Nonlinear Observer Design with an Appropriate Riemannian Metric

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***Abstract*— An observer whose state lives in the same space as the one of the given system and which guarantees a vanishing estimation error exhibits necessarily a symmetric matrix field which is related to the local observability information. A direct construction of this matrix field is possible by solving off-line ordinary differential equations. Using this symmetric matrix field as a Riemannian metric, we prove that geodesic convexity of the level sets of the output function is sufficient to allow the construction of an observer that contracts the geodesic distance between the estimated state and the system's state, globally in the estimated state and semi-globally in the estimation error.**

I. Introduction

For a complete nonlinear system of the form

$$\dot{x} \;=\; f(x) \quad , \qquad y \;=\; h(x) \;, \tag{1}$$

with $x \in \mathbb{R}^n$ being the system's state and $y \in \mathbb{R}$ the measured system's output, we consider the problem of obtaining an estimate $\hat{x}$ of the state x by means of the dynamical system, called *observer*,

$$\dot{\chi} \;=\; F(\chi, y) \quad , \qquad \hat{x} \;=\; H(\chi, y), \tag{2}$$

with $\chi \in \mathbb{R}^p$ being the observer's state, and $\hat{x} \in \mathbb{R}^n$ the observer's output, used as the system's state estimate. More precisely, we consider the following problem:

($\star$) Given functions f and h, design functions F and H such that, for the interconnection of systems (1) and (2), the set

$$\{(x, \chi) \in \mathbb{R}^n \times \mathbb{R}^p \;\mid x = H(\chi, h(x))\} \tag{3}$$

is globally asymptotically stable (see Section II for a definition).

This note focuses on the particular case where the state χ of the observer evolves in the same space as the system's state x, i.e., they both belong to $\mathbb{R}^n$. In such a case, we can pick the observer output function H trivial, i.e., pick

$$p \;=\; n \;, \qquad \hat{x} \;=\; \chi \;. \tag{4}$$

Many contributions from different points of view have been made to address this problem. While a summary of the very rich literature on the topic is out of the scope of this note, it is important for us to point out the interest of exploiting a possible non expansivity property of the flow generated by the observer which emerged from [10]. Study of non expansive flows has a very long history and has been proposed independently by several authors; see, e.g., [9], [6], [4], [11] (see also [8] for a historical discussion). Indeed, as we report in this note, when problem ($\star$) has a solution then there is necessarily a symmetric matrix field involved. It is then very tempting to use it as a Riemannian metric to measure the distance between system's state x and its estimation $\hat{x}$, and therefore, characterize the non expansivity of the observer flow.

Riemannian metrics have already been used in the context of observers in [1], [2] for instance. In these papers, the authors consider systems whose dynamics follow from a principle of least action, which involves such a metric, such as Euler-Lagrange systems with a Lagrangian that is quadratic in the generalized velocities. The Riemannian metric used in such observer designs depends only on the system vector field f. This is a key difference with the approach taken in this paper: the proposed metric depends on the pair (f, h), i.e., it incorporates the observability property of the system.

The paper contains three main parts. In Section II we show that an observer whose state χ lives in the same space as the state x of the given system and which guarantees a vanishing estimation error exhibits necessarily a symmetric matrix field which is related to the local observability information. In Section III we establish a relationship between the necessary condition in Section II and a local observability of system (1), as well as provide a construction of a symmetric matrix field satisfying the necessary conditions in Section II. Finally, in Section IV, using the above symmetric matrix field as a Riemannian metric, we propose a set of sufficient conditions for the construction of an observer guaranteeing contraction of the Riemannian distance between system's state and estimated state.

From our knowledge of the literature, we believe that the ideas which follow are new, although they can be seen as extension of what was proposed in [12] under the restriction of existence of a quadratic Lyapunov function depending only on the estimation error. For the sake of simplicity, all along this paper we work under, not always written, restrictions like, for instance, time independence, completeness of the given system, functions differentiable sufficiently many times, single output, $\mathbb{R}^n$ as system state manifold, among others. Further extensions relaxing some of these assumptions are in fact possible.

II. A Necessary Condition

Let $e = \hat{x} - x$ be the estimation error. The interconnection of system (1) and observer (2) under the conditions in (4)

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admits (x, e) as state with dynamics given by

$$\dot{x} = f(x) \quad , \qquad \dot{e} = F(x+e, h(x)) - f(x) \ . \tag{5}$$

In this context, the set to be rendered globally asymptotically stable (GAS) takes the form

$$\mathcal{A} = \{(x, e) \in \mathbb{R}^n \times \mathbb{R}^n \ | \ e = 0\} \ . \tag{6}$$

By GAS of this particular set, we mean that there exists a class-$\mathcal{KL}$ function β such that for all pairs (x, e) in $\mathbb{R}^n \times \mathbb{R}^n$, the solution $(X((x, e), t), E((x, e), t))$ of (5), issued from (x, e), is right maximally defined on $[0, +\infty)$ and satisfies

$$|E((x, e), t)| \ \leq \ \beta(|e|, t) \qquad \forall t \geq 0 \ .$$

To state the following proposition, we need to introduce the Lie derivative $\mathcal{L}_f P$ of the symmetric covariant tensor field P of order 2 (see [3, Exercise V.2.8]). In x coordinates, it satisfies:

$$\begin{aligned} & v^\top \mathcal{L}_f P(x)\, v \\ & \quad = \ \frac{\partial}{\partial x}\left(v^\top P(x)\, v\right) f(x) \ + \ 2\, v^\top P(x) \left(\frac{\partial f}{\partial x}(x)\, v\right) \ . \end{aligned} \tag{7}$$

Proposition 2.1: If the set $\mathcal{A}$ is GAS for (5), then there exists a C^∞ function $P : \mathbb{R}^n \to \mathbb{R}^{n*n}$ with nonnegative symmetric matrix values satisfying

$$v^\top \mathcal{L}_f P(x)\, v \leq -v^\top P(x)\, v \quad \forall (x, v) : \ \frac{\partial h}{\partial x}(x)\, v = 0. \tag{8}$$

Proof: Since the set $\mathcal{A}$ is GAS, there exists (see, for instance, [15, Theorem 3.2]) a C^∞ function $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfying for all x in $\mathbb{R}^n$

$$\begin{gathered} V(x, 0) \ = \ 0 \quad \frac{\partial V}{\partial e}(x, 0) \ = \ 0 \quad \frac{\partial^2 V}{\partial e \partial x}(x, 0) \ = \ 0, \\ P(x) \ := \ \frac{\partial^2 V}{\partial e^2}(x, 0) \ \geq \ 0, \end{gathered} \tag{9}$$

$$\begin{aligned} & \frac{\partial V}{\partial x}(x, e) f(x) \ + \ \frac{\partial V}{\partial e}(x, e)\left(F(x+e, h(x)) - f(x)\right) \\ & \qquad \leq \ -V(x, e) \qquad \forall (x, e) \in \mathbb{R}^n \times \mathbb{R}^n \ . \end{aligned} \tag{10}$$

