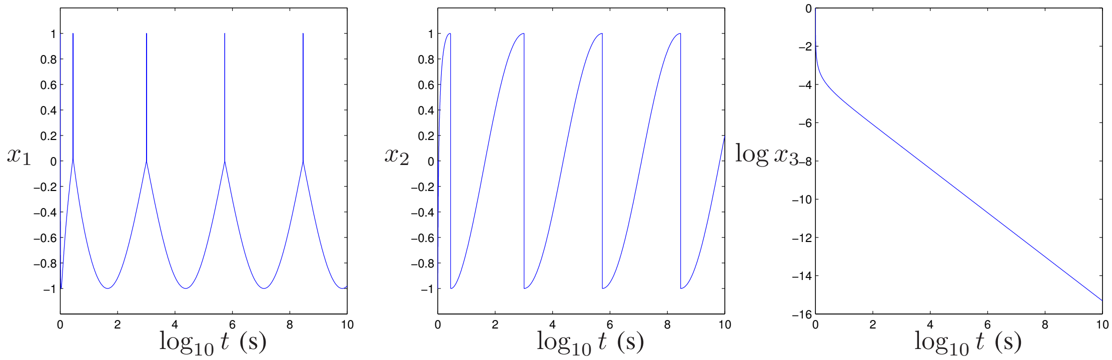

Fig. 2. Time histories of the states of the system (17) with $p = 2$, $q = 3$ and $x(0) = [1\ 0\ 1]'$. Note that the states x_1 and x_2 undergo fast transients.

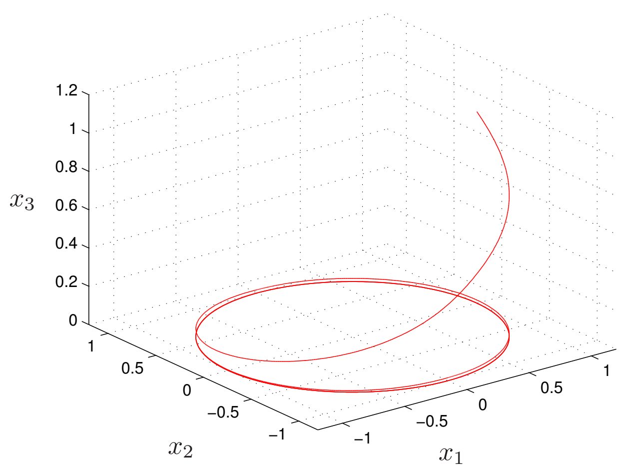

Fig. 1. The trajectory of the system (17), with $p = 2$, $q = 3$ and $x(0) = [1\ 0\ 1]'$.

does not contradict the result in [11], which highlights (among other things, and similarly to what done in this paper) the importance of asymptotic stability (of an equilibrium, or of a set) to establish asymptotic properties of solutions.

Remark 5: The result of the above discussion has to be interpreted on the basis of the results in [12]. Therein it is shown that the ω-limit set of any solution of the sub-system

$$\dot{x}_1 = x_{1+}^2 x_2, \qquad \dot{x}_2 = -x_{1+}^2 x_1$$

is chain recurrent. This says *approximately* that any point of this set can be made a point of an homoclinic orbit when solutions are taken as limit, for $\varepsilon \to 0$, of solutions, on $[0, +\infty)$, of the perturbed system

$$\dot{x}_1 = x_{1+}^2 x_2 + u_1(t), \qquad \dot{x}_2 = -x_{1+}^2 x_1 + u_2(t),$$

where (u_1, u_2) has L^∞ and L^2-norm smaller than ε. For system (17) the *perturbations* are given by

$$u_1(t) = x_3(t)^2 x_2(t), \qquad u_2(t) = -x_3(t)^2 x_1(t).$$

These perturbations are such that

$$\begin{aligned} u_1(t)^2 + u_2(t)^2 &= x_3(t)^4 (x_1(0)^2 + x_2(0)^2) \\ &\leq x_3(0)^4 (x_1(0)^2 + x_2(0)^2), \end{aligned}$$

