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Asymptotic tracking of a state trajectory by output-feedback for a class of non linear systems

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Abstract— **We consider the problem of tracking a reference trajectory with an output feedback for a class of nonlinear systems. We solve this problem by combining the techniques of dynamic scaling and homogeneity in the bi-limit.**

I. INTRODUCTION

We address the following problem : *Given a system* $\dot{\eta} = f(\eta, u)$ *with output* $y = h(\eta)$*, and a bounded reference trajectory* (η_r, u_r)*, exact solution of* $\dot{\eta}_r = f(\eta_r, u_r)$*, design an output feedback* $u = \varphi(w, y, \eta_r)$*,* $\dot{w} = \theta(w, y, \eta_r)$ *which ensures global convergence of* η *to* η_r*.*

To illustrate our contribution we consider the system[1] :

$$\left\{ \begin{array}{rcl} \dot{z} &=& -z + x_2 \ , \\ \dot{x}_1 &=& x_2 \ , \\ \dot{x}_2 &=& u + x_2^{1+d} + z \ , \\ y &=& x_1 \ . \end{array} \right. \tag{1}$$

where y in $\mathbb{R}$ is the available measurement, u in $\mathbb{R}$ is the control input and d a real number in $[-1, 1)$. Consider a bounded reference trajectory $(z_r, x_{1r}, x_{2r}, u_r)$, exact solution of (1), namely :

$$\left\{ \begin{array}{rcl} \dot{z}_r &=& -z_r + x_{2,r} \ , \\ \dot{x}_{r,1} &=& x_{r,2} \ , \\ \dot{x}_{r,2} &=& u_r + x_{r,2}^{1+d} + z_r \ . \end{array} \right. \tag{2}$$

The problem is to find an output feedback for u such that (z, x_1, x_2) converges to $(z_r, x_{r,1}, x_{r,2})$. This can be rephrased as finding $\tilde{u} = u_r - u$, depending on $(z_r, \tilde{x}_1, x_{r,1}, x_{r,2}, u_r)$, rendering the origin of the error system :

$$\left\{ \begin{array}{rcl} \dot{\tilde{z}} &=& -\tilde{z} + \tilde{x}_2 \ , \\ \dot{\tilde{x}}_1 &=& \tilde{x}_2 \ , \\ \dot{\tilde{x}}_2 &=& \tilde{u} + x_{r,2}^{1+d} - (x_{r,2} + \tilde{x}_2)^{1+d} + \tilde{z} \ , \end{array} \right.$$

globally attractive.

To solve this problem we follow a domination approach based on homogeneity. This leads to regard the term $x_{r,2}^{1+d} - (x_{r,2} + \tilde{x}_2)^{1+d} + \tilde{z}$ as a perturbation which can be upper-bounded as :

$$\begin{array}{l} |x_{r,2}^{1+d} - (x_{r,2} + \tilde{x}_2)^{1+d} + \tilde{z}| \\ \qquad \leq \ (1 + d)\, |x_{r,2}|^d \, |\tilde{x}_2| \ + \ |\tilde{x}_2|^{1+d} \ + \ |\tilde{z}| \ . \end{array} \tag{3}$$

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[1] For any real numbers w and r, w^r denotes $\text{sign}(w)\,|w|^r$.

The term $(1 + d)\,|x_{r,2}|^d$ is a known function which multiplies a linear function of the error. To deal with this kind of term, we follow an idea introduced in [14] and design a high-gain output feedback with a dynamic scaling with a gain updated from the reference signal $x_{r,2}$.

The second term, namely $|\tilde{x}_2|^{1+d}$, is a power of the norm of the error $|\tilde{x}_2|$. To deal with this term, we use the homogeneous in the bi-limit output feedback design tool we have introduced in [1] and further developed in [2] (see the Appendix).

Finally, the term $|\tilde{z}|$ depends on the zero dynamics. We deal with this one by imposing a minimum phase assumption and invoking a small gain argument.

In conclusion, our solution to this tracking problem is based on a domination approach and combines high-gain with dynamic scaling and homogeneity in the bi-limit.

When compared with what can be found in the textbooks [13], [12] for instance, our approach allows us to deal with some polynomial terms in the unmeasured state components. This type of extension is already present in [8] where a practical tracking result is obtained.

In section II the main results of the paper are stated and commented. In Section III, we introduce an extension of the homogeneous in the bi-limit output feedback design for a chain of integrator introduced in [1] to render it compatible with the use of dynamic scaling. This tool is then exploited in Sections IV to introduce the dynamic high-gain output-feedback.

II. MAIN RESULT OF THE PAPER

Consider a system whose dynamics are :

$$\left\{ \begin{array}{l} \dot{z} = F(z, x) \ , \\ \dot{x}_1 = x_2 \ + \ \delta_1(z, x) \ , \\ \dot{x}_2 = x_3 \ + \ \delta_2(z, x) \ , \\ \quad \vdots \\ \dot{x}_n = u \ + \ \delta_n(z, x) \ , \end{array} \right. \qquad y = x_1 \ . \tag{4}$$

where $x = (x_1, \dots x_n)$ in $\mathbb{R}^n$, y is the output in $\mathbb{R}$, u is the input in $\mathbb{R}$ and z in $\mathbb{R}^{n_z}$ is the state of the inverse dynamics. This state can be "neglected" provided the inverse dynamics with δ_i as output and x as input are incremental ISS (see [3]). Specifically, we make the following assumption.