Since $\mathcal{A}$ is stable, it is also forward invariant. Then, the solutions to (5) with $e = 0$ as initial condition remain in $\mathcal{A}$ for all $t \geq 0$. In other words, $e = 0$ is an equilibrium point of

$$\dot{e} = F(x(t) + e, h(x(t))) - f(x(t))$$

for any C^1 function $t \mapsto x(t)$. This establishes

$$F(x, h(x)) = f(x) \qquad \forall x \in \mathbb{R}^n \ . \tag{11}$$

By differentiating this identity with respect to x, we get

$$\frac{\partial F}{\partial x}(x, h(x)) \ + \ \frac{\partial F}{\partial y}(x, h(x)) \frac{\partial h}{\partial x}(x) \ = \ \frac{\partial f}{\partial x}(x)$$

and therefore

$$\begin{aligned} & \frac{\partial F}{\partial x}(x, h(x))\, v = \frac{\partial f}{\partial x}(x)\, v \\ & \qquad \forall (x, v) \in \mathbb{R}^n \times \mathbb{R}^n \ : \ \frac{\partial h}{\partial x}(x)\, v = 0 \end{aligned} \tag{12}$$

Setting $e = rv$, with $r \in \mathbb{R}$, in (10), we obtain

$$\begin{aligned} & \lim_{r \to 0} \frac{1}{r^2} \frac{\partial V}{\partial x}(x, rv) f(x) \\ & \quad + \lim_{r \to 0} \frac{\frac{\partial V}{\partial e}(x, rv)}{r} \frac{F(x + rv, h(x)) - f(x))}{r} \\ & \qquad \leq \ -\lim_{r \to 0} \frac{1}{r^2} V(x, rv) \end{aligned} \tag{13}$$

With (9) and (11), we have that, for all (x, v) in $\mathbb{R}^n \times \mathbb{R}^n$

$$\begin{aligned} & \lim_{r \to 0} \frac{\frac{\partial V}{\partial e}(x, rv)}{r} \frac{F(x + rv, h(x)) - f(x))}{r} \\ & \qquad = v^\top \frac{\partial^2 V}{\partial e^2}(x, 0) \frac{\partial F}{\partial x}(x, h(x)) v \ . \end{aligned} \tag{14}$$

To compute the limit of the first term of (13), note that the Taylor expansion of $\frac{\partial V}{\partial x}(x, e)$ around $e = 0$ is given by

$$\begin{aligned} \frac{\partial V}{\partial x}(x, e) = {} & \frac{\partial V}{\partial x}(x, 0) + \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial e}(x, 0)\, e\right) \\ & + \frac{1}{2} \frac{\partial}{\partial x}\left(e^\top \frac{\partial^2 V}{\partial e^2}(x, 0)\, e\right) + O_x(|e|^3). \end{aligned}$$

With (9), it follows that

$$\lim_{r \to 0} \frac{1}{r^2} \frac{\partial V}{\partial x}(x, rv) = \frac{1}{2} \frac{\partial}{\partial x}\left(v^\top P(x)\, v\right) \ . \tag{15}$$

Similarly, we get

$$\lim_{r \to 0} \frac{1}{r^2} V(x, rv) \ = \ \frac{1}{2} v^\top P(x)\, v \ . \tag{16}$$

Then, combining (14), (15), and (16), and using equation (12), we have that equation (13) becomes

$$\begin{aligned} & v^\top P(x) \frac{\partial f}{\partial x}(x)\, v + \frac{1}{2} \frac{\partial}{\partial x}\left(v^\top P(x)\, v\right) f(x) \leq -\frac{1}{2} v^\top P(x)\, v \\ & \qquad \forall (x, v) \in \mathbb{R}^n \times \mathbb{R}^n \ : \ \frac{\partial h}{\partial x}(x)\, v = 0 \ . \end{aligned}$$

∎

III. A Link with Local Observability

The necessary condition in (8) is linked to properties of the family of linear time-varying systems obtained from linearizing (1) along its solutions. We denote by $X(x, t)$ a solution to (1) at time t issued from x. Since (1) is assumed to be complete, for each x, $t \mapsto X(x, t)$ is defined on $(-\infty, +\infty)$. The linearization of f and h evaluated along a solution $X(x, t)$ gives the functions

$$A_x(t) \ = \ \frac{\partial f}{\partial x}(X(x, t)) \quad , \qquad C_x(t) \ = \ \frac{\partial h}{\partial x}(X(x, t)) \ .$$

They allow us to define the following family of linear time-varying systems:

$$\dot{\xi} \ = \ A_x(t)\, \xi \quad , \qquad \eta \ = \ C_x(t)\, \xi, \tag{17}$$

with state $\xi \in \mathbb{R}^n$ and output $\eta \in \mathbb{R}$. Systems (17) are parameterized by the initial condition x of the chosen solution $X(x, t)$. Their state transition matrices are denoted as Φ_x for a given initial condition $x \in \mathbb{R}^n$.

To state our following proposition, we need two definitions.

Definition 3.1:

1) Given $x \in \mathbb{R}^n$, system (17) is said to be uniformly detectable if there exists a continuous function $t \mapsto K_x(t)$ such that the origin of

$$\dot{\xi} \;=\; (A_x(t) - K_x(t)C_x(t))\,\xi \tag{18}$$

is uniformly exponentially stable.

2) The family of systems (17) is said to be reconstructible uniformly in x if there exist strictly positive real numbers τ and ϵ such that we have, for all x in $\mathbb{R}^n$,

$$\int_{-\tau}^{0} \Phi_x(s,0)^\top C_x(s)^\top C_x(s)\Phi_x(s,0)ds \;\geq\; \epsilon\, I\ . \tag{19}$$

Proposition 3.2:

1) Suppose there exist strictly positive real numbers p_i and p_s, and a function $P : \mathbb{R}^n \to \mathbb{R}^{n*n}$ with positive symmetric matrix values satisfying condition (8) and

$$0 \;<\; p_i\, I \;\leq\; P(x) \leq\; p_s\, I \quad \forall x \in \mathbb{R}^n, \tag{20}$$

Then, for each $x \in \mathbb{R}^n$, the linear time-varying system (17) is detectable.

2) Conversely, suppose that the family of systems (17) is reconstructible uniformly in x. Furthermore, assume that the functions f and h have bounded differential. Then, there exist a strictly positive real number λ and a continuous function $P : \mathbb{R}^n \to \mathbb{R}^{n*n}$ satisfying (20) such that the system

$$\begin{aligned} \dot{\Pi} &= -\Pi\,\frac{\partial f}{\partial x}(x) \;-\; \frac{\partial f}{\partial x}(x)^\top \Pi \\ &\qquad +\; \frac{\partial h}{\partial x}(x)^\top \frac{\partial h}{\partial x}(x) \;-\; \lambda\,\Pi, \\ \dot{x} &= f(x) \end{aligned} \tag{21}$$

admits the set $\{(x,\Pi) \in \mathbb{R}^n \times \mathbb{R}^{n*n} \;:\; \Pi = P(x)\}$ as an invariant manifold.

