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Practical Output Regulation without High-Gain

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Abstract— **The main goal of the paper is to complement the theory of nonlinear output regulation without immersion presented in [7] with results useful for practical design of the regulator. In a general framework relying only on an assumption about stabilizability of the zero dynamics of the controlled plant, we present explicit expressions of the regulator and a practical design procedure leading to a regulator achieving practical regulation uniformly in the local gain of the stabilizer and in the dimension of the internal model.**

I. Introduction

This paper complements the theory of output regulation presented in [7] with additional results regarding the practical implementation of the regulator. In [7] the problem of output regulation has been addressed in a fairly general framework consisting of a class of controlled systems and exosystems required to satisfy only an appropriate assumption about stabilizability of the zero dynamics by output feedback. The emphasis in that work was placed on the fact that the so-called immersion assumption, characterizing in different forms all the previous literature on this topic and limiting in a substantial way the applicability of the theory in a nonlinear context, was not necessary to solve a problem of output regulation. The design methodology underlying [7], induced by the approach pioneered in [2] (see also [3]), is based on the reformulation of the problem of output regulation into a problem of output feedback stabilization of compact attractors and on the use of the theory of nonlinear observers for internal model design. In [8] the design methodology behind [7] has been also extended to a more general framework regarding output feedback stabilization of compact attractors for nonlinear systems. In plain words the main achievement in [8] has been to show that the steady state input rendering invariant a compact attractor to be stabilized by output feedback can be dynamically generated, in a robust framework, by an appropriately designed regulator without any specific condition on this input (required, on the contrary, in the past through the so-called immersion assumption). In achieving this result a big role has been played by the theory of nonlinear observers developed in [11], [1].

The developments in [7] and [8] was deliberately focused on issues regarding the existence of the regulator and no special attention was given on real design aspects. In this paper we aim to fill this lack. More specifically we present explicit expressions of the regulator and we address possible practical implementations of it for real design purposes. In proposing practical implementation of the regulator we implicitly solve a problem of practical output regulation namely the design of a regulator achieving arbitrarily small asymptotic regulation error. Several attempts have been done in the past literature along this direction. One way of approaching the problem of practical output regulation is the one pursued, besides others, in [5] and [11] (see also [4], section 2.5). Here the idea is to use polynomial approximation and/or power series expansion of the so-called regulator equations to identify an approximation of the desired steady state control input, with a degree of accuracy depending on the bound of the residual error, which can be dynamically reproduced by means of a linear internal model. The main drawback in pursuing this strategy is that the dimension of the internal model is, in general, dependent on the desired bound of the regulation error and tends to growth indefinitely as the desired bound tends to zero. A different control philosophy to steer regulation error to arbitrarily small values is to use high-gain error feedback (see [6] and related literature). In general this kind of strategies, not relying upon the use of internal models and applicable to a wide class of reference/disturbance signals, present the typical problems linked to high-gain control structures, such as sensitivity to measurement noise and minimum-phase constraints, which substantially limit the range of applications. On the contrary, in this paper, we present practical design procedures leading to a regulator achieving practical regulation uniformly in the local gain of the stabilizer and in the dimension of the internal model.

For reasons of space in this paper we limit ourself to present the main results without providing detailed proofs which can be found, along with extra discussions, results and examples, in the expanded journal version which is under submission.

Notation. For $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean norm and, for $\mathcal{C}$ a closed subset of $\mathbb{R}^n$, $|x|_{\mathcal{C}} = \min_{y\in\mathcal{C}} |x-y|$ denotes the distance of x from $\mathcal{C}$. For $\mathcal{S}$ a subset of $\mathbb{R}^n$, int$\mathcal{S}$ is the interior of $\mathcal{S}$ respectively. For a locally Lipschitz function $V(t)$ we define the Dini's derivative of V at t as

$$D^+V(t) = \lim_{h\to 0^+} \sup \frac{1}{h}[V(t+h) - V(t)]\,.$$

By extension, when $V(t)$ is obtained by evaluating V along

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a solution $x(t, x_0)$, we denote also

$$D^+V(x_0) = \lim_{h\to 0^+} \sup \frac{1}{h}[V(x(h, x_0)) - V(x_0)] . \qquad (1)$$