Assumption 1 : There exist a real number d_∞ *in* $\left[0, \frac{1}{n-1}\right)$*, positive real numbers* υ*,* c_∞*,* p *and* q*, non negative* C^1 *functions* Z_i*, non negative continuous functions* Ω*,* γ_i *and* μ_i*, a function* α *of class* $\mathcal{K}_\infty$ *and a continuous function* ω*, with strictly positive values such that, for all* i*,* *1.1* μ_i^p *is* C^1*, convex and satisfies* $s\mu_i^{p\prime}(s) \leq q\mu_i^p(s)$, $\mu_i(0) = 0$,

1.2 $\alpha(|\tilde{z}|)\,\omega(|z|) \;\leq\; \sum_{i=1}^{n} \mu_i(Z_i(z,\tilde{z}))\,,$

1.3 $$\frac{\partial Z_i}{\partial z}(z,\tilde{z})\,F(z,x) + \frac{\partial Z_i}{\partial \tilde{z}}(z,\tilde{z})\,\left[F(z+\tilde{z},x+\tilde{x}) - F(z,x)\right] \leq\; -Z_i(z,\tilde{z}) \,+\, \gamma_i(\tilde{x})\,,$$

1.4 $$|\delta_i(z+\tilde{z},x+\tilde{x}) - \delta_i(z,x)| \leq\; \Omega(x,z)\sum_{j=1}^{i}|\tilde{x}_j| \;+\; c_\infty \sum_{j=1}^{i}|\tilde{x}_j|^{\frac{1-d_\infty(n-i-1)}{1-d_\infty(n-j)}} + \mu_i(Z_i(z,\tilde{z}))\,.$$

1.5 $$\mu_i((1+v)\,\gamma_i(\tilde{x})) \leq c_\infty \sum_{j=1}^{i}|\tilde{x}_j| + |\tilde{x}_j|^{\frac{1-d_\infty(n-i-1)}{1-d_\infty(n-j)}}\;.$$

In Section IV, we prove :

Theorem 1 (Main result): Consider system (4) with Assumption 1. Then the controller given in (16) solves the tracking problem for any bounded reference trajectory $t \mapsto (z_r(t), x_r(t), u_r(t))$ which is a particular solution of (4).

For the system (1), the condition 1.4 of Assumption 1 follows from inequality (3) by setting $d_\infty = d$ and picking $\Omega(x_r,z_r) = (1+d)\,|x_{r,2}|^d$, $Z_2(z,z_r) = |\tilde{z}|^2$, $\mu_2(s) = \sqrt{s}$, $\gamma_2(\tilde{x}) = |\tilde{x}_2|^2$. Consequently, system (1) belongs to the class of systems satisfying Assumption 1 and Theorem 1 applies.

Assumption 1 is to be compared with the one in [8] where standard homogeneity and domination is also used. In that contribution, there is no z dynamics and the objective is only practical tracking. This allows the authors to work with an assumption weaker than 1.4 since it is needed only for $(x,z,\tilde{x},\tilde{z}) = (0,0,\tilde{x},\tilde{z})$. (See footnote 4). Here we get exact tracking but under the more restrictive assumption that the reference trajectory is an exact solution of the system to be controlled. [4] and [11] are two other contributions where such a more restrictive assumptions is not needed, but on the other hand they do not allow polynomial growth in the δ_i.

Another important point is that we do not need to know in advance the whole reference trajectory or even a bound on it to design the controller. This is to be opposed for instance to the controller proposed in [8]. Compare also with [11].

III. HOMOGENEOUS TOOLS FOR A CHAIN OF INTEGRATOR

Throughout this section we consider a chain of integrator, with state $\mathfrak{X} = (x_1,\ldots,x_n)$ in $\mathbb{R}^n$ described by :

$$\dot{\mathfrak{X}} \;=\; \mathcal{S}\,\mathfrak{X} \,+\, B\,u \quad , \qquad y = x_1\;, \tag{5}$$

where $B = (0,\ldots,1)^T$ and $\mathcal{S}$ denotes the left shift matrix of order n, i.e.

$$\mathcal{S}\,\mathfrak{X} = (x_2,\ldots,x_n,0)^T\;.$$

We deal now with homogeneity in the bi-limit. This notion and its properties are studied in [2]. We give in the Appendix a brief summary.

Selecting arbitrary degrees $d_0 \leq d_\infty$ in $\left(-1, \frac{1}{n-1}\right)$, homogeneity in the bi-limit is obtained for system (5) provided the weights $r_0 = (r_{0,1},\ldots,r_{0,n})$ and $r_\infty = (r_{\infty,1},\ldots,r_{\infty,n})$ are :

$$r_{0,i} = 1 \,-\, d_0\,(n-i)\;, \quad r_{\infty,i} = 1 \,-\, d_\infty\,(n-i)\;. \tag{6}$$

A. Homogeneous in the bi-limit observer

In [1], [2], we have proposed an observer for system (5) given by :

$$\dot{\hat{\mathfrak{X}}} \;=\; \mathcal{S}\,\hat{\mathfrak{X}} \,+\, B\,u \,+\, K(\hat{x}_1 - x_1)\;, \tag{7}$$

where $\hat{\mathfrak{X}} = (\hat{x}_1,\ldots,\hat{x}_n)$ in $\mathbb{R}^n$, and K is a homogeneous in the bi-limit vector field with weights r_0 and r_∞, and degrees d_0 and d_∞. Setting :

$$E \;=\; (e_1\ldots,e_n)^T \;=\; \hat{\mathfrak{X}} - \mathfrak{X}$$

yields the error system :

$$\dot{E} \;=\; S\,E + K(e_1)\;. \tag{8}$$

The design of K is done recursively in such a way that there exists a homogeneous in the bi-limit Lyapunov function W of degree d_W satisfying, for some real number c_1,

$$\frac{\partial W}{\partial E}(E)\,(\mathcal{S}\,E + K(e_1)) \leq -c_1\left(W(E)^{\frac{d_W+d_0}{d_W}} + W(E)^{\frac{d_W+d_\infty}{d_W}}\right)\;, \tag{9}$$