Proof: We show item 1) first. We start by establishing the existence of a continuous function $k : \mathbb{R}^n \to \mathbb{R}$ satisfying for all x in $\mathbb{R}^n$

$$\mathcal{L}_f P(x) \;\leq\; k(x)\,\frac{\partial h}{\partial x}(x)^\top \frac{\partial h}{\partial x}(x) \;-\; \frac{1}{2}\,P(x)\ . \tag{22}$$

For this, let C_i be the compact set

$$C_i = \{x \in \mathbb{R}^n \;:\; i \leq |x| \leq i+1\}.$$

There exists a real number k_i such that for all $k \geq k_i$, we have for all x in C_i

$$\mathcal{L}_f P(x) \;\leq\; k\,\frac{\partial h}{\partial x}(x)^\top \frac{\partial h}{\partial x}(x) \;-\; \frac{1}{2}\,P(x)\ . \tag{23}$$

Indeed, if not, with $\mathbb{S}^n$ being the unit sphere in $\mathbb{R}^n$, there is a sequence (x_n, v_n) in $C_i \times \mathbb{S}^n$ satisfying :

$$v_n^\top \mathcal{L}_f P(x_n)\, v_n \;+\; \frac{1}{2}\, v_n^\top P(x_n)\, v_n \;\geq\; n\,\left|\frac{\partial h}{\partial x}(x_n)\, v_n\right|^2\ .$$

Since the functions $\mathcal{L}_f P$ and P are continuous, the left hand side is bounded. So any accumulation point (x_ω, v_ω), known to exist by compactness of $C_i \times \mathbb{S}^n$, satisfies

$$\begin{aligned} v_\omega^\top \mathcal{L}_f P(x_\omega)\, v_\omega \;+\; \tfrac{1}{2}\, v_\omega^\top P(x_\omega)\, v_\omega \;\geq\; 0 \quad , \\ \left|\frac{\partial h}{\partial x}(x_\omega)\, v_\omega\right|^2 \;=\; 0\ . \end{aligned} \tag{24}$$

Since from (20) $v_\omega^\top P(x_\omega)\, v_\omega$ is strictly positive, (24) contradicts (8). Then, (23) holds. It follows that any continuous function k satisfying

$$k(x) \;\geq\; k_i \qquad \forall x \in C_i\ , \quad \forall i$$

satisfies the claim.

Now, to any $x \in \mathbb{R}^n$, we associate the functions :

$$\Pi_x(t) \;=\; P(X(x,t))\ , \quad \mathcal{V}_x(\xi, t) \;=\; \xi^\top \Pi_x(t)\, \xi\ ,$$
$$K_x(t) \;=\; \frac{k(X(x,t))}{2}\,\Pi_x(t)^{-1}\, C_x(t)^\top\ .$$

We have :

$$p_i\,|\xi|^2 \;\leq\; \mathcal{V}_x(\xi,t) \;\leq\; p_s\,|\xi|^2 \qquad \forall (x,t,\xi) \tag{25}$$

and, with (22) and (7) :

$$\begin{aligned} &\frac{d}{dt}\left(v^\top \Pi_x(t)\, v\right) \\ &\quad= \frac{\partial}{\partial x}\left(v^\top P(x)\, v\right)\, f(x)\Big|_{x=X(x,t)}\ , \\ &\quad\leq -v^\top \Pi_x(t)\, v \;-\; 2\, v^\top \Pi_x(t)\,(A_x(t) - K_x(t)C_x(t))\, v\ . \end{aligned}$$

So, for $\dot{\xi}$ given by (18), we get :

$$\frac{d}{dt}\mathcal{V}_x(\xi,t) \;\leq\; -\mathcal{V}_x(\xi,t)\ .$$

The conclusion follows with (25).

To show item 2), note that, by boundedness of $\frac{\partial f}{\partial x}$ and $\frac{\partial h}{\partial x}$, there exist scalars M_1 and M_2 such that

$$|\Phi_x(t,s)| \leq \exp(M_1|t-s|) \quad , \qquad |C_x(t)| \leq M_2$$
$$\forall (x,t,s) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}\ .$$

Let $\lambda > 2M_1$. Define

$$P(x) \;=\; \lim_{T \to -\infty} \int_{T}^{0} \exp(\lambda s)\Phi_x(s,0)^\top C_x(s)^\top C_x(s)\Phi_x(s,0)ds\ . \tag{26}$$

Since, for each x, we have

$$\begin{aligned} &|\exp(\lambda s)\Phi_x(s,0)^\top C_x(s)^\top C_x(s)\Phi_x(s,0)| \\ &\qquad\leq\; M_2^2\, \exp([\lambda - 2M_1]s) \qquad \forall s \leq 0\ , \end{aligned}$$

and, for each s, the function $x \mapsto X(x,s)$ and therefore the functions $x \mapsto (\Phi_x(s,0), C_x(s))$ are continuous, P is a continuous function on $\mathbb{R}^n$. Using (19), it satisfies, for all x in $\mathbb{R}^n$,

$$\begin{aligned} P(x) \;&\geq\; \int_{-\tau}^{0} \exp(\lambda s)\Phi_x(s,0)^\top C_x(s)^\top C_x(s)\Phi_x(s,0)ds \\ &\geq\; \epsilon\exp(-\lambda\tau)\, I \qquad \forall x \in \mathbb{R}^n, \end{aligned}$$

$$P(x) \leq M_2^2 \int_{-\infty}^0 \exp(\lambda s)\, \exp(2M_1|s|) ds\, I = \frac{M_2^2}{\lambda - 2M_1} I$$

Then, the bounds in (20) hold with $p_i = \epsilon \exp(-\lambda\tau)$ and $p_s = \frac{M_2^2}{\lambda - 2M_1}$.

