A new observer for an unknown harmonic oscillator

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***Abstract*— In this paper we consider the problem of estimating amplitude, phase and frequency of a pure sinusoid following the nonlinear observer theory presented in [7] and [1]. We show how the estimation can be carried out by processing, through a static nonlinear function, the state of an Hurwitz system of suitable dimension. Simulation results are also presented showing the effectiveness of the method also in presence of high frequency noise superimposed to the estimated sinusoid.**

I. INTRODUCTION

This paper deals with the problem of asymptotically estimating amplitude, frequency and phase of a sinusoidal signal by adopting the theory of observers proposed in [7] and further investigated in [1] (see also [8] and [6]). The theory in question leads to an observer which is given by a linear Hurwitz system of suitable dimension whose state is processed by a nonlinear map to yield the desired estimates.

The problem of frequency, amplitude and phase estimation of a sinusoidal signal has attracted a remarkable research attention in the past and current literature (see [9]). The reasons of this interest rely on several engineering applications where an effective and robust solution to this problem is crucial. To mention few, it is worth mentioning problems of harmonic disturbance compensation in automatic control, design of phase-looked loop circuits in telecommunication, adaptive filtering in signal processing, etc.

In [3] the authors propose an adaptive notch filter for global estimation of the frequency of a sinusoidal signal. The problem can also be addressed by means of classical adaptive control techniques as, e.g., in[5], [4]. This is motivated by the fact that a signal consisting of a finite sum of sinusoids with unknown frequencies can be thought as generated by the output of a linear system with uncertain parameters. In this framework, the problem of estimation of frequencies can be cast as problem of parameters estimation and, as expected, the theory of adaptive observers can be successfully proposed as a tool. Recently, in [2], a global dynamic estimator of frequency and amplitude of a single sinusoidal signal has been presented. As shown by the author, the proposed solution can be cast in terms of adaptive observers and it is potentially very sensitive to measurement noise. To reduce noise sensitivity, in [2] an higher order estimator making use of filtered variables is proposed.

The goal of this paper is to suggest a further contribution to this rich scenario by showing how to solve the problem at hand through the observer's theory proposed in [7] (see also [1]). In this method the observer is given by a memoryless transformation of the state of an Hurwitz system driven by the measured signal. The apparent advantage in pursuing this strategy is that the Hurwitz system is able to filter the effect of measurement noise by thus showing certain robustness features to high-frequency additive disturbances. In this paper these robustness features are exhibited by simulation investigation. The proposed results represent only preliminary achievements which must be improved in several directions. First of all the proposed method is inherently limited to the estimation of a single sinusoid. In this respect future attempts will be directed to identify a possible interconnection of elementary units of the kind presented in this paper to estimate periodic/quasi-periodic signals given by the superimposition of different harmonics. Furthermore the explicit observer form is given under the assumption that the actual frequency lies within a known (possibly large) compact set. Global frequency estimation results are still missing and will be investigated in the future. Finally the aforementioned robustness properties with respect to additive noise is not yet proved formally but only supported by intuition and simulation results. This is a further theoretical improvement, which will be addressed in the future, involving the investigation of the regularity properties of the memoryless output transformation which characterizes the proposed observer. Moreover, because of the very simple structure of the system and the many solutions proposed to solve the observation problem, it could very well be that our observer is nothing but an already available one (or a variation of it) but expressed in a completely different way. A very deep and specific analysis is needed to check this point.

The paper is organized as follows. In the next section the nonlinear Luenberger observer theory as originally proposed in [7] and refined in [1] is briefly recalled. Section III presents the main result of the paper, namely the explicit form of the observer in the case of amplitude, phase and frequency estimation of a pure sinusoid. The result is proved in Section IV while simulation results are shown in Section V. Finally Section VI with final remarks and future developments.

II. THE NONLINEAR LUENBERGER OBSERVER

In this section we briefly review the structure of the observer proposed in [1]. Given an observed system

$$\dot{x} = f(x) \qquad y = h(x) \tag{1}$$

with state $x \in \mathcal{R} \subseteq \mathbb{R}^n$, and output $y \in \mathbb{R}$, the observer is chosen as

$$\dot{\xi} = F\xi + Gy \qquad \hat{x} = \gamma(\xi) \tag{2}$$

in which F is an Hurwitz matrix of suitable dimension $m \times m$, (F, G) is a controllable pair, $\gamma : \mathbb{R}^m \mapsto \mathbb{R}^n$ is a continuous map and $\hat{x} \in \mathbb{R}^n$ represents an estimate of x. Following [1], the intuition behind this choice is to design the map $\gamma(\cdot)$ to be a left inverse of a continuous map $T : \mathcal{R} \mapsto \mathbb{R}^m$ satisfying[1]