To combine this tool with dynamic scaling we need to establish a specific property on the error Lyapunov function W. This property is a homogeneous in the bi-limit version of the one given in [14, equation (16)] or in [9, Lemma A1]. Namely, given the diagonal matrix

$$D = \mathrm{diag}(d_1,\ldots,d_n)\;,$$

with $d_i > 0$, the function K has to be selected such that the associated error Lyapunov function W satisfies (9), and also :

$$\frac{\partial W}{\partial E}(E)\,D\,E \;\geq\; c_2\,W(E)\;, \tag{10}$$

for some positive real number c_2.

Such a property can be obtained by modifying[2] the recursive procedure given in [1], [2] as claimed in the following statement the proof of which is omitted.

Theorem 2: Let d_W be a positive real number satisfying $d_W \geq 2\max_{1\leq j\leq n} r_{0,j} + d_\infty$ and $D = \mathrm{diag}(d_1,\ldots,d_n)$ with $d_j > 0$. There exists a homogeneous in the bi-limit vector field $K : \mathbb{R} \to \mathbb{R}^n$, with associated triples (r_0, d_0, K_0) and $(r_\infty, d_\infty, K_\infty)$ and a positive definite, proper and C^1 function $W : \mathbb{R}^n \to \mathbb{R}_+$, homogeneous in the bi-limit with associated triples (r_0, d_W, W_0) and $(r_\infty, d_W, W_\infty)$, such that :

1) The functions W_0 and W_∞ are positive definite and proper and the functions $\frac{\partial W}{\partial e_j}$ are homogeneous in the bi-limit

[2] We multiply W_{i+1} by a sufficiently large number before using it in the definition of W_i

2) There exist two positive real numbers c_1 and c_2 such that (9) and (10) are satisfied.

B. Homogeneous in the bi-limit state feedback

In the same spirit as the homogeneous in the bi-limit observer design introduced in the previous section, we modify[3] the recursive state feedback design introduced in [1] to make it compatible with dynamic scaling. The result is expressed in the following statement.

Theorem 3: Let d_V be a positive real number satisfying $d_V > 2\max 1 \leq j \leq n r_{0,j}$, and $D = (d_1, \ldots, d_n)$, with $d_j > 0$. There exist a homogeneous in the bi-limit function $\phi : \mathbb{R}^n \to \mathbb{R}$, with associated triples $(r_0, 1+d_0, \phi_0)$ and $(r_\infty, 1+d_\infty, \phi_\infty)$ and a positive definite, proper and C^1 function $V : \mathbb{R}^n \to \mathbb{R}_+$ homogeneous in the bi-limit with associated triples (r_0, d_V, V_0) and $(r_\infty, d_V, V_\infty)$, such that :

1) The functions V_0 and V_∞ are positive definite and proper and the functions $\frac{\partial V}{\partial x_j}$ are homogeneous in the bi-limit
2) There exists $c_3 > 0$ such that, for all $\mathfrak{X}$ in $\mathbb{R}^n$,
$$\frac{\partial V}{\partial \mathfrak{X}}(\mathfrak{X})\,(\mathcal{S}\,\mathfrak{X} + B\,\phi(\mathfrak{X})) \leq -c_3\left(V(\mathfrak{X})^{\frac{d_V+d_0}{d_V}} + V(\mathfrak{X})^{\frac{d_V+d_\infty}{d_V}}\right) . \tag{11}$$
3) There exists $c_4 > 0$ such that, for all $\mathfrak{X}$ in $\mathbb{R}^n$,
$$\frac{\partial V}{\partial \mathfrak{X}}(\mathfrak{X})\,D\,\mathfrak{X} \;\geq\; c_4\,V(\mathfrak{X}) . \tag{12}$$

IV. Proof of Theorem 1

A. Rephrasing the problem as a stabilization problem

Setting
$$\tilde{u} = u - u_r \ , \quad \tilde{x} \;=\; x - x_r \ , \quad \tilde{z} \;=\; z - z_r \ ,$$
we obtain :
$$\begin{cases} \dot{\tilde{z}} = F(z_r+\tilde{z}, x_r+\tilde{x}) \;-\; F(z_r, x_r) \ , \\ \dot{\tilde{x}}_1 = \tilde{x}_2 \;+\; \delta_1(z_r+\tilde{z}, x_r+\tilde{x}) \;-\; \delta_1(z_r, x_r) \ , \\ \quad\vdots \\ \dot{\tilde{x}}_n \;=\; \tilde{u} \;+\; \delta_n(z_r+\tilde{z}, x_r+\tilde{x}) \;-\; \delta_1(z_r, x_r) \ . \end{cases} \tag{13}$$
The objective is now to find $\tilde{u}$ depending on the output $\tilde{x}_1 = y - x_{r,1}$ and on the reference trajectory (z_r, x_r) such that the solution $\tilde{x} = 0$ is globally attractive[4].

B. Output feedback using homogeneous in the bi-limit tools and dynamic scaling

The output feedback is obtained from the functions K and ϕ given by Theorems 2 and 3 and by introducing an extra dynamically updated gain L.

The first step consists in selecting the parameter D in Theorem 2 and 3. Following [14], let :
$$D \;=\; \mathrm{diag}(b, 1+b, \ldots, n-1+b)^T \ , \tag{14}$$
where b is a positive real number satisfying[5],
$$\frac{1-d_\infty(n-i-1)}{1-d_\infty(n-j)} < \frac{i+b}{j-1+b} < \frac{i}{j-1} \ , \tag{15}$$
for all $1 \leq j \leq i \leq n$ and with d_∞ as given in Assumption 1.