and

$$\begin{aligned} \int_0^t [u_1(s)^2 + u_2(s)^2] ds &= (x_1(0)^2 + x_2(0)^2) \int_0^t x_3(s)^4 ds \\ &\leq (x_1(0)^2 + x_2(0)^2) \frac{x_3(0)^2}{2}, \end{aligned}$$

hence, selecting $x_3(0) = o(\sqrt{\varepsilon})$, the constraints on (u_1, u_2) are satisfied. The consequence is that the ω-limit set of any solution of the (x_1, x_2)-subsystem with $x_3 = 0$, which is composed of equilibrium points located on the portion of a circle centered at the origin and having the x_1-coordinate non-positive, can be transformed into the full circle by such *small* perturbations.

B. The cases $p = 2$, $q = 1$ and $p = 4$, $q = 3$

Consider system (17) with $p = 2$, $q = 1$ or $p = 4$, $q = 3$. In this case we have

$$\dot{V}_2 = -(x_{1+}^2 + x_3^2) x_1 = -\left(x_{1+}^2 + x_3^p\right) x_{1+} - \dot{V}_1 |x_{1-}|,$$

yielding

$$\dot{V}_2 + k\dot{V}_1 \leq -\left(x_{1+}^2 + x_3^p\right) x_{1+}, \tag{20}$$

for all (x_1, x_2, x_3) such that $x_{1-} > -k$. Recall now that all trajectories are bounded, *i.e.* remain in a compact set. Hence it is possible to select k (as a function of the initial conditions) such that the inequality (20) holds for all $t \geq 0$. By LaSalle invariance principle we conclude that all solutions converge to the largest invariant set contained in a level set of V_1 and in the set $\{(x_1, x_2, x_3) : x_{1+} = x_3 = 0\}$. This convergence result is illustrated in Figure 3, which shows a trajectory with initial state $x(0) = [1\ 0\ 1]'$. Similarly to the case in Section IV-A, the x_1 and x_2 states show an oscillatory behavior but, unlike the case in Section IV-A, $x_1(t)$ and $x_2(t)$ converge to a point such that $x_{1+} = 0$.

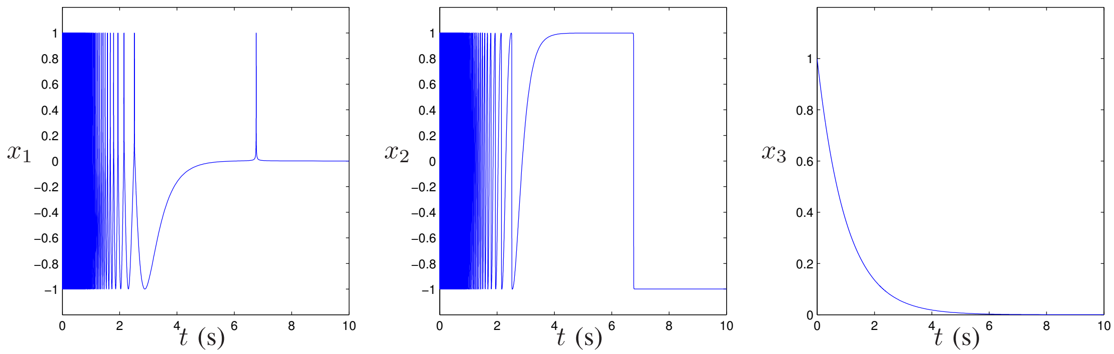

Fig. 3. Time histories of the states of the system (17) with $p = 2$, $q = 1$ and $x(0) = [1\ 0\ 1]'$. Note that the states x_1 and x_2 undergo fast transients.

V. Conclusions

A LaSalle version of Matrosov theorem has been stated and proved. It has been shown that stability plays a crucial role in the study of the asymptotic behavior of trajectories and that a *naive* application of ideas borrowed from LaSalle invariance principle may yield wrong conclusions. The theory has been illustrated by means of two simple examples.

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