To conclude our proof, it remains to show that $(X(x,t), P(X(x,t))$ is a solution of (21). For this it is sufficient to prove that $\frac{d}{dt}P(X(x,t))$ exists and satisfies

$$\begin{aligned}\frac{d}{dt}P(X(x,t)) \;=\; & -P(X(x,t))A_x(t) - A_x(t)^\top P(X(x,t)) \\ & + C_x(t)^\top C_x(t) \;-\; \lambda\, P(X(x,t))\ . \end{aligned} \tag{27}$$

Evaluating (26) along $X(x,t)$ gives

$$\begin{aligned}P(X(x,t)) = \lim_{T\to-\infty} \int_T^0 & \exp(\lambda s)\Phi_{X(x,t)}(s,0)^\top C_{X(x,t)}(s)^\top \\ & \times C_{X(x,t)}(s)\Phi_{X(x,t)}(s,0) ds\end{aligned}$$

where, by definition,

$$\begin{aligned} C_{X(x,t)}(s) &= \frac{\partial h}{\partial x}(X(X(x,t),s)) \\ &= \frac{\partial h}{\partial x}(X(x,t+s)) \;=\; C_x(t+s)\ . \end{aligned} \tag{28}$$

Similarly, we have

$$A_{X(x,t)}(s) \;=\; A_x(t+s) \tag{29}$$

and therefore

$$\Phi_{X(x,t)}(s,0) \;=\; \Phi_x(s+t,0)\,\Phi_x(0,t)\ . \tag{30}$$

Note finally that

$$\Phi_x(s,r)\,\Phi_x(r,s) \;=\; I \qquad \forall (s,r)$$

implies

$$\frac{\partial \Phi_x}{\partial r}(s,r) \;=\; -\Phi_x(s,r)\,A_x(r) \qquad \forall (s,r)\ . \tag{31}$$

Using the expressions above, and changing $s+t$ into s inside the integral, $P(X(x,t))$ can be expressed as

$$\begin{aligned} & P(X(x,t)) \\ &= \lim_{T\to-\infty}\int_T^0 \exp(\lambda s)\Phi_x(0,t)^\top \Phi_x(s+t,0)^\top C_x(t+s)^\top \\ &\qquad\qquad \times C_x(t+s)\Phi_x(s+t,0)\,\Phi_x(0,t) ds\ , \\ &= \exp(-\lambda t)\Phi_x(0,t)^\top \\ &\quad \times \left[\lim_{T\to-\infty}\int_T^t \exp(\lambda s)\Phi_x(s,0)^\top C_x(s)^\top \right. \\ &\qquad\qquad \left. \times\ C_x(s)\Phi_x(s,0)\, ds\right] \Phi_x(0,t)\ . \end{aligned}$$

We conclude that (27) holds by taking the limit of $\frac{P(X(x,t+r)) - P(X(x,t))}{r}$ when r goes to 0, with the help of (31). ∎

Item 1) in Proposition 3.2 indicates that the existence of P satisfying (1) is closely related to the local observability information of (1). The second item, and more specifically the expression (26), suggests a method to approximate $P(x)$. Given a point $x \in \mathbb{R}^n$ where we want to evaluate P, we compute the solution $X(x,t)$ to $\dot{x} = f(x)$ backward in time from the initial condition x, at time $t = 0$, up to negative time $t = -T$, for some $T > 0$ such that $\exp(-\lambda T)$ is sufficiently small. Then, $P(x)$ is given by $\Pi(0)$, the solution at time $t = 0$ of

$$\dot{\Pi} \;=\; -\Pi A_x(t) \;-\; A_x(t)^\top \Pi \;+\; C_x(t)^\top C_x(t) \;-\; \lambda\,\Pi$$

with initial condition $\Pi(-T) = 0$ at time $t = -T$.

IV. Sufficient conditions

In this section, we employ a symmetric matrix field P satisfying

$$\mathcal{L}_f P(x) \;-\; \frac{\partial h}{\partial x}(x)^\top \frac{\partial h}{\partial x}(x) \;<\; 0 \qquad \forall x \in \mathbb{R}^n$$

to design the function F of the observer (2). To that end, we use P as a Riemannian metric on $\mathbb{R}^n$. Then, define the length of a C^1 path γ between points x_1 and x_2 as

$$L(\gamma)\Big|_{s_1}^{s_2} \;=\; \int_{s_1}^{s_2} \sqrt{\frac{d\gamma}{ds}(s)^\top P(\gamma(s))\frac{d\gamma}{ds}(s)}\, ds$$

where

$$\gamma(s_1) \;=\; x_1 \quad , \qquad \gamma(s_2) \;=\; x_2\ .$$

The Riemannian distance $d(x_1,x_2)$ between two such points is then the minimum of $L(\gamma)\Big|_{s_1}^{s_2}$ among all possible piecewise C^1 path γ between x_1 and x_2. With the Hopf-Rinow Theorem (see [3, Lemma VII.7.8]), we know that, if every geodesic can be maximally extended to $\mathbb{R}$, then the minimum of $L(\gamma)\Big|_{s_1}^{s_2}$ is actually given by the length of a (maybe nonunique) geodesic, which is called a minimal geodesic. In the following, γ^* denotes such a minimal geodesic.

Lemma 4.1: Suppose that a function $P : \mathbb{R}^n \to \mathbb{R}^{n*n}$ with symmetric values satisfies

$$0 \;<\; P(x) \quad \forall x \in \mathbb{R}^n \quad , \qquad \lim_{r\to\infty} r^2 p_i(r) \;=\; +\infty\ , \tag{32}$$

where, for any positive real number r,

$$p_i(r) \;=\; \min_{x:|x|\leq r}\ \min_{v:|v|=1} v^\top P(x) v\ .$$

Then, with P as Riemannian metric, any geodesic can be maximally extended to $\mathbb{R}$.

Proof: Let x_1 and x_2 be any point in the ball B_r centered at the origin and with radius r. The Euclidean distance $|x_1 - x_2|$ satisfies

$$\int_{s_1}^{s_2} \left|\frac{d\gamma}{ds}(s)\right| ds \;\geq\; |x_1 - x_2|,\ .$$

where γ is any C^1 path between x_1 and x_2. This implies :

$$L(\gamma)\Big|_{s_1}^{s_2} \geq \sqrt{p_i(r)}\int_{s_1}^{s_2}\left|\frac{d\gamma}{ds}(s)\right| ds \;\geq\; \sqrt{p_i(r)}\,|x_1 - x_2|\,. \tag{33}$$

Let γ be any geodesic maximally defined on (σ_-, σ_+). By definition, it satisfies

$$\frac{d\gamma}{ds}(s)^\top P(\gamma(s))\,\frac{d\gamma}{ds}(s) \;=\; 1 \qquad \forall s \in (\sigma_-,\sigma_+)\ . \tag{34}$$

Let $[s_1, s_2]$ be any closed interval contained in (σ_-, σ_+). The function $\gamma : [s_1, s_2] \to \mathbb{R}^n$ is bounded (with the Euclidean norm). We denote

$$r_{[s_1,s_2]} \;=\; \max_{s\in[s_1,s_2]} |\gamma(s)| \;.$$

By continuity, there exists s_{12} in $[s_1, s_2]$ satisfying

$$r_{[s_1,s_2]} \;=\; |\gamma(s_{12})|.$$

Then, from (33), we obtain

$$\sqrt{p_i(r_{[s_1,s_2]})}\; |\gamma(s_{12}) - \gamma(s_2)| \;\le\; L(\gamma)\Big|_{s_{12}}^{s_2} \;=\; |s_{12} - s_2| \;. \tag{35}$$