Selecting $d_0 = 0$ and $d_W = d_V = d_U$ sufficiently large, we apply Theorem 2 and 3 to construct the homogeneous in the bi-limit vector field K, the state feedback ϕ and the Lyapunov functions W and V such that (9), (10), (11) and (12) hold.

Following [1], the output feedback is given by :
$$u = u_r + L^{n+b}\phi(\mathfrak{L}^{-1}\hat{x}) \ , \tag{16}$$
$$\dot{\hat{x}} = \mathcal{S}\,\hat{x} + B\,L^{n+b}\phi(\mathfrak{L}^{-1}\hat{x}) + L\,\mathfrak{L}\,K\left(\tfrac{\hat{x}_1-(y-x_{r,1})}{L^b}\right) \ ,$$
with :
$$\mathfrak{L} \;=\; \mathrm{diag}(L^b, \ldots, L^{n+b-1})$$
and L satisfying :
$$\begin{aligned} \dot{L} \;=\; & -a_1 a_{22}\,L \\ & + L\,\max\{0, a_1(a_{21}-L) + a_3\,\Omega(z_r, x_r)\} \ , \end{aligned} \tag{17}$$
where a_1, a_{21}, a_{22} and a_3 are positive real numbers to be defined with $a_2 = a_{21} - a_{22} > 0$.

This update law of the gain L is an elaboration from the original one introduced in [14, (24)] and modified in [5, (3.12)] or [10, (134)]. Note that the driving term depends only on the reference trajectory (x_r, z_r). Since this trajectory is bounded, L is bounded along any closed-loop solution. Also we see that, if initialized to a value larger than a_2, L remains larger than a_2 along any closed-loop solution. Moreover the presence of the term $-a_1 a_{22}\,L$ allows to recover the main property of [14, (24)], i.e. L "follows" its driving term. Specifically the Dini derivative of $\left|L - \frac{a_1 a_2 + a_3\Omega(z_r, x_r)}{a_1}\right|$ satisfies
$$D^+\left|L - \tfrac{a_1 a_2 + a_3\Omega(z_r,x_r)}{a_1}\right| \leq \tfrac{a_3}{a_1}\left|\dot{\overline{\Omega(z_r,x_r)}}\right| - a_1\min\{a_{22}, a_2\}\left|L - \frac{a_1 a_2 + a_3\Omega(z_r,x_r)}{a_1}\right| .$$
Hence along the solution of the closed loop system, we have
$$\limsup_{t\to+\infty}\left|L(t) - \frac{a_1 a_2 + a_3\Omega(z_r(t), x_r(t))}{a_1}\right| \leq \leq \frac{a_3}{\min\{a_{22}, a_2\}}\limsup_{t\to+\infty}\left|\dot{\overline{\Omega(z_r(t), x_r(t))}}\right| .$$

Properties of the closed loop system : Let
$$E = (e_1, \ldots, e_n)^T \ , \quad \hat{\mathfrak{X}} = (\hat{x}_1, \ldots, \hat{x}_n)^T$$
and τ be scaled quantities defined as :

[3] We multiply V_i by a sufficiently large number before using it in the definition of V_{i+1}

[4] In [8], $\tilde{x}$ is defined as $\tilde{x} = ((x_1 - y_r, x_2, \ldots, x_n)$. This leads to the presence of $\dot{y}_r$ as a disturbance in the $\tilde{x}$ dynamics explaining why practical tracking is obtained and why a weaker assumption can be invoked.

[5] This choice is always possible since, for $1 \leq j \leq i \leq n$, we have :
$$\tfrac{i+b}{j-1+b} < \tfrac{i}{j-1} \quad \forall\, b > 0,$$
and
$$1 \leq \tfrac{1-d_\infty\,(n-i-1)}{1-d_\infty\,(n-j)} < \tfrac{i}{j-1} \quad \forall\, d_\infty \in [0, \tfrac{1}{n-1}) \ .$$

$$\underbrace{\begin{aligned}\frac{dE}{d\tau} &= \mathcal{S}\,E\,+\,K(e_1)\\ \frac{d\hat{\mathfrak{X}}}{d\tau} &= \mathcal{S}\,\hat{\mathfrak{X}}\,+\,B\,\phi(\hat{\mathfrak{X}})\,+\,K(e_1)\end{aligned}}_{\text{Chain of integrator part}} \begin{aligned}&-\\&-\end{aligned} \underbrace{\begin{aligned}&L^{-1}\,\frac{dL}{d\tau}\,D\,E\\ &L^{-1}\,\frac{dL}{d\tau}\,D\,\hat{\mathfrak{X}}\end{aligned}}_{\text{Dynamic Scaling}} \begin{aligned}&-\\&\end{aligned} \underbrace{\begin{aligned}&\mathfrak{D}(L)\ ,\\&\end{aligned}}_{\text{Nonlinearities = Disturbances}} \qquad (19)$$

$$E = \mathfrak{L}^{-1}(\hat{x} - \tilde{x}) \quad , \quad \hat{\mathfrak{X}} = \mathfrak{L}^{-1}\hat{x} \quad , \quad \frac{d}{d\tau} = \frac{1}{L}\frac{d}{dt} \ . \qquad (18)$$

The closed loop dynamics can be described by equation (17), the z dynamics and equations[6] (19) at the top of this page, where

$$\mathfrak{D}(L) = L^{-1}\mathfrak{L}^{-1}(\delta(z_r + \tilde{z}, x_r + \tilde{x}) - \delta(z_r, x_r))$$

is regarded as a perturbation.