$$\frac{dT(x)}{dx} f(x) = FT(x) + Gh(x)\,. \tag{3}$$

As a matter of fact, it is easy to show that, if T is any map satisfying (3) for all $x \in \mathcal{R}$, so long as the trajectory of (1) exists and it is finite,

$$\lim_{t\to\infty} |\xi(t) - T(x(t))| = 0\,.$$

Designing the map γ so that it is uniformly continuous on some open neighborhood of $\mathrm{cl}\, T(\mathcal{R})$ and such that

$$\gamma \circ T(x) = x \qquad \forall\, x \in \mathcal{R}\,,$$

it turns out that $\hat{x}$ asymptotically converges to the true state x and (2) qualifies as a possible state observer.

The crucial result in this context (see [1]) is that if m – the dimension of the observer – is chosen sufficiently large, and certain *observability conditions* are satisfied (see [1] for more details), the map T is injective and hence possesses a left-inverse. The big practical obstacle, though, in the design of such an observer is the actual construction of the map γ. This will be addressed in the next part of the paper in the specific case system (1) is an uncertain oscillator.

III. The result

The main goal of the paper is use the approach of [1] in the specific case in which the observed system (1) is a system of the form

$$\begin{aligned} \dot{x}_1 &= -x_2 \\ \dot{x}_2 &= x_3 x_1 \\ \dot{x}_3 &= 0 \\ y &= x_1 \end{aligned} \tag{4}$$

with state $x := (x_1, x_2, x_3) \in \mathbb{R}^3$ and measured output $y \in \mathbb{R}$. The output of this system, generated from an initial condition $(x_{10}, x_{20}, x_{30}) \in \mathbb{R}^3$, is given by the sinusoidal signal $y(t) = A \sin(\omega t + \Phi)$ with amplitude

$$A = \sqrt{(x_{10}^2 x_{30} + x_{20}^2)/x_{30}}\,,$$

frequency and phase respectively given by

$$\omega = \sqrt{x_{30}}\,, \qquad \Phi = \tan^{-1}(\sqrt{x_{30}}\, x_{10}/x_{20})\,.$$

[1]It turns out that a possible expression of T satisfying (3) is given by

$$T(x) = \int_{-\infty}^{0} e^{-Fs} G h(\varphi_x(s, x)) ds\,,$$

in which $\varphi_x(s, x)$ denotes the value of the trajectory of (1) at time s passing through x at time $s = 0$.

In this way, the problem of estimating amplitude, frequency and phase of a sinusoidal signal can be translated into the problem of state observation of the system (4).

According to the results sketched in the previous section, we consider an observer of the form (2) in which (F, G) are chosen as

$$F = -\mathrm{diag}\,(\lambda_1,\, \lambda_2,\, \ldots\, \lambda_m) \qquad G = (1,\, 1,\, \ldots\, 1)$$

in which λ_i, $i = 1, \ldots, m$ are positive design parameters. In this specific case the expression of the map T satisfying (3) can be proved to be

$$T(x) = \begin{pmatrix} T_1(x) & T_2(x) & \ldots & T_m(x) \end{pmatrix}^{\mathrm{T}} \tag{5}$$

in which

$$\begin{array}{rcccl} T_i & : & \mathbb{R}^3 & \to & \mathbb{R} \qquad i = 1, \ldots, m \\ & & x & \mapsto & \dfrac{\lambda_i x_1 + x_2}{\lambda_i^2 + x_3}\,. \end{array} \tag{6}$$

The result which will be proved in the paper is that the map T defined in (5)-(6) turns out to be injective on any compact subset of the open set

$$\mathcal{O} := \{x \in \mathbb{R}^3 \quad : \quad x_1^2 + x_2^2 \neq 0\,,\ x_3 > 0\} \tag{7}$$

provided that $m \geq 4$ and the λ_i's are mutually distinct. It is worth noting that the requirement that $x \in \mathcal{O}$, qualifying as a *persistence of excitation condition*, represents, in this specific setting, the required observability condition needed to have $T(\cdot)$ injective and, as a consequence, to design the map $\gamma(\cdot)$. As far as the latter is concerned, the crucial result which will be proved in the paper is that if the initial condition of x_3 is known to range in a bounded set $[\underline{x}_3,\ \bar{x}_3]$ with $\underline{x}_3$ and $\bar{x}_3$ arbitrary positive numbers, and the dimension m is chosen equal to $m = 4$, a possible expression of the map γ is given by

$$\gamma(\xi) = \begin{cases} \Phi(\xi)^{-1} L^{\mathrm{T}}(\xi) F^2 \xi & \text{if } \xi \in \mathbb{R}^4 \setminus \{0\} \\ \gamma_0 & \text{if } \xi = 0 \end{cases} \tag{8}$$

in which γ_0 is an arbitrary real number,

$$\begin{aligned} \Phi(\xi) &= \left(L^{\mathrm{T}}(\xi) L(\xi) + \mu(\xi)\, I\right) \\ L(\xi) &= \begin{pmatrix} \lambda & \mathbf{1} & -\xi \end{pmatrix} \end{aligned}$$

with

$$\begin{aligned} \lambda &= \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \end{pmatrix}^{\mathrm{T}} \\ \mathbf{1} &= \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}^{\mathrm{T}}, \end{aligned}$$