Because (σ_-, σ_+) is the maximal interval of definition of γ, if σ_- is finite, we must have

$$\lim_{s_1\to\sigma_-} \left|\left(\gamma(s_1), \frac{d\gamma}{ds}(s_1)\right)\right| \;=\; +\infty \;.$$

But, by definition of s_{12}, we have the implication

$$\lim_{s_1\to\sigma_-} |\gamma(s_1)| \;=\; +\infty \quad\Rightarrow\quad \lim_{s_1\to\sigma_-} |\gamma(s_{12})| \;=\; +\infty$$

and therefore, with assumption (32),

$$\lim_{s_1\to\sigma_-} \sqrt{p_i(r_{[s_1,s_2]})}\; |\gamma(s_{12}) - \gamma(s_2)| \;=\; +\infty \;.$$

This contradicts (35). Then, we must have

$$\lim_{s_1\to\sigma_-} \left|\frac{d\gamma}{ds}(s_1)\right| \;=\; +\infty \;.$$

But, again, this contradicts (34) since we just established that γ is bounded on (σ_-, σ_+), which, with (32), implies that $P\circ\gamma$ is bounded away from 0.

The same arguments apply to show that $\sigma_+ = +\infty$. ■

In the following the function P is assumed to satisfy the conditions of Lemma 4.1. Consequently, the Riemannian distance is given by the length of minimal geodesics.

With these preliminaries, our problem is now to define the observer vector field $\hat{x} \mapsto F(\hat{x}, y)$ so that it makes the Riemannian distance $d(\hat{x}, x)$ between estimated state $\hat{x}$ and system state x to decrease along solutions. More precisely, let γ^* be a minimal geodesic satisfying

$$\gamma^*(0) \;=\; x \quad,\qquad \gamma^*(\hat{s}) \;=\; \hat{x} \;.$$

Then, the Riemannian distance $d(\hat{x}, x)$ is

$$d(\hat{x}, x) \;=\; L(\gamma^*)\Big|_0^{\hat{s}} \;=\; |\hat{s}| \;.$$

As already remarked in the proof of Proposition 2.1, a necessary condition for having the set $\mathcal{A}$ in (6), which is also the set of pairs $(\hat{x}, x)$ with $d(\hat{x}, x) = 0$, stable is

$$F(x, h(x)) \;=\; f(x) \qquad \forall x\in\mathbb{R}^n \;. \tag{36}$$

This is a first constraint we impose on F. It implies that the observer contains also all solutions to (1). Then to study how the distance $d(\hat{x}, x)$ evolves along the solutions, we define a C^1 function Γ as a solution of

$$\frac{\partial\Gamma}{\partial t}(s,t) \;=\; F(\Gamma(s,t), h(x)) \quad,\qquad \Gamma(s,0) \;=\; \gamma^*(s) \;.$$

From the first order variation formula (see [14, Theorem 6.14] or [7, Theorem 5.7] for instance), we get

$$\begin{aligned}\frac{d}{dt}d(\hat{x}, x) \;=\;& \frac{d\gamma^*}{ds}(\hat{s})^{\top} P(\gamma^*(\hat{s}))\, F(\gamma^*(\hat{s}), y) \\ &-\frac{d\gamma^*}{ds}(0)^{\top} P(\gamma^*(0))\, F(\gamma^*(0), y) \;.\end{aligned} \tag{37}$$

Since the last term on the right-hand side is imposed by (36), to obtain $\frac{d}{dt}d(\hat{x}, x)$ nonpositive we are left with choosing F so that $\frac{d\gamma^*}{ds}(\hat{s})^{\top} P(\gamma^*(\hat{s}))F(\gamma^*(\hat{s}), y)$ is negative enough to dominate that last term. Satisfying this requirement would not be a problem if $\frac{d\gamma^*}{ds}(\hat{s})$ were known. Indeed, by definition, since

$$\gamma^*(\hat{s}) \;=\; \hat{x},$$

it would be sufficient to choose, at least when $h(\hat{x})$ is far from y,

$$F(\hat{x}, y) \;=\; -k(\hat{x}, y)\, P(\hat{x})^{-1}\, \frac{d\gamma^*}{ds}(\hat{s})$$

with $k : \mathbb{R}^n\times\mathbb{R} \to \mathbb{R}_{\ge 0}$ an arbitrary C^1 function. But $\frac{d\gamma^*}{ds}(\hat{s})$ represents the direction in which the state estimate $\hat{x}$ "sees" the system state x along a minimal geodesic. Unfortunately, such a direction is unknown and we know only that x belongs to the following level set of the output function

$$\mathfrak{H}(y) \;=\; \{\bar{x} : h(\bar{x}) = y\} \;.$$

Then, to satisfy the above requirement, we need the property: given $\hat{x}$ and y, the level set of the output function $\mathfrak{H}(y)$ is "seen" from $\hat{x}$ within a cone whose aperture is less than π. This property implies that $\mathfrak{H}(y)$ is weakly geodesically convex.

Definition 4.2: A subset S of $\mathbb{R}^n$ is said to be weakly geodesically convex if, for any pair of points $(x_1, x_2) \in S$, there exists a minimal geodesic γ^* satisfying

$$\gamma^*(s_1) \;=\; x_1 \quad,\qquad \gamma^*(s_2) \;=\; x_2 \quad, \\ \gamma^*(s) \in S \quad \forall s\in[s_1, s_2] \;.$$

Lemma 4.3: Let P be a Riemannian metric, a subset S of $\mathbb{R}^n$ such that, for any $\hat{x}$ in $\mathbb{R}^n \setminus S$, there exists a unit vector $v_{\hat{x}}$ such that, for any minimal geodesic γ^* satisfying

$$\gamma^*(0) \in S \quad,\qquad \gamma^*(\hat{s}) \;=\; \hat{x} \;,$$

we have

$$\frac{d\gamma^*}{ds}(\hat{s})^{\top} P(\hat{x})\, v_{\hat{x}} \;<\; 0 \;,$$

is weakly geodesically convex.