II. THE FRAMEWORK

We focus on the class of systems

$$\begin{array}{rcl} \dot z &=& f(w, z, \zeta) \\ \dot\zeta &=& q(w, z, \zeta) + u \\ e &=& h(w, z, \zeta) \\ y &=& k(w, z, \zeta) . \end{array} \qquad (2)$$

with state $(z, \zeta) \in I\!R^n \times I\!R$, control input $u \in I\!R$, regulated output $e \in I\!R$, measured output $y \in I\!R$ and exogenous (disturbance) input $w \in I\!R^r$ generated by an exosystem

$$\dot w = s(w) . \qquad (3)$$

The functions $f(w, z, \zeta), q(w, z, \zeta)$ and $s(w)$ are C^k functions (for some large k) of their arguments. The initial conditions of (2) range on a set $Z \times E$, in which Z and E are fixed *compact* subsets of $I\!R^n$ and, respectively, $I\!R$. The initial conditions of the exosystem (3) range on a *compact* subset W of $I\!R^r$ which, according to standard output regulation theory, is supposed to be *(forward and backward) invariant* for (3). The problem of output regulation for system (2)-(3), which amounts in designing an output (y) feedback controller achieving boundedness of the closed-loop trajectories and steering the regulation error to zero asymptotically, has been solved in [7] under the following assumption involving stabilizability of the zero dynamics in appropriate sense.

Assumption. There exists a bounded subset B of $W \times I\!R^n$, a C^k function $\alpha : I\!R^r \times I\!R^n \to I\!R$ and a C^k map $\Phi : I\!R^p \to I\!R$ such that:

($\mathbf{a}_1$) the set B contains the positive orbit of the set $W \times Z$ under the flow of

$$\begin{array}{rcl} \dot w &=& s(w) \\ \dot z &=& f(w, z, \alpha(w, z)) . \end{array} \qquad (4)$$

Moreover, the consequent ω-limit set $\omega(W \times Z)$ of the set $W \times Z$ of initial conditions (see [2]), is such that there exists a number $d_1 > 0$ such that

$$\begin{array}{c} (w, z) \in W \times I\!R^n \\ |(w, z)|_{\omega(W\times Z)} \le d_1 \end{array} \quad\Rightarrow\quad (w, z) \in W \times Z .$$

($\mathbf{a}_2$) $h(w, z, \alpha(w, z)) = 0$ for all $(w, z) \in \omega(W \times Z)$.

($\mathbf{a}_3$) $\Phi(k(w, z, \zeta)) = \zeta - \alpha(w, z)$ for all $(w, z, \zeta) \in I\!R^r \times I\!R^n \times I\!R$.

We refer the reader to [7] and related literature for comments to this assumption. From now on we denote

$$\mathcal{A}_0 := \omega(W \times Z) .$$

The regulator proposed in [7] is of the form

$$\begin{array}{rcl} \dot\xi &=& F\xi + Gu \\ u &=& \gamma(\xi) + v \\ v &=& -\kappa(y) , \end{array} \qquad (5)$$

in which $(F, G) \in I\!R^{m\times m} \times I\!R^m$ is a controllable pair with F Hurwitz and $\gamma : I\!R^m \to I\!R$ and $\kappa : I\!R \to I\!R$ are suitable continuous maps. In particular a crucial role in (5) is played by the function $\gamma(\cdot)$ which is required to satisfy the design formula

$$\mathbf{q}_0(\mathbf{z}) + \gamma \circ \tau(\mathbf{z}) = 0 \qquad \forall\, \mathbf{z} \in \mathcal{A}_0 \qquad (6)$$

having defined $\mathbf{z} = \mathrm{col}(w, z)$,

$$\mathbf{q}_0(\mathbf{z}) = q(w, z, \alpha(w, z)) + \frac{\partial\alpha}{\partial w} s(w) + \frac{\partial\alpha}{\partial z} f(w, z, \alpha(w, z))$$

and with the function $\tau : \mathcal{A}_0 \to I\!R^m$ solution of the PDE

$$\frac{d\tau(\mathbf{z})}{d\mathbf{z}} \mathbf{f_0}(\mathbf{z}) = F\tau(\mathbf{z}) - G\mathbf{q}_0(\mathbf{z}) \qquad \mathbf{z} \in \mathcal{A}_0 \qquad (7)$$