I the 4×4 identity matrix and

$$\mu(\xi) = \max\{\ell |\xi|^2 - 2|M\xi|^2\,,\ 0\} \tag{9}$$

with

$$M = \begin{pmatrix} 0 & \lambda_3 - \lambda_4 & \lambda_4 - \lambda_2 & \lambda_2 - \lambda_3 \\ \lambda_4 - \lambda_3 & 0 & \lambda_1 - \lambda_4 & \lambda_3 - \lambda_1 \\ \lambda_2 - \lambda_4 & \lambda_4 - \lambda_1 & 0 & \lambda_1 - \lambda_2 \\ \lambda_3 - \lambda_2 & \lambda_1 - \lambda_3 & \lambda_2 - \lambda_1 & 0 \end{pmatrix}.$$

and ℓ obtained by solving the following minimization problem

$$\ell \;=\; \inf_{x \in \mathcal{O}_r :\, |T(x)|=1} |MT(x)|^2 \tag{10}$$

having denoted

$$\mathcal{O}_r = \left\{ x \in \mathbb{R}^3 : \, x_1^2 + x_2^2 \neq 0 \,, \; x_3 \in [\underline{x}_3 \,, \; \bar{x}_3] \right\} . \qquad (11)$$

The map γ so defined will be proved to be continuous on $\mathbb{R}^4 \setminus \{0\}$ and to satisfy $\gamma \circ T(x) = x$ for any $x \in \mathcal{O}_r$. Thus, according to the previous discussion, the proposed observer asymptotically estimates the state of (4) provided that its initial condition is constrained on the *invariant set* $\mathcal{R} \subset \mathcal{O}_r$ defined as

$$\mathcal{R} = \left\{ x \in \mathbb{R}^3 : \, r \leq x_1^2 + x_2^2 \leq R \,, \; x_3 \in [\underline{x}_3 \,, \; \bar{x}_3] \right\} .$$

with r, R arbitrary positive real numbers. Details regarding the proof of this result are given in the next section.

IV. Proof

To prove that the expression of the function T is the one given in (6), note that, being the dynamics of the i-th a component of ξ given by

$$\dot{\xi}_i \;=\; -\lambda_i \, \xi_i \;+\; x_1 \,, \qquad (12)$$

in view of the linearity in x_1 and x_2 of (4) and (12), the T_i's are linear in those variables, i.e.

$$T_i(x) \;=\; \alpha_i(x_3) \, x_1 \;+\; \beta_i(x_3) \, x_2 \; .$$

Since, by (3), T_i must satisfy

$$\dot{T}_i \;=\; -\lambda_i \, T_i \;+\; x_1 \;,$$

the auxiliary functions α_i and β_i are solutions of

$$\begin{aligned} -x_2 \, \alpha_i(x_3) \;+\; x_3 \, x_1 \, \beta_i(x_3) \;=& \\ -\lambda_i \left[\alpha_i(x_3) \, x_1 + \beta_i(x_3) \, x_2\right] \;+\; x_1 \end{aligned}$$

or equivalently

$$\begin{aligned} x_3 \, \beta_i(x_3) \;\;&=\;\; -\lambda_i \, \alpha_i(x_3) \;+\; 1 \;, \\ \alpha_i(x_3) \;\;&=\;\; \lambda_i \beta_i(x_3) \;, \end{aligned}$$

which, solved for α_i and β_i, yield (6).

We prove now that if $m \geq 4$ the map $x \mapsto T = (T_1, \ldots, T_n)$ is Lipschitz injective[2] on any compact subset of $\mathcal{O}$ defined in (7). For, let x^{a} and x^{b} be two points in $\mathbb{R}^3$. We have

$$\begin{aligned} & T_i(x^{\mathrm{a}}) - T_i(x^{\mathrm{b}}) = \\ & \frac{[\lambda_i^2 + x_3^{\mathrm{b}}]\,[\lambda_i x_1^{\mathrm{a}} + x_2^{\mathrm{a}}] - [\lambda_i^2 + x_3^{\mathrm{a}}]\,[\lambda_i x_1^{\mathrm{b}} + x_2^{\mathrm{b}}]}{[\lambda_i^2 + x_3^{\mathrm{b}}][\lambda_i^2 + x_3^{\mathrm{a}}]} \end{aligned}$$

yielding

$$T(x^{\mathrm{a}}) - T(x^{\mathrm{b}}) = \frac{\mathfrak{V} \, \mathrm{v}(x_1^{\mathrm{a}}, x_1^{\mathrm{b}}, x_2^{\mathrm{a}}, x_2^{\mathrm{b}}, x_3^{\mathrm{a}}, x_3^{\mathrm{b}})}{[\lambda_i^2 + x_3^{\mathrm{b}}][\lambda_i^2 + x_3^{\mathrm{a}}]}$$