We consider now the homogeneous in the bi-limit Lyapunov function :

$$U(\hat{\mathfrak{X}}, E) \;=\; V(\hat{\mathfrak{X}}) + \ell\, W(E)\ , \qquad (20)$$

where ℓ is a positive real number to be specified and V and W are given by Theorems 2 and 3. Along the trajectories of system (19), the Lyapunov function satisfies in the time τ :

$$\frac{d}{d\tau}U(\hat{\mathfrak{X}}, E) \;\leq\; T_{CI} \;+\; T_{DS} \;+\; T_{Dist}\ , \qquad (21)$$

with

$$\begin{aligned} T_{CI} &= \tfrac{\partial V}{\partial \hat{\mathfrak{X}}}(\hat{\mathfrak{X}})\left(\mathcal{S}\hat{\mathfrak{X}} + B\,\phi(\hat{\mathfrak{X}}) + K(e_1)\right)\\ &\qquad\qquad + \ell\tfrac{\partial W}{\partial E}(E)\,(\mathcal{S}\,E + K(e_1))\ ,\\ T_{DS} &= -L^{-1}\,\tfrac{dL}{d\tau}\left(\tfrac{\partial V}{\partial \hat{\mathfrak{X}}}(\hat{\mathfrak{X}})D\hat{\mathfrak{X}} + \ell\,\tfrac{\partial W}{\partial E}(E)DE\right)\ ,\\ T_{Dist} &= -\ell\,\tfrac{\partial W}{\partial E}(E)\,\mathfrak{D}(L)\ . \end{aligned}$$

Bound on the term T_{CI}. Let η and γ be functions defined as :

$$\begin{aligned} \eta(\hat{\mathfrak{X}}, E) &= \tfrac{\partial V}{\partial \hat{\mathfrak{X}}}(\hat{\mathfrak{X}})\left(\mathcal{S}\,\hat{\mathfrak{X}}_n + B\,\phi_n(\hat{\mathfrak{X}}_n) + K(e_1)\right)\ ,\\ \gamma(E) &= \tfrac{\partial W}{\partial E}(E)\,(\mathcal{S}\,E + K(e_1))\ . \end{aligned}$$

These two functions are continuous and homogeneous in the bi-limit with associated triples $((r_0, r_0), d_U, \eta_0)$, $((r_\infty, r_\infty), d_U + d_\infty, \eta_\infty)$ and $((r_0, r_0), d_U, \gamma_0)$, $((r_\infty, r_\infty), d_U + d_\infty, \gamma_\infty)$, where γ_0, γ_∞ and η_0, η_∞ are continuous functions. Furthermore, by (9), $\gamma(E)$ is non positive for all E in $\mathbb{R}^n$ and by (11), we have, for all $(\hat{\mathfrak{X}}, E) \neq (0, 0)$,

$$\begin{aligned} \gamma(E) = 0 &\quad\Rightarrow\quad \eta(\hat{\mathfrak{X}}, E) < 0\ ,\\ \gamma_0(E) = 0 &\quad\Rightarrow\quad \eta_0(\hat{\mathfrak{X}}, E) < 0\ ,\\ \gamma_\infty(E) = 0 &\quad\Rightarrow\quad \eta_\infty(\hat{\mathfrak{X}}, E) < 0\ . \end{aligned}$$

Consequently, by Claim A.2 in the Appendix, we can select ℓ such that T_{CI} is negative definite and the same holds for the homogeneous approximation in the 0-limit and in the ∞-limit. Then, Claim A.3, yields a positive real number c_5 such that :

$$T_{CI} \;\leq\; -c_5\left(U(\hat{\mathfrak{X}}, E) + U(\hat{\mathfrak{X}}, E)^{\frac{d_\infty + d_U}{d_U}}\right) \qquad (22)$$

Note that ℓ and c_5 have been fixed.

Bound on the term T_{DS}. From Claim A.1, the function $(\hat{\mathfrak{X}}, E) \mapsto \frac{\partial V}{\partial \hat{\mathfrak{X}}}(\hat{\mathfrak{X}})\,D\,\hat{\mathfrak{X}} + \ell\,\frac{\partial W}{\partial E}(E)\,D\,E$ is homogeneous in the bi-limit with associated weights (r_0, r_0) and (r_∞, r_∞) and degrees d_U and d_U. Hence Claim A.3, as well as equations (10) and (12) yield positive real numbers c_6 and c_7 such that :

$$c_6 \geq \frac{\frac{\partial V}{\partial \hat{\mathfrak{X}}}(\hat{\mathfrak{X}})\,D\,\hat{\mathfrak{X}} + \ell\,\frac{\partial W}{\partial E}(E)\,D\,E}{U(\hat{\mathfrak{X}}, E)} \;\geq\; c_7\ .$$

Finally, using the expression of $\dot{L}$ in (17), this gives :

$$T_{DS} \;\leq\; -c_7\,a_3\,\tfrac{\Omega(z_r, x_r)}{L}\,U(\hat{\mathfrak{X}}, E) + c_6\,a_1\,U(\hat{\mathfrak{X}}, E)\ . \qquad (23)$$