where $\mathbf{f}_0(\mathbf{z}) := \mathrm{col}(s(w), f(w, z, \alpha(w, z))$. The motivation behind (6), is that, after the change of variables $\zeta \mapsto \chi := \zeta - \alpha(w, z)$ and $\eta \mapsto x := \eta - G\zeta$, the closed-loop system (2), (3), (5) can be written in the form

$$\begin{array}{rcl} \dot w &=& s(w) \\ \dot p &=& M(w, p) + N(w, p, \chi) \\ \dot\chi &=& H(w, p) + K(w, p, \chi) + v \end{array} \qquad (8)$$

$p = \mathrm{col}(z, x)$, $N(w, p, \chi)$ and $K(w, p, \chi)$ are suitably defined functions vanishing at $\chi = 0$ for any $p \in I\!R^{n+m}$ and $w \in W$ and

$$M(w, p) = \begin{pmatrix} f(w, z, \alpha(w, z)) \\ Fx - G\mathbf{q}_0(\mathbf{z}) \end{pmatrix}$$

$$H(w, p) = \mathbf{q}_0(\mathbf{z}) + \gamma(x) .$$

The crucial property exhibited by this system is that, by the fact that the set $\mathcal{A}_0$ is invariant for (4) and by (6), (7), the set

$$\begin{array}{l} \mathrm{graph}(\tau) \times \{0\} = \{(\mathbf{z}, x, \chi) \in \mathcal{A}_0 \times I\!R^m \times I\!R \;\; : \\ \qquad x = \tau(\mathbf{z}) , \chi = 0\} \end{array} \qquad (9)$$

is an invariant set for (8) (with $v \equiv 0$) on which, by assumption ($\mathbf{a_2}$), the regulation error e is identically zero. According to this, the design of the regulator (5) can be completed by taking any output feedback $v = \kappa(y)$ rendering the set (9) asymptotically stable with a domain of attraction containing the set of initial conditions. To this purpose, in [7], it has been shown that a stabilizer of the form $v = -k\chi = -k\Phi(y)$ with k sufficiently large succeeds in the stabilization goal provided that the set $\mathcal{A}_0$ is locally exponentially stable for (4) and the function γ satisfying (6) is locally Lipschitz. In the case these two additional assumptions fail, the results in [8] can be used to prove the existence of a continuous function $v = \kappa(\chi) = \kappa(\Phi(y))$, not necessarily linear at the origin, rendering the set (9) asymptotically stable[1]. In general, any design procedure leading to an output feedback control law able to asymptotically stabilize (9) can be adopted to successfully complete the regulator design.

[1]The stabilization procedures described in [7] and generalized in [8] adopt small gain arguments to prove the desired results using the fact that system (8) has relative degree one (with respect to the output χ and input v) and zero dynamics characterized by an asymptotically stable compact attractor.

The important result established in [7] is that a function γ satisfying (6) always exists provided that the dimension m of F and its eigenvalues are properly chosen. Instrumental in proving this result is the following proposition, proven in detail in [8], which details the properties required to F to have the function τ satisfying a partial injectivity condition and, in turn, to guarantee the existence of γ satisfying (6).

Proposition 1: Set $m \geq 2(r+n)+2$. There exist an $\ell > 0$ and a set $\mathcal{S} \subset \mathbb{C}$ of zero Lebesgue measure such that if $\sigma(F) \subset \{\lambda \in \mathbb{C} \ : \ \mathrm{Re}\lambda \leq -\ell\} \setminus \mathcal{S}$ then the function $\tau(\cdot) : \mathcal{A}_0 \to \mathbb{R}^m$ defined as

$$\tau(\mathbf{z}) = -\int_{-\infty}^{0} e^{-Fs} G\mathbf{q}_0(\mathbf{z}(s,\mathbf{z}))ds\,, \qquad (10)$$

in which $\mathbf{z}(t,\mathbf{z}_0)$ denotes the solution at time t of $\dot{\mathbf{z}} = \mathbf{f}_0(\mathbf{z})$ passing through $\mathbf{z}_0$ at time $t = 0$, is differentiable, is the *unique* solution of (7), and satisfies the *partial injectivity condition*

$$|\mathbf{q}_0(\mathbf{z}_1) - \mathbf{q}_0(\mathbf{z}_2)| \leq \varrho(|\tau(\mathbf{z}_1) - \tau(\mathbf{z}_2)|) \quad \text{for all } \mathbf{z}_1, \mathbf{z}_2 \in \mathcal{A}_0 \qquad (11)$$

where $\varrho(\cdot)$ is a class-$\mathcal{K}$ function.