[2]A function $f : A \to B$ on metric spaces A and B is said Lipschitz injective if there exists a positive real number L such that, for any pair (a, b) in A, we have:

$$d_A(a, b) \;\leq\; L \, d_B(f(a), f(b)).$$

where

$$\mathrm{v}(x_1^{\mathrm{a}}, x_1^{\mathrm{b}}, x_2^{\mathrm{a}}, x_2^{\mathrm{b}}, x_3^{\mathrm{a}}, x_3^{\mathrm{b}}) = \begin{pmatrix} x_1^{\mathrm{a}} - x_1^{\mathrm{b}} \\ x_2^{\mathrm{a}} - x_2^{\mathrm{b}} \\ x_3^{\mathrm{b}} x_1^{\mathrm{a}} - x_3^{\mathrm{a}} x_1^{\mathrm{b}} \\ x_3^{\mathrm{b}} x_2^{\mathrm{a}} - x_3^{\mathrm{a}} x_2^{\mathrm{b}} \end{pmatrix}$$

and where $\mathfrak{V}$ denotes the *Vandermonde matrix*

$$\mathfrak{V} \;=\; \begin{pmatrix} \lambda_1^3 & \lambda_1^2 & \lambda_1 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_m^3 & \lambda_m^2 & \lambda_m & 1 \end{pmatrix} .$$

Provided the λ_i's are mutually distinct, this matrix is left invertible for $m \geq 4$. In this case, we have

$$\begin{aligned} & \left|(\mathfrak{V}^T \mathfrak{V})^{-1} \mathfrak{V}^T [T(x^{\mathrm{a}}) - T(x^{\mathrm{b}})]\right| \;\leq \\ & \frac{1}{\sqrt{\lambda_{\min}(\mathfrak{V}^T \mathfrak{V})^{-1}}} \left|T(x^{\mathrm{a}}) - T(x^{\mathrm{b}})\right| \; . \end{aligned}$$

On the other hand, the triangle inequality leads to

$$\begin{aligned} & |x_3^{\mathrm{b}} x_1^{\mathrm{a}} - x_3^{\mathrm{a}} x_1^{\mathrm{b}}| \;+\; \tfrac{|x_3^{\mathrm{a}}| + |x_3^{\mathrm{b}}|}{2} \, |x_1^{\mathrm{a}} - x_1^{\mathrm{b}}| \;\geq \\ & \frac{|x_1^{\mathrm{a}}| + |x_1^{\mathrm{b}}|}{2} \, |x_3^{\mathrm{b}} - x_3^{\mathrm{a}}| \; . \end{aligned}$$

yielding, for $m \geq 4$ and mutually distinct λ_i's,

$$\begin{aligned} & |x_1^{\mathrm{a}} - x_1^{\mathrm{b}}| \;+\; |x_2^{\mathrm{a}} - x_2^{\mathrm{b}}| \;+\; \mathrm{c}(x_1^{\mathrm{a}}, x_1^{\mathrm{b}}, x_2^{\mathrm{a}}, x_2^{\mathrm{b}}) \, |x_3^{\mathrm{b}} - x_3^{\mathrm{a}}| \\ & \leq \; \mathrm{d}(x_3^{\mathrm{a}}, x_3^{\mathrm{b}}) \; \left|T(x^{\mathrm{a}}) - T(x^{\mathrm{b}})\right| \end{aligned}$$

where

$$\mathrm{c}(x_1^{\mathrm{a}}, x_1^{\mathrm{b}}, x_2^{\mathrm{a}}, x_2^{\mathrm{b}}) = \frac{|x_1^{\mathrm{a}}| + |x_2^{\mathrm{a}}| + |x_1^{\mathrm{b}}| + |x_2^{\mathrm{b}}|}{2}$$

$$\mathrm{d}(x_3^{\mathrm{a}}, x_3^{\mathrm{b}}) = \frac{\left[\max_i |\lambda_i|^2 + \frac{|x_3^{\mathrm{a}}| + |x_3^{\mathrm{b}}|}{2}\right]^2 \left[1 + \frac{|x_3^{\mathrm{a}}| + |x_3^{\mathrm{b}}|}{2}\right]}{\sqrt{\lambda_{\min}(\mathfrak{V}^T \mathfrak{V})^{-1}}}$$

This proves the desired result, namely that the function T is Lipschitz injective on any compact subset of $\mathcal{O}$. Note that it is not Lipschitz-injective on $\mathcal{O}$ as its Lipschitz constant would tend to infinity as x_3 would go to infinity or (x_1, x_2) would go to the origin.