Bound on the term T_{Dist}. Let

$$\tilde{x}_j \;=\; L^{b+j-1}\,(x_j - e_j)\ ,$$

By Assumption 1.4 and equation (6), we have, for all i,

$$\begin{aligned} &|\mathfrak{D}_i(L)|\\ &\quad= L^{-i-b}|\delta_i(z_r + \tilde{z}, x_r + \tilde{x}) - \delta_i(z_r, x_r)|\ ,\\ &\quad\leq \Omega(z_r, x_r)\sum_{j=1}^{i} L^{j-1-i}|\hat{x}_j - e_j|\\ &\qquad\quad + c_\infty\,L^{-i-b}\sum_{j=1}^{i}|L^{b+j-1}(\hat{x}_j - e_j)|^{\frac{r_{\infty,i}+d_\infty}{r_{\infty,j}}}\\ &\qquad\qquad\qquad\qquad + L^{-i-b}\mu_i(Z_i)\ . \end{aligned}$$

Inequalities (15) imply the existence of a real number $\epsilon > 0$ such that

$$L^{-\epsilon} \geq L^{(b+j-1)\frac{r_{\infty,i}+d_\infty}{r_{\infty,j}} - i - b} \qquad \forall L \geq 1\ .$$

The condition $L \geq 1$ holds along all closed loop solutions if $a_2 \geq 1$. Consequently, for all $(\hat{\mathfrak{X}}, E)$ in $\mathbb{R}^{2n}$ and $L \geq a_2 \geq 1$,

$$\begin{aligned} |\mathfrak{D}_i(L)| \leq\ & \frac{\Omega(z_r, x_r)}{L}\sum_{j=1}^{i}|\hat{x}_j - e_j|\\ & + c_\infty\,a_2^{-\epsilon}\sum_{j=1}^{i}|\hat{x}_j - e_j|^{\frac{r_{\infty,i}+d_\infty}{r_{\infty,j}}} + L^{-i-b}\mu_i(Z_i)\ . \end{aligned}$$

On the other hand, the function $\left|\frac{\partial W}{\partial e_i}(E)\right|\,|\hat{x}_j - e_j|$ (respectively $\left|\frac{\partial W}{\partial e_i}(E)\right||\hat{x}_j - e_j|^{\frac{r_{\infty,i}+d_\infty}{r_{\infty,j}}}$) is homogeneous in the bi-limit with weights (r_0, r_0) and (r_∞, r_∞) and degrees d_U and $d_U + r_{\infty,j} - r_{\infty,i}$ (respectively $d_U - 1 + \frac{r_{\infty,i}+d_\infty}{r_{\infty,j}}$ $(\geq d_U)$ and $d_\infty + d_U$). Hence, Claim A.3 yields positive real numbers c_8 and c_9, such that :

[6]Note that $\dot{\overbrace{\mathfrak{L}^{-1}}} = -L^{-1}\,\dot{L}\,D\,\mathfrak{L}^{-1}$

$$\frac{|T_{Dist}|}{\ell} \leq c_8 \frac{\Omega(z_r, x_r)}{L} U(\hat{\mathfrak{X}}, E) + c_\infty a_2^{-\epsilon} c_9 \left(U(\hat{\mathfrak{X}}, E) + U(\hat{\mathfrak{X}}, E)^{\frac{d_U + d_\infty}{d_U}} \right) + \sum_{i=1}^{n} \left| \frac{\partial W}{\partial e_i}(E) \right| L^{-i-b} \mu_i(Z_i) . \tag{24}$$

Hence, by (22) and (23), selecting a_1 sufficiently small and a_2 and a_3 sufficiently large guarantees the existence of a positive real number c_{10} such that (21) becomes :

$$\frac{dU(\mathfrak{X}, E)}{d\tau} \leq -2 c_{10} \left(U(\hat{\mathfrak{X}}, E) + U(\hat{\mathfrak{X}}, E)^{\frac{d_U + d_\infty}{d_U}} \right) + \sum_{i=1}^{n} \left| \frac{\partial W}{\partial e_i}(E) \right| L^{-i-b} \mu_i(Z_i) . \tag{25}$$

Small-gain arguments. Let ζ_i and θ be the homogeneous in the bi-limit functions defined as

$$\zeta_i(s) = |s|^{d_U} + |s|^{\frac{d_U + d_\infty}{r_{\infty,i} + d_\infty}} , \ \theta(s) = |s| + |s|^{\frac{d_U + d_\infty}{d_U}} .$$

We use Claim A.2 with the functions

$$-c_{10}\, \theta(U(\hat{\mathfrak{X}}, E)) + \textstyle\sum_{i=1}^{n} \left| \frac{\partial W}{\partial e_i}(E) \right| s_i \quad \text{and} \quad \textstyle\sum_{i=1}^{n} \zeta_i(s_i),$$

which are homogeneous in the bi-limit with weights (r_∞, r_∞) and (r_0, r_0) for $(\hat{\mathfrak{X}}, E)$, and $(1, r_{\infty,i} + d_\infty)$ for s_i and degrees d_U and $d_\infty + d_U$. This gives a positive real number c_{11} satisfying :

$$-c_{10}\, \theta(U(\hat{\mathfrak{X}}, E)) + \sum_{i=1}^{n} \left| \frac{\partial W}{\partial e_i}(E) \right| s_i \leq c_{11} \sum_{i=1}^{n} \zeta_i(s_i) .$$

Thus, inequality (25) (in time t) implies :

$$\dot{U}(\mathfrak{X}, E) \leq -c_{10}\, L\, \theta(U(\hat{\mathfrak{X}}, E)) + c_{11}\, L \textstyle\sum_{i=1}^{n} \zeta_i(L^{-i-b} \mu_i(Z_i(z_r, \tilde{z}))) .$$

Since, along solutions, L is lower bounded by a_2, this implies the existence of a class $\mathcal{KL}$ function β_u such that we have, for all t in the domain of existence of the closed loop solution and for all $s \leq t$

$$U(t) \leq \beta_u(U(s), t-s) + \sup_{r \in [s,t)} \sum_{i=1}^{n} \theta^{-1} \left(\frac{2c_{11}}{c_{10}} \zeta_i(L(r)^{-i-b} \mu_i(Z_i(r))) \right),$$

where to simplify the notation, we have defined

$$U(t) = U(\mathfrak{X}(t), E(t)) \ , \ Z_i(t) = Z_i(z_r(t), \tilde{z}(t))$$

and we have used the property $\theta(a) + \theta(b) \leq \theta(a+b)$.