Remark Following the proof of Proposition 3 in [8], it turns out the requirement of choosing F, besides Hurwitz, with a certain stability margin fixed in the previous proposition by the positive real number ℓ, represents only a technical assumption needed to guarantee differentiability of the function τ. In this sense the assumption in question must be not confused with an "high gain" requirement on the choice of F. In other words any choice of F such that (10) is differentiable is an appropriate choice. $\triangleleft$

By the previous discussion it turns out that the bottleneck in the design procedure proposed in [7] is the design of γ satisfying (6). Apart the existence result, the practical design of the function γ is an issue left open in [7] which will be dealt with in this paper.

III. Expressions of γ

In this part we present possible expressions of γ satisfying (6) in terms of the function τ fulfilling the PDE (7).

Our first expression is strongly inspired by [10] (see, in particular, Lemma 4 in the quoted reference). It can be written upon the assumption that the set $\mathcal{A}_0$ is not locally thin[2]. To give it, it is appropriate to associate to each $x \in \mathbb{R}^m$, one of the closest points in $\tau(\mathcal{A}_0)$, namely we associate

$$\bar{\mathbf{z}}_x \in \arg\min_{z \in \mathcal{A}_0} |x - \tau(z)|\,.$$

Also, we introduce a function $\omega : \mathcal{A}_0 \times \mathbb{R}^m \setminus \tau(\mathcal{A}_0) \to \mathbb{R}_+$ as :

$$\omega(\xi, x) = \frac{1}{|x - \tau(\xi)|^{r+1}}\,.$$

with $r = s + n$.

Proposition 2: Let τ be a function which is Lipschitz on $\mathcal{A}_0$ and satisfies (11). Assume the set $\mathcal{A}_0$ be not locally thin. Then

$$\gamma(x) = \begin{cases} -\dfrac{\displaystyle\int_{\mathcal{A}_0} \mathbf{q}_0(\xi)\,\omega(\xi,x)\,\mathrm{d}\Xi}{\displaystyle\int_{\mathcal{A}_0} \omega(\xi,x)\,\mathrm{d}\Xi} & \forall x \in \mathbb{R}^m \setminus \tau(\mathcal{A}_0) \\ -\mathbf{q}_0(\bar{\mathbf{z}}_x) & \forall x \in \tau(\mathcal{A}_0)\,, \end{cases} \qquad (12)$$

with Ξ the Lebesgue measure pf $\mathbb{R}^r$, defines properly a continuous function on $\mathbb{R}^m$ which satisfies (6).

We present now a result, inspired by [9], showing a different expression of γ. In formulating the expression of γ it is argued that the class-$\mathcal{K}$ function $\varrho(\cdot)$ satisfies

$$\varrho(|x_3 - x_1|) \leq \varrho(|x_3 - x_2|) + \varrho(|x_1 - x_2|) \quad \forall (x_1, x_2, x_3) \in \mathbb{R}^{3m}\,. \qquad (13)$$

This, indeed, can be assumed without loss of generality as it can be shown by means of computations omitted for reasons of space.

Proposition 3: Let τ be fulfilling (11) with a function ϱ satisfying (13). Then the function $\gamma : \mathbb{R}^m \to \mathbb{R}$ defined by

$$\gamma(x) = \inf_{\mathbf{z} \in \mathcal{A}} -\mathbf{q}_0(\mathbf{z}) + \varrho(|\tau(\mathbf{z}) - x|) \qquad (14)$$

is continuous and such that (6) is satisfied.