Proof: Assume that S is not weakly geodesically convex. Then we can find pair (x_1, x_2) of points of S such that for any minimal geodesic γ_1^* satisfying :

$$\gamma_1^*(0) \;=\; x_1 \quad,\qquad \gamma_1^*(s_2) \;=\; x_2$$

there exists $\hat{s}_1$ in $(0, s_2)$ for which $\gamma_1^*(\hat{s}_1)$ is not in S. Let

$$\hat{x} \;=\; \gamma_1^*(\hat{s}_1) \;.$$

γ_1^* is a minimal geodesic satisfying

$$x_1 = \gamma_1^*(0) \in S \quad , \qquad \gamma_1^*(\hat{s}_1) \;=\; \hat{x} \notin S \ .$$

Similarly

$$\gamma_2^*(s) \;=\; \gamma_1^*(s_2 - s)$$

defines a minimal geodesic satisfying, with $\hat{s}_2 = s_2 - \hat{s}_1$,

$$x_2 = \gamma_2^*(0) \in S \quad , \qquad \gamma_2^*(\hat{s}_2) \;=\; \hat{x} \notin S \ .$$

With our assumption we know there exists a unit vector $v_{\hat{x}}$ satisfying

$$\frac{d\gamma_1^*}{ds}(\hat{s}_1)^\top P(\hat{x})\, v_{\hat{x}} \;<\; 0 \quad , \qquad \frac{d\gamma_2^*}{ds}(\hat{s}_2)^\top P(\hat{x})\, v_{\hat{x}} \;<\; 0 \ .$$

But this impossible since we have :

$$\frac{d\gamma_1^*}{ds}(\hat{s}_1) \;=\; -\frac{d\gamma_2^*}{ds}(\hat{s}_2) \ .$$

∎

This lemma motivates our restriction to consider the level set of the output function $\mathfrak{H}(y)$ as being weakly geodesically convex for any y in $\mathbb{R}$. Actually, we ask for the stronger property that $\mathfrak{H}(y)$ is an invariant set for the geodesic flow which implies the weak geodesic convexity.

Definition 4.4: We say that $\mathfrak{H}(y)$ is maximally geodesically convex for any y in $\mathbb{R}$ if, for any pair (x,v) in $\mathbb{R}^n \times \mathbb{R}^n$ satisfying :

$$\frac{\partial h}{\partial x}(x)\, v \;=\; 0 \quad , \qquad v^\top P(x)\, v \;=\; 1 \ ,$$

the geodesic γ satisfying :

$$\gamma(0) \;=\; x \quad , \qquad \frac{d\gamma}{ds}(0) \;=\; v$$

is defined on $(-\infty, +\infty)$ and takes its values in $\mathfrak{H}(h(x))$.

Remark 4.5: The maximal geodesic convexity of $\mathfrak{H}(y)$ for any y in $\mathbb{R}$ holds if we have

$$\frac{\partial^2 h}{\partial x_i \partial x_j}(x) \;-\; \sum_{m=1}^{n} \frac{\partial h}{\partial x_m}(x)\, \Gamma_{ij}^m(x) \;=\; g_i(x) \frac{\partial h}{\partial x_j}(x) \;+\; g_j(x) \frac{\partial h}{\partial x_i}(x) \qquad \forall (i,j) \ , \quad \forall x$$

where Γ_{ij}^m are the Christoffel symbols and g_i are arbitrary functions.

More about geodesic convexity can be found in [13] for instance.

Proposition 4.6: Let $P : \mathbb{R}^n \to \mathbb{R}^{n*n}$ be a sufficiently many time differentiable function with symmetric matrix values satisfying, for all x in $\mathbb{R}^n$,

$$0 \;<\; p_i\, I \;\le\; P(x) \;\le\; p_s\, I \tag{38}$$

$$\mathcal{L}_f P(x) \;-\; \frac{\partial h}{\partial x}(x)^\top \frac{\partial h}{\partial x}(x) \;\le\; -q_i\, I \;<\; 0 \ . \tag{39}$$

Assume the set $\mathfrak{H}(y)$ is maximally geodesically convex for any y in $\mathbb{R}$. Under these conditions, for any positive real number E there exists a continuous function $k_E : \mathbb{R}^n \to \mathbb{R}$ such that the observer given by

$$F(\hat{x}, y) \;=\; f(\hat{x}) \;+\; k_E(\hat{x})\, P(\hat{x})^{-1} \frac{\partial h}{\partial x}(\hat{x})\, [y - h(\hat{x})] \tag{40}$$

renders the set $\mathcal{A}$ asymptotically stable with domain of attraction containing the set $\{(x,\hat{x}) : |\hat{x} - x| < E\}$.

Proof: First we observe, with the help of Lemma 4.1, that our assumptions on P imply that the Riemannian distance $d(x_1, x_2)$ is given by the length of minimal geodesics between x_1 and x_2 and satisfies :

$$p_i\, |x_1 - x_2| \;\le\; d(x_1, x_2) \;\le\; p_s\, |x_1 - x_2| \ . \tag{41}$$

This implies that we have the inclusion :

$$\{(x,\hat{x}) : |\hat{x} - x| < E\} \;\subset\; \{(x,\hat{x}) : d(\hat{x}, x) < p_s E\} \ . \tag{42}$$

(38) and (39) imply also that we have

$$\mathcal{L}_f P(x) \;-\; \frac{\partial h}{\partial x}(x)^\top \frac{\partial h}{\partial x}(x) \;\le\; -\frac{q_i}{p_s}\, P(x) \quad \forall x \in \mathbb{R}^n \ . \tag{43}$$

Now, for any pair $(\hat{x}, x)$ in $\mathbb{R}^n \times \mathbb{R}^n$ and any minimal geodesic γ^* satisfying :

$$\gamma^*(0) \;=\; x \quad , \qquad \gamma^*(\hat{s}) \;=\; \hat{x} \ ,$$

(40) with $y = h(x)$, gives :

$$\begin{aligned} &\frac{d\gamma^*}{ds}(\hat{s})^\top P(\gamma^*(\hat{s}))\, [F(\gamma^*(\hat{s}), y) - f(\gamma^*(\hat{s}))] \\ &\qquad - \frac{d\gamma^*}{ds}(0)^\top P(\gamma^*(0))\, [F(\gamma^*(0), y) - f(\gamma^*(0))] \\ &\qquad\qquad = \; -k_E(\hat{x})\, \frac{d\, h \circ \gamma^*}{ds}(\hat{s})\, (h \circ \gamma^*(\hat{s}) - h(x)) \ . \end{aligned}$$

On the other hand, we have :

$$\begin{aligned} &\left[\frac{d\gamma^*}{ds}(\hat{s})^\top P(\hat{x}) f(\hat{x}) - \frac{d\gamma^*}{ds}(0)^\top P(x) f(x)\right] \\ &\qquad = \; \int_0^{\hat{s}} \frac{d}{ds}\left(\frac{d\gamma^*}{ds}(s)^\top P(\gamma^*(s))\, f(\gamma^*(s))\right) ds \ , \end{aligned}$$

But, with the definition of the Lie derivative $\mathcal{L}_f P$ and (43), we have :

$$\begin{aligned} &\frac{d}{ds}\left(\frac{d\gamma^*}{ds}(s)^\top P(\gamma^*(s))\, f(\gamma^*(s))\right) \\ &\qquad = \; \frac{1}{2}\, \frac{d\gamma^*}{ds}(s)^\top \mathcal{L}_f P(\gamma^*(s)) \frac{d\gamma^*}{ds}(s) \ , \\ &\qquad \le \; \frac{1}{2}\, \frac{d\, h \circ \gamma^*}{ds}(s)^2 \;-\; \frac{q_i}{2 p_s} \ , \end{aligned}$$

where, in the last equation, we have used :

$$\frac{d\gamma^*}{ds}(s)^\top P(\gamma^*(s)) \frac{d\gamma^*}{ds}(s) \;=\; 1 \ .$$