We prove now that the map $\xi \mapsto \gamma(\xi)$ defined in (8) represents a left inverse of T on $T(\mathcal{O})$. To this purpose note that the problem of computing a left inverse of T amounts, given $\xi = \mathrm{col}(\xi_1, \xi_2, \xi_3, \xi_4)$, to find $x = \mathrm{col}(x_1, x_2, x_3)$ possibly solution of

$$\xi_i \;=\; \frac{\lambda_i x_1 + x_2}{\lambda_i^2 + x_3} \qquad i = 1, \ldots, 4$$

or equivalently, as $\lambda_i^2 + x_3 > 0$ on $\mathcal{O}$,

$$\xi_i \left(\lambda_i^2 + x_3\right) \;-\; \lambda_i x_1 \;-\; x_2 \;=\; 0 \qquad i = 1, \ldots, 4 \; .$$

In compact form the previous set of equations rewrites as

$$L(\xi) x - F^2 \xi = 0 \, . \qquad (13)$$

Lipschitz injectivity of the function T guarantees that this set of equations has a solution if $\xi \in T(\mathcal{O})$. In particular it turns out that $L(\xi)$ is left invertible on $T(\mathcal{O})$ and the solution of (13) is given by

$$x = (L^{\mathrm{T}}(\xi)L(\xi))^{-1}L^{\mathrm{T}}(\xi)F^2\xi \qquad \xi \in T(\mathcal{O})\,.$$

To extend the solution outside $T(\mathcal{O})$, we look for a solution of (13) in a mean square sense. Specifically, we look for the vector $\hat{x} = \mathrm{col}(\hat{x}_1, \hat{x}_2, \hat{x}_3)$ minimizing the function

$$J_1 \;=\; \left|L(\xi)\hat{x} - F^2\xi\right|^2 + \mu(\xi)\hat{x}^{\mathrm{T}}\hat{x}$$

where $\mu(\cdot)$ is an arbitrary positive function satisfying

$$\mu(\xi) = 0 \qquad \forall \xi \in T(\mathcal{O}) \tag{14}$$

designed to extend the solution outside $T(\mathcal{O})$. In particular, as the minimizer is solution of

$$\left[L^{\mathrm{T}}(\xi)L(\xi) + \mu(\xi)I\right]\hat{x} \;=\; L^{\mathrm{T}}(\xi)F^2\xi \tag{15}$$

we design $\mu(\xi)$ satisfying (14) so that the matrix $\Phi(\xi) = L^{\mathrm{T}}(\xi)L(\xi) + \mu(\xi)I$ is invertible for all $\xi \in \mathbb{R}^4 \setminus \{0\}$. Indeed, in the next part of the proof, we prove that (9) is a possible choice fulfilling the previous requirements. For, we first note that the matrix $L(\xi)$ is not left invertible, namely $L^{\mathrm{T}}(\xi)L(\xi)$ is singular, if and only if ξ satisfies $M\xi = 0$.

Furthermore, in case $\det L^{\mathrm{T}}(\xi)L(\xi) = 0$, namely $M\xi = 0$, then necessarily there exist a and b such that

$$\xi = a\lambda + b\mathtt{1}\,.$$

With this and the definition (11) in mind, we show now that if ξ is in $T(\mathcal{O}_r)$ (i.e. satisfies (13)), a lower bound for $|M\xi|^2$ in the form $\ell|\xi|^2$ with ℓ, strictly positive, defined in (10). To prove this claim, we first observe that (13) implies (change (x_1, x_2) in (ax_1, ax_2))

$$\xi \in T(\mathcal{O}_r) \qquad \Longrightarrow \qquad a\xi \in T(\mathcal{O}_r) \quad \forall a\;.$$

Furthermore, from (10), we have

$$\ell \;\leq\; \frac{|M\xi|^2}{|\xi|^2} \qquad \forall\, \xi \in T(\mathcal{O}_r) \setminus \{0\}\;.$$