Now, with p as in Assumption 1, let

$$Y_i = L^{-p(i+b)} \mu_i(Z_i(z_r, \tilde{z})))^p .$$

We get :

$$\dot{Y}_i \leq -p(i+b) \frac{\dot{L}}{L} Y_i + L^{-p(i+b)} \mu_i^{p\prime}(Z_i(z_r, \tilde{z}))) \left[-Z_i(z_r, \tilde{z})) + \gamma_i(\tilde{x}) \right] .$$

Since $\dot{L} + a_1 a_{22} L$ is non-negative, using property 1.1 of μ_i and considering the two cases $(1+v)\gamma_i(\tilde{x}) \leq Z_i(z_r, \tilde{z}))$ and $Z_i(z_r, \tilde{z})) \leq (1+v)\gamma_i(\tilde{x})$, we obtain

$$\dot{Y}_i \leq -\left[\frac{v}{1+v} - p(i+b) a_1 a_{22} \right] Y_i + q L^{-p(i+b)} \mu_i((1+v)\gamma_i(\tilde{x}))^p .$$

We remark that the functions $\zeta_i^{-1}\left(\frac{c_{10}}{2c_{11}} \theta(U(\hat{\mathfrak{X}}, E)) \right)$ and $|x_j - e_j| + |x_j - e_j|^{\frac{r_{\infty,i} + d_\infty}{r_{\infty,j}}}$ are homogeneous in the bi-limit with weights (r_0, r_0), (r_∞, r_∞) and degree 1 and $r_{\infty,i} + d_\infty$. Hence, using Claim A.3, we obtain the existence of a positive number c_{12} such that :

$$c_\infty \textstyle\sum_{j=1}^{i} |x_j - e_j| + |x_j - e_j|^{\frac{r_{\infty,i} + d_\infty}{r_{\infty,j}}} \leq c_{12}\, \zeta_i^{-1}\left(\frac{c_{10}}{2c_{11}} \theta(U(\hat{\mathfrak{X}}, E)) \right) .$$

Moreover, since $L \geq a_2 \geq 1$,

$$L^{-(i+b)} \sum_{j=1}^{i} |\tilde{x}_j| + |\tilde{x}_j|^{\frac{r_{\infty,i} + d_\infty}{r_{\infty,j}}} \leq a_2^{-\epsilon} \sum_{j=1}^{i} |x_j - e_j| + |x_j - e_j|^{\frac{r_{\infty,i} + d_\infty}{r_{\infty,j}}} .$$

By Assumption 1.5, it follows that :

$$\dot{Y}_i \leq -\left[\tfrac{v}{1+v} - p(i+b) a_1 a_{22} \right] Y_i + q \left[a_2^{-\epsilon} c_{12}\, \zeta_i^{-1} \left(\frac{c_{10}}{2c_{11}} \theta(U(\hat{\mathfrak{X}}, E)) \right) \right]^p .$$

Hence by selecting a_1 and a_{22} to satisfy

$$a_1 a_{22} < \tfrac{v}{p(1+v)(n+b)} ,$$

we obtain the existence of a class $\mathcal{KL}$ function β_z and a real number c_{13} such that, for all i, all t in the domain of existence of the closed loop solution and all $s \leq t$,

$$L(t)^{-(i+b)} \mu_i(Z_i(t))) \leq \beta_z(L(s)^{-(i+b)} \mu_i(Z_i(s)), t-s) + \sup_{r \in [s,t)} a_2^{-\epsilon} c_{13}\, \zeta_i^{-1} \left(\frac{c_{10}}{2c_{11}} \theta(U(r)) \right) .$$

Since, by choosing a_2 large enough the small gain condition [6, condition (14)] is satisfied (see also [7]), we conclude there exists a function $\overline{\beta}_u$ of class $\mathcal{KL}$ such that, on the domain of existence of the closed loop solution, and for all i,

$$U(t) \leq \overline{\beta}_u(U(0) + \sum_{i=1}^{n} L(0)^{-(i+b)} \mu_i(Z_i(0)) \, , \, t) .$$

By Assumptions 1.2 and 1.3, there exists another function $\overline{\beta}_z$ of class $\mathcal{KL}$ which depends on the bounds $\sup_t L(t)$ and $\sup_t |z_r(t)|$ such that, on the domain of existence of the closed loop solution,

$$\alpha(|\tilde{z}(t)|) \leq \overline{\beta}_z(U(0) + \sum_{i=1}^{n} \mu_i(Z_i(0)) \, , \, t) .$$

This implies that the domain of existence is $[0, +\infty)$, and the global attractiveness of $\tilde{x} = E = 0$ and $\tilde{z} = 0$.

V. Conclusion

We have solved a state trajectory tracking problem for minimum phase non linear systems which admit globally a strict normal form and in such a way that the nonlinearities satisfy power growth. This has been achieved by exploiting the tools of domination, homogeneity in the bi-limit and dynamic scaling gain.

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APPENDIX
ON HOMOGENEITY IN THE BI-LIMIT

For details on the notion of homogeneity in the bi-limit refer to [2]. We only give the definition and state the main properties used in this paper.