So, with (37), we have obtained :

$$\frac{d}{dt} d(\hat{x}, x) \;\le\; \frac{1}{2} \int_0^{\hat{s}} \frac{d\, h \circ \gamma^*}{ds}(s)^2 \;-\; \frac{q_i}{2 p_s}\, \hat{s}$$

$$- k_E(\hat{x}) \frac{d\, h\circ\gamma^*}{ds}(\hat{s})\, (h\circ\gamma^*(\hat{s}) - h(x)) \ .$$

By integrating by parts and using the fact that $d(\hat{x},x) = \hat{s}$, this yields:

$$\begin{aligned}\frac{d}{dt} d(\hat{x},x) \ \leq\ & -\frac{q_i}{2p_s}\, d(\hat{x},x) \qquad (44)\\ & - \left[k_E(\hat{x}) - \frac{1}{2}\right] \frac{d\, h\circ\gamma^*}{ds}(\hat{s})\, (h(\hat{x}) - h(x))\\ & - \frac{1}{2}\int_0^{\hat{s}} \left[\frac{d^2\, h\circ\gamma^*}{ds^2}(s)\, (h\circ\gamma^*(s) - h(x))\right] ds \ .\end{aligned}$$

In view of (41) and (42), to complete our proof, it suffices to show the existence of k_E such that we have :

$$\frac{d}{dt} d(\hat{x},x) \ \leq\ -\frac{q_i}{4p_s}\, d(\hat{x},x) \qquad \forall (x,\hat{x}) \in \mathcal{D}$$

where $\mathcal{D}$ is the open set

$$\mathcal{D} \ =\ \{(x,\hat{x}) \,:\, d(\hat{x},x) < p_s E\} \ .$$

As a preliminary step for this, we observe that, for any pair $(\hat{x},x)$ in $\mathbb{R}^n\times\mathbb{R}^n$ satisfying :

$$h(\hat{x}) \ \neq\ h(x)$$

and any geodesic γ satisfying :

$$\gamma(0) \ =\ x \quad , \qquad \gamma(\hat{s}) \ =\ \hat{x} \ ,$$

we have :

$$\frac{d\, h\circ\gamma}{ds}(s)\, (h\circ\gamma(s) - h(x)) \ >\ 0 \qquad \forall s\neq 0 \ . \qquad (45)$$

Indeed assume there exists s satisfying :

$$\begin{aligned}\frac{d\, h\circ\gamma}{ds}(s) \ =\ \frac{\partial h}{\partial x}(\gamma(s))\, \frac{d\gamma}{ds}(s) \ =\ 0 \quad ,\\ \frac{d\gamma}{ds}(s)^\top P(\gamma(s))\, \frac{d\gamma}{ds}(s) \ =\ 1 \ .\end{aligned}$$

Then the maximal geodesic convexity of $\mathfrak{H}(h(\gamma(s))$ implies :

$$h\circ\gamma(s) \ =\ h\circ\gamma(0) \ =\ h(x) \qquad \forall s$$

which contradicts :

$$h\circ\gamma(\hat{s}) \ =\ h(\hat{x}) \ \neq\ h(x) \ .$$

So $\dfrac{d\, h\circ\gamma}{ds}$ has a constant sign. But, since we have :

$$h(\hat{x}) - h(x) \ =\ \int_0^{\hat{s}} \frac{d\, h\circ\gamma}{ds}(s)\, ds \ ,$$

this sign must be the same as the one of as $h(\hat{x}) - h(x)$.

Now, for each integer i, we introduce the compact set :

$$D_i \ =\ \{(x,\hat{x}) \,:\, d(\hat{x},x) \leq p_s E \ ,\ i \leq |\hat{x}| \leq i+1\}$$

and we remark that, for a given geodesic γ satisfying $\gamma(0) = x$, the functions

$$\begin{aligned} s \ &\mapsto\ \frac{\frac{d\, h\circ\gamma}{ds}(s)\, (h\circ\gamma(s) - h(x_n))}{s} \ ,\\ s \ &\mapsto\ \frac{1}{2s}\int_0^{s} \left[\frac{d^2\, h\circ\gamma}{ds^2}(s)\, (h\circ\gamma(s) - h(x))\right] ds\end{aligned}$$

are well defined and continuous and satisfy :

$$\lim_{s\to 0} \frac{1}{2s}\int_0^{s} \left[\frac{d^2\, h\circ\gamma}{ds^2}(s)\, (h\circ\gamma(s) - h(x))\right] ds \ =\ 0 \ .$$

Let us prove the existence of a real number k_i such that, if the function k_E satisfies :

$$k_E(\hat{x}) \ \geq\ k_i \qquad \forall \hat{x} \,:\, i \leq |\hat{x}| \leq i+1 \ ,$$

then, for any pair $(\hat{x},x)$ in D_i and any minimal geodesic γ^* satisfying :

$$\gamma^*(0) \ =\ x \quad , \qquad \gamma^*(\hat{s}) \ =\ \hat{x} \ ,$$

we have :

$$\begin{aligned}\frac{q_i}{4p_s} \ +\ & k_E(\hat{x}) \frac{\frac{d\, h\circ\gamma^*}{ds}(\hat{s})\, (h\circ\gamma^*(\hat{s})) - h(x))}{\hat{s}}\\ & \geq\ -\frac{1}{2\hat{s}}\int_0^{\hat{s}} \left[\frac{d^2\, h\circ\gamma^*}{ds^2}(s)\, (h\circ\gamma^*(s) - h(x))\right] ds\end{aligned}$$

If k_i would not exist, we could find a sequence $(\hat{s}_n, x_n, \hat{x}_n, \gamma_n^*)$, with $(x_n, \hat{x}_n)$ in D_i and γ_n^* a minimal geodesic satisfying

$$\gamma_n^*(0) \ =\ x_n \quad , \qquad \gamma_n^*(\hat{s}_n) \ =\ \hat{x}_n \ ,$$

such that

$$\begin{aligned}\frac{q_i}{4p_s} \ +\ & \left[n - \frac{1}{2}\right] \frac{\frac{d\, h\circ\gamma_n^*}{ds}(\hat{s}_n)\, (h\circ\gamma_n^*(\hat{s}_n)) - h(x_n))}{\hat{s}_n} \qquad (46)\\ & \leq\ -\frac{1}{2\hat{s}_n}\int_0^{\hat{s}_n} \left[\frac{d^2\, h\circ\gamma_n^*}{ds^2}(s)\, (h\circ\gamma_n^*(s) - h(x))\right] ds \ .\end{aligned}$$