For the sake of obtaining a contradiction, assume $\ell = 0$. This implies the existence of a sequence ξ_n in $T(\mathcal{O}_r)$ and with unit norm such that $M\xi_n$ goes to zero. The sequence being in the unit sphere, it has an accumulation point $\xi^\star$ which, by continuity, satisfies $|\xi^\star| = 1$ and $M\xi^\star = 0$. So there exist a and b and a sequence x_n in $\mathcal{O}_r$ such that we have

$$\xi^\star \;=\; a\,\lambda \;+\; b\,\mathtt{1} \quad , \qquad \xi_{in} \;=\; \frac{\lambda_i x_{1n} + x_{2n}}{\lambda_i^2 + x_{3n}}\;,$$

and, for any ε, we can find N such that for all $n \geq N$, we have

$$\sum_i |\xi_{in} - (a\lambda_i + b)|^2 \;\leq\; \varepsilon\;. \tag{16}$$

Since x_{3n} is in $[\underline{x}_3, \bar{x}_3]$ there is a finite accumulation point $x_3^\star$ in this interval. In this case, for any $\varepsilon > 0$, there exists n (large enough), such that we have

$$\sum_i \left|\frac{\lambda_i[x_{1n} - ax_3^\star] + [x_{2n} - bx_3^\star] - \lambda_i^3 a - \lambda_i^2 b}{\lambda_i^2 + x_3^\star}\right|^2 \;\leq\; 2\varepsilon\;. \tag{17}$$

Because the λ_i are mutually distinct, the left hand side is a positive definite quadratic form in (x_{1n}, x_{2n}) with a unique global minimizer. Because of the Vandermonde structure, its global minimum can be smaller than 2ε, with ε arbitrary only if $a = b = 0$. $\xi^\star$ having norm 1, this leads to a contradiction.

From this, we conclude that if $|M\xi|^2$ is strictly smaller than $\ell|\xi|^2$, then ξ cannot be in $T(\mathcal{O}_r)$. Hence, the choice (9) satisfies (14) and yields a matrix $\Phi(\xi)$ which is positive definite for all $\xi \in \mathbb{R}^4 \setminus \{0\}$.

V. Simulation Results

We have implemented the proposed observer in order to check the reliability of the estimation in presence of high frequency noise superimposed to a low frequency pure sinusoid. The observer has been implemented with the values of λ_i set to $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$ and $\lambda_4 = 4$. It turns out that the bandwidth of the linear system (F, G) is approximately 1 rad/sec. The nonlinear output map γ in (8) has been tuned with $\ell = 5$ according to a trial and error procedure to satisfy (10). The observer has been driven with a “low frequency” sinusoid with amplitude 2 and frequency 3 rad/sec perturbed with an higher frequency harmonic with amplitude 0.2 (10% of the main amplitude) and frequency set to 10 rad/sec, 10^2 rad/sec and 10^3 rad/sec in three subsequent experiments. The high frequency components have been chosen to be respectively 1, 2 and 3 decades after the bandwidth of the filter. Simulation results regarding the amplitude and frequency estimation are shown respectively in figures 1,2,3 and figures 4,5,6 in the three different noise scenarios described above. From these figures it can be noted that the steady state estimation error is approximately 10%, 1% and 0.1% of the respective nominal values with mean and standard deviation steady state values specified in the caption of the figures.

In order to compare the performances of the proposed observer with the ones of recently-proposed frequency estimators, we have implemented the 7th order adaptive observer proposed in [2]. This adaptive observer is given by the filters

$$\begin{aligned} \dot{\xi}_1 &= -\lambda\xi_1 + 3\lambda y \\ \dot{\xi}_1 &= -\lambda\xi_1 - 2\lambda y^2\,, \end{aligned}$$

introduced for measurement noise reduction, and by the dynamics (of order 5) described by

$$\begin{aligned} \dot{\hat{z}}_1 &= \hat{z}_2 + \xi^{\mathrm{T}}\hat{\theta} + (1+\alpha\lambda)(\lambda y^2/2 - \hat{z}_1) \\ \dot{\hat{z}}_2 &= \lambda\xi^{\mathrm{T}}\hat{\theta} + \alpha(\lambda y^2/2 - \hat{z}_1) \\ \dot{\hat{\theta}} &= \Gamma\xi(\lambda y^2/2 - \hat{z}_1) \end{aligned}$$

with $\Gamma = \mathrm{diag}(\gamma_0, \gamma_1, \gamma_2)$ and $\gamma_0, \gamma_1, \gamma_2$, λ and α positive design parameters. The numerical values of the latters have been chosen as proposed in the simulation section of [2]. The adaptive observers have been tested in the three noise scenarios described above and the results (only regarding the frequency estimation given by $\sqrt{\hat{\theta}_3}$) are shown in figures 7,8 and 9 below.