Given a vector $r = (r_1, \ldots, r_n)$ in $(\mathbb{R}_+/\{0\})^n$, we define the dilation of a vector x in $\mathbb{R}^n$ as

$$\lambda^r \diamond x \;=\; (\lambda^{r_1}\, x_1, \,\ldots,\, \lambda^{r_n}\, x_n)^T \quad .$$

Definition 1 (Homogeneity in the 0-limit):

- A continuous function $\phi \;:\mathbb{R}^n \to \mathbb{R}$ is said homogeneous in the 0-limit with associated triple (r_0, d_0, ϕ_0), where r_0 in $(\mathbb{R}_+/\{0\})^n$ is the weight, d_0 in $\mathbb{R}_+$ the degree and $\phi_0 \;:\mathbb{R}^n \to \mathbb{R}$ the approximating function, respectively, if ϕ_0 is continuous and not identically zero and, for each compact set C in $\mathbb{R}^n \setminus \{0\}$ and each $\varepsilon > 0$, there exists λ^* such that we have :
$$\max_{x \in C}\left|\frac{\phi(\lambda^{r_0}\diamond x)}{\lambda^{d_0}} - \phi_0(x)\right| \;\le\; \varepsilon \quad \forall \lambda \in (0, \lambda^*].$$
- A vector field $f = \sum_{i=1}^n f_i \frac{\partial}{\partial x_i}$ is said homogeneous in the 0-limit with associated triple (r_0, d_0, f_0), where $f_0 = \sum_{i=1}^n f_{0,i} \frac{\partial}{\partial x_i}$, if, for each i in $\{1, \ldots, n\}$, the function f_i is homogeneous in the 0-limit with associated triple $(r_0, d_0 + r_{0,i}, f_{0,i})$[7].

Definition 2 (Homogeneity in the ∞-limit):

- A continuous function $\phi \;:\mathbb{R}^n \to \mathbb{R}$ is said homogeneous in the ∞-limit with associated triple $(r_\infty, d_\infty, \phi_\infty)$ where r_∞ in $(\mathbb{R}_+/\{0\})^n$ is the weight, d_∞ in $\mathbb{R}_+$ the degree and $\phi_\infty \;:\mathbb{R}^n \to \mathbb{R}$ the approximating function, respectively, if ϕ_∞ is continuous and not identically zero and, for each compact set C in $\mathbb{R}^n \setminus \{0\}$ and each $\varepsilon > 0$, there exists λ^* such that we have :
$$\max_{x \in C}\left|\frac{\phi(\lambda^{r_\infty}\diamond x)}{\lambda^{d_\infty}} - \phi_\infty(x)\right| \;\le\; \varepsilon \quad \forall \lambda \in [\lambda^*, +\infty) \;.$$
- A vector field $f = \sum_{i=1}^n f_i \frac{\partial}{\partial x_i}$ is said homogeneous in the ∞-limit with associated triple $(r_\infty, d_\infty, f_\infty)$, with $f_\infty = \sum_{i=1}^n f_{\infty,i} \frac{\partial}{\partial x_i}$, if, for each i in $\{1, \ldots, n\}$, the function f_i is homogeneous in the ∞-limit with associated triple $(r_\infty, d_\infty + r_{\infty,i}, f_{\infty,i})$.

Definition 3 (Homogeneity in the bi-limit): A continuous function $\phi \;:\mathbb{R}^n \to \mathbb{R}$ (or a vector field f) is said homogeneous in the bi-limit if it is homogeneous in the 0-limit and homogeneous in the ∞-limit.

Claim : Let η and γ be two continuous homogeneous in the bi-limit functions with weights r_0, r_∞, degrees $d_{\eta,0}, d_{\eta,\infty}$ and $d_{\gamma,0}, d_{\gamma,\infty}$, and continuous approximating functions η_0, η_∞, γ_0, γ_∞.

A.1. The function $x \mapsto \eta(x)\gamma(x)$ is homogeneous in the bi-limit with associated triples $(r_0, d_{\eta,0} + d_{\gamma,0}, \eta_0\,\gamma_0)$ and $(r_\infty, d_{\eta,\infty} + d_{\gamma,\infty}, \eta_\infty\,\gamma_\infty)$.

A.2. If the degrees satisfy $d_{\eta,0} \ge d_{\gamma,0}$ and $d_{\eta,\infty} \le d_{\gamma,\infty}$, and $\gamma(x) \ge 0$, and for $x \neq 0$

$$\begin{aligned} \gamma(x) = 0 \quad &\Rightarrow \quad \eta(x) < 0 \;, \\ \gamma_0(x) = 0 \quad &\Rightarrow \quad \eta_0(x) < 0 \;, \\ \gamma_\infty(x) = 0 \quad &\Rightarrow \quad \eta_\infty(x) < 0 \;, \end{aligned}$$

then there exists a real number k^* such that, for all $k \ge k^*$, and for all non zero x in $\mathbb{R}^n$:

$$\eta(x) < k\gamma(x)\;,\; \eta_0(x) < k\gamma_0(x)\;,\; \eta_\infty(x) < k\gamma_\infty(x)\;.$$

A.3. If the degrees satisfy $d_{\eta,0} \ge d_{\gamma,0}$ and $d_{\eta,\infty} \le d_{\gamma,\infty}$ and the functions γ, γ_0 and γ_∞ are positive definite then there exists a positive real number c satisfying $\eta(x) \;\le\; c\,\gamma(x)$ for all x in $\mathbb{R}^n$.

[7] In the case of a vector field the degree d_0 can be negative as long as $d_0 + r_{0,i} \ge 0$, for all $1 \le i \le n$.