Because $(x_n, \hat{x}_n)$ is in D_i, and γ_n^* is a minimal geodesic, we have :

$$p_i\, |\gamma_n^*(s) - x| \ \leq\ d(\gamma_n^*(s), x) \ \leq\ \hat{s}^n \ \leq\ p_s E \quad \forall s\in[0,\hat{s}_n] \ .$$

This implies that $\gamma_n^* : [0, p_s E] \to \mathbb{R}^s$ takes its values in a compact set independent of n. γ_n^* being also a solution of the geodesic equation, it follows (see [5, Theorems 5, §1] for instance) that there is subsequence denoted n_1 and a quadruple $(\hat{s}_\omega, x_\omega, \hat{x}_\omega, \gamma_\omega)$ such that :

$$\lim_{n_1\to\infty} (\hat{s}_{n_1}, x_{n_1}, \hat{x}_{n_1}) \ =\ (\hat{s}_\omega, x_\omega, \hat{x}_\omega),$$

$$\lim_{n_1\to\infty} \gamma_{n_1}^*(s) \ =\ \gamma_\omega(s) \qquad \text{uniformly in } s\in[0,p_s E]$$

where γ_ω is a solution of the geodesic equation and therefore a geodesic satisfying $\gamma_\omega(0) = x_\omega, \gamma_\omega(\hat{s}_\omega) = \hat{x}_\omega$. With (45), this convergence implies also :

$$\begin{aligned} -\frac{q_i}{4p_s} & \qquad (47)\\ \geq\ & \frac{1}{2\hat{s}_\omega}\int_0^{\hat{s}_\omega} \left[\frac{d^2\, h\circ\gamma_\omega}{ds^2}(s)\, (h\circ\gamma_\omega(s) - h(x))\right] ds \ .\end{aligned}$$

On the other hand, again since $\gamma_n^* : [0, p_s E] \to \mathbb{R}^s$ takes its values in a compact set independent of n, the functions h, $\frac{\partial h}{\partial x}$ and $\frac{\partial^2}{\partial x^2}$ restricted to this compact set are continuous and bounded. The same hold, from the geodesic equation and completeness, for γ_n^*, $\frac{d\gamma_n^*}{ds}$ and $\frac{d^2\gamma_n^*}{ds^2}$ restricted to $[0, p_s E]$.

This implies that the right hand side of (46) is bounded, say by B. Consequently, we have :

$$\frac{d\, h\circ\gamma_n^*}{ds}(\hat{s}_n)\,(h\circ\gamma_n^*(\hat{s}_n)) - h(x_n)) \;\leq\; \frac{Bp_sE}{n-\frac{1}{2}}\;.$$

With (45), this implies :

$$\frac{d\, h\circ\gamma_\omega}{ds}(\hat{s}_\omega)\,(h\circ\gamma_\omega(\hat{s}_\omega)) - h(x_\omega)) \;=\; 0\;.$$

We must have :

$$h(\hat{x}_\omega) - h(x_\omega) \;=\; h\circ\gamma_\omega(\hat{s}_\omega) - h(x_\omega) \;=\; 0\;. \qquad (48)$$

Indeed, if not, we get :

$$\gamma_\omega(\hat{s}_\omega) \;=\; \hat{x}_\omega\;, \quad \frac{d\gamma_\omega}{ds}(\hat{s}_\omega) \;=\; v\;,$$
$$\frac{\partial h}{\partial x}(\hat{x}_\omega)\,v \;=\; 0\;, \quad v^\top P(\hat{x}_\omega)\,v \;=\; 1\;.$$

With the maximal geodesic convexity of $\mathfrak{H}(y)$ for any y, this implies :

$$h(\gamma_\omega(s)) \;=\; h(\hat{x}_\omega) \qquad \forall s$$

which is a contradiction for $s = 0$.

So now either

$$\hat{s}_\omega \;=\; 0 \qquad (49)$$

or we have

$$h(\gamma_\omega(s)) \;=\; h(x_\omega) \qquad \forall s\;. \qquad (50)$$

To prove the latter, we remark that, if $\hat{s}_\omega \neq 0$, then (48) implies that the function $h\circ\gamma_\omega$ has a stationary point in the interval $[0, \hat{s}_\omega]$, i.e. there exists s_0 satisfying :

$$0 \;=\; \frac{d\, h\circ\gamma_\omega}{ds}(s_0) \;=\; \frac{\partial h}{\partial x}(x_0)\,v_0$$

with the notations

$$x_0 \;=\; \gamma_\omega(s_0) \quad , \qquad v_0 \;=\; \frac{d\gamma_\omega}{ds}(s_0)\;.$$

So, as above, the maximal geodesic convexity of $\mathfrak{H}(y)$ for any y implies (50).

But whether we have (49) or (50), we obtain :

$$\frac{1}{2\hat{s}_\omega}\int_0^{\hat{s}_\omega}\left[\frac{d^2\, h\circ\gamma_\omega}{ds^2}(s)\,(h\circ\gamma_\omega(s) - h(x))\right]ds \;=\; 0$$

which contradicts (47).

Hence we have established the existence of k_i by contradiction.

The proof is completed by picking k_E as any continuous function satisfying :

$$k_E(\hat{x}) \;\geq\; k_i \qquad \forall\hat{x}:\; i \leq |\hat{x}| \leq i+1\;.$$

■

According to Proposition 4.6, we are able to design an observer provided we have a function P satisfying (38) and (39) but also making the level set $\mathfrak{H}(y)$ maximally geodesically convex for any y in $\mathbb{R}$. It is interesting to note that this condition is satisfied each time there exists an observer whose convergence can be established with a quadratic Lyapunov function which depends only on the estimation error (see [12]) and with one of the state components as measured output.

V. Conclusion

We showed that if the observer problem can be solved for the system (1), then there exists a symmetric matrix field P satisfying the property (8). We showed also that the satisfaction of such property is related to the observability of the linear time-varying systems obtained from linearizing (1) along its solutions.

Conversely, from the data of a symmetric matrix field satisfying (8) and under geodesic convexity of the level sets of the output function, we constructed an observer which gives convergence to 0 of the estimation error e, globally in the estimated state $\hat{x}$ but semi-globally in the error e. To prove this result, we use the symmetric matrix field as a Riemannian metric. We have also established that the geodesic convexity is somehow necessary if we want to be able to make the Riemannian distance between estimated state and system state to decrease along the solutions. But, at this time, it is not known whether it is necessary for the observer problem to be solvable.

The main reason that the proposed observer provides semiglobal rather than global stability of the set $\hat{x} = x$ seems to be its elementary form: a copy of the system plus a correction term vanishing when $h(\hat{x}) = y$. We expect that more appropriate choices of the observer can be made to obtain global asymptotic stability.

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