VI. Conclusions and future developments

Following the observer theory pioneered in [7] and developed in [1], we have presented a new observer for an uncertain oscillator. We have shown how amplitude, phase and frequency of a pure harmonic can be estimated by a memoryless transformation of the steady state of an Hurwitz system, of dimension at least 4, driven by the estimated signal. Simulation results showing the effectiveness of the proposed method and its robustness with respect to high frequency noise have been also presented. Future developments will be mainly focused on extending the proposed theory to estimate a multi-frequency signal and to achieve global convergence in the frequency component. Further attempts will be directed to rigorously characterize the robustness of the proposed observer to high frequency noise by comparing its properties with respect to the ones of phase-looked loop circuits presented in literature.

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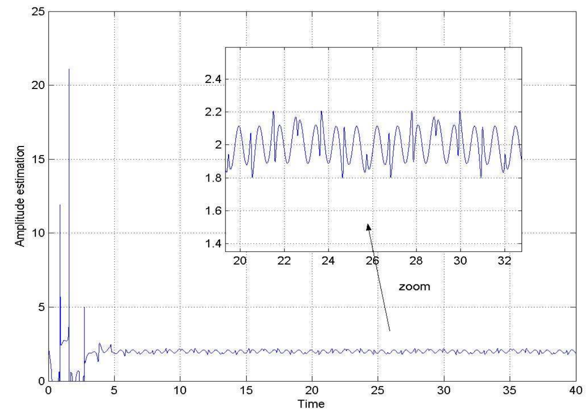

Fig. 1. Amplitude estimation in case of high frequency noise at 10 rad/sec. The mean and standard deviation steady state values of the estimate are respectively equal to 1.9999 and 0.0947.

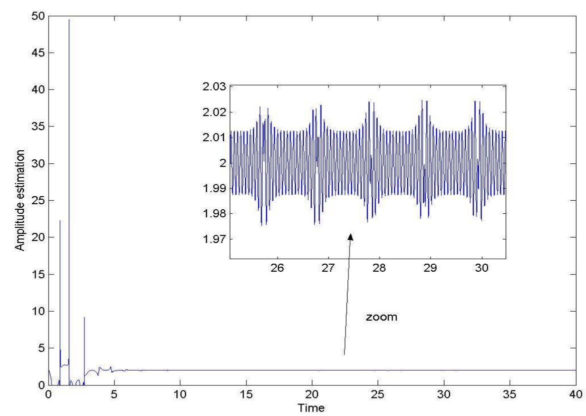

Fig. 2. Amplitude estimation in case of high frequency noise at 10^2 rad/sec. The mean and standard deviation steady state values of the estimate are respectively equal to 2.0 and 0.0108.

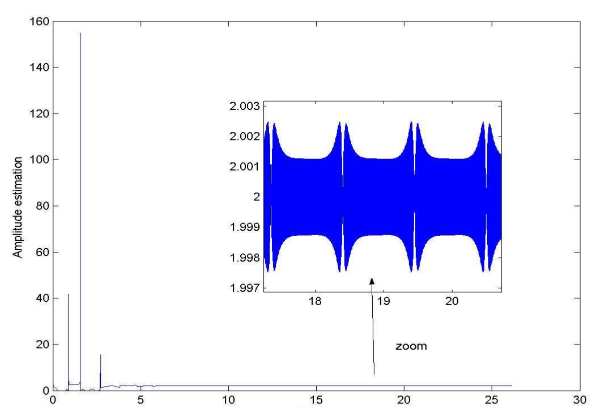

Fig. 3. Amplitude estimation in case of high frequency noise at 10^3 rad/sec. The mean and standard deviation steady state values of the estimate are respectively equal to 2.0 and 0.0011.

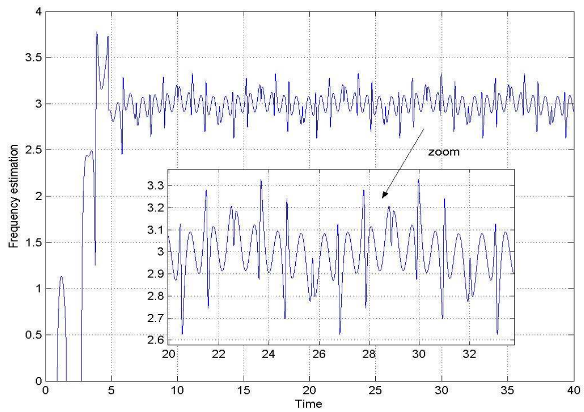

Fig. 4. Frequency estimation in case of high frequency noise at 10 rad/sec. The mean and standard deviation steady state values of the estimate are respectively equal to 2.9956 and 0.1164.

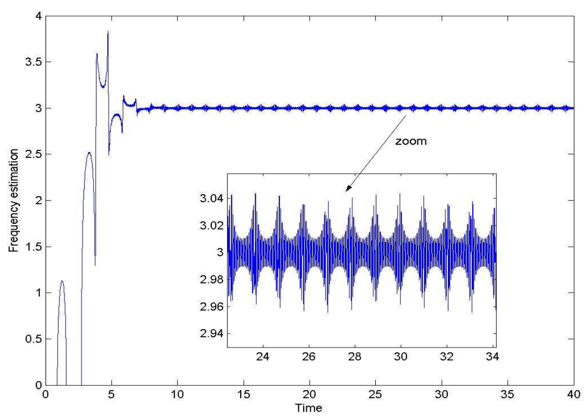

Fig. 5. Frequency estimation in case of high frequency noise at 10^2 rad/sec. The mean and standard deviation steady state values of the estimate are respectively equal to 3.0 and 0.0142.

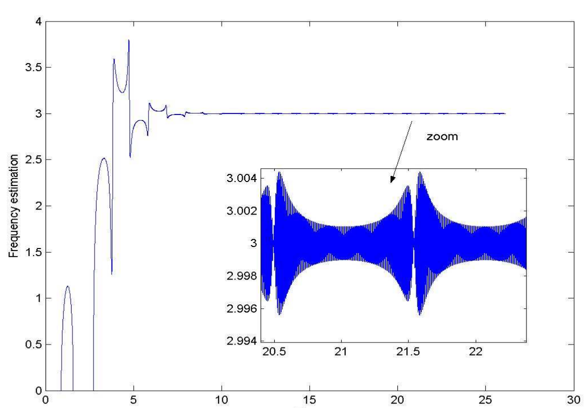

Fig. 6. Frequency estimation in case of high frequency noise at 10^3 rad/sec. The mean and standard deviation steady state values of the estimate are respectively equal to 3.0 and 0.0014.

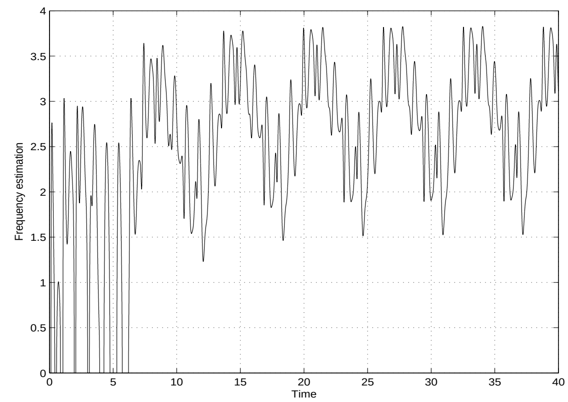

Fig. 7. 7th order adaptive observer described in [2]. Frequency estimation in case of measurement noise at 10 rad/sec. The mean and standard deviation steady state values of the estimate are respectively equal to 2.6677 and 0.8463.

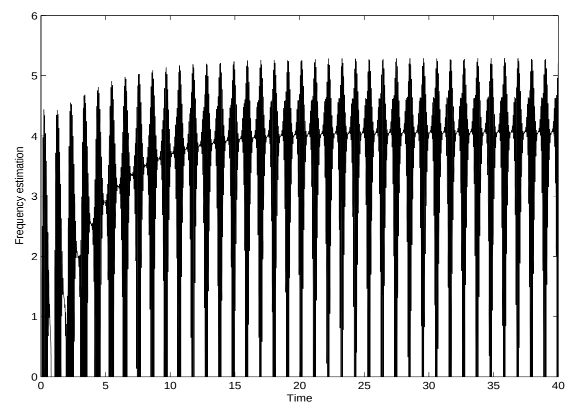

Fig. 8. 7th order adaptive observer described in [2]. Frequency estimation in case of measurement noise at 10^2 rad/sec. The mean and standard deviation steady state values of the estimate are respectively equal to 3.7028 and 1.1004.

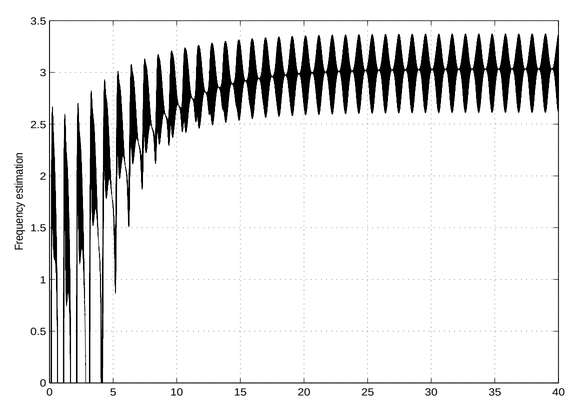

Fig. 9. 7th order adaptive observer described in [2]. Frequency estimation in case of measurement noise at 10^3 rad/sec. The mean and standard deviation steady state values of the estimate are respectively equal to 2.8141 and 0.5285.