IV. Approximated design and practical regulation

The expressions (12) and (14) represent formulas to design γ which are explicit if we know the set $\mathcal{A}_0$, the function τ solution of (7) and the volume integrals characterizing (12) and the inf characterizing (14). This, indeed, may be a difficult task even in simple cases. For this reason, in this section, we look for an approximated expression of γ which results in a practical regulator yielding an arbitrarily small regulation error. In order to properly state the problem, we argue the existence of an *ideal regulator*

$$\begin{aligned} \dot{\eta} &= F\eta + Gu \\ u &= \gamma(\eta) + v \qquad v = \kappa(y) \end{aligned} \qquad (15)$$

yielding exact regulation. In particular, following the discussion in section 2, the function γ is supposed to satisfy condition (6) while the function κ is supposed to asymptotically stabilize the set (9). In this framework we suppose the function κ given and we look for an approximation of the function γ in order to have and arbitrarily small regulation error. The next proposition details the properties required to the approximated γ in order to obtain a practical regulator. In this we denote by X the compact set characterizing the initial condition of the state variable x.

[2] $\mathcal{A}_0$ is said to be not locally thin if there exist positive constants c and ε_0 such that $\int_{\mathcal{A}_0 \cap \mathcal{B}_\varepsilon(\xi)} \mathrm{d}\Xi \geq c \int_{\mathcal{B}_\varepsilon(\xi)} \mathrm{d}\Xi$ for all $\xi \in \mathcal{A}_0$ and $\varepsilon \in (0, \varepsilon_0]$, in which $\mathcal{B}_\varepsilon(\xi) := \{s \in \mathbb{R}^r \ : \ |s - \xi| < \varepsilon\}$ (see [10]).

Proposition 4: There exists a compact set $\hat{X} \supset X$ and, for any $\epsilon > 0$, there exists an $\delta_\epsilon > 0$, such that if $\gamma_{\rm a} : I\!\!R^m \to I\!\!R$ is a continuous function satisfying

$$|\gamma_a(x) - \gamma(x)| \le \delta_\epsilon \qquad \forall\, x \in \hat{X}$$

the trajectories of (2) in closed loop with the controller

$$\dot{\eta} = F\eta + Gu \qquad u = \gamma_a(\eta) + \kappa(y) \tag{16}$$

are bounded and such that

$$\lim_{t\to\infty} \sup |(w(t), p(t), \chi(t))|_{\text{gr}(\tau)\times\{0\}} \le \epsilon\,.$$

Proof: The system obtained by (2) in closed loop with (16) has the form

$$\begin{array}{rcl} \dot{w} & = & s(w) \\ \dot{p} & = & M(w,p) + N(w,p,\chi) \\ \dot{\chi} & = & H(w,p) + K(w,p,\chi) + \kappa(y) + \Delta\gamma(p) \end{array} \tag{17}$$

in which

$$\Delta\gamma(p) := \gamma_\epsilon(x) - \gamma(x)\,.$$

By assumption the set gr$(\tau) \times \{0\}$ is asymptotically stable for system (17) in the case $\Delta\gamma(p) \equiv 0$. From this the result of the proposition follows by the total stability result reported in the appendix. ■

Remark As $h(w, z, \zeta)$ defining the regulation error e is a continuous function vanishing on the set (9), it turns out that a practical implementation of the regulators allows one to solve a problem of practical output regulation. In particular it turns out that for any $\nu > 0$ there exists a $\epsilon^\star > 0$ such that for any positive $\epsilon \le \epsilon^\star$ the closed-loop trajectories obtained by implementing (16) are such that $\lim_{t\to\infty} |e(t)| \le \nu$. In this respect note that, by the fact that the stabilizer κ in (16) is constrained to be equal to the one of the ideal regulator (15), the solution of a practical output regulation problem does not necessarily rely upon stabilizers which are high-gain local to the zero error manifold unlike the cases in which also the ideal regulator relies upon a not locally Lipschitz stabilizer. $\triangleleft$

Motivated by this in the next part we look for an approximated expressions of γ in the sense of the previous proposition. For reasons of space we limit ourself to present an approximated expression of (14). The first instrumental step amounts to approximate the set $\mathcal{A}_0$. To this purpose, the idea is to introduce a grid of points $\{\mathbf{z}_i\}_{i\in I}$, $I := \{1, 2, \dots, N\}$, satisfying the property

$$\begin{array}{l} |\mathbf{z}_i|_{\mathcal{A}_0} \le \nu \quad \forall\, i \in I \qquad \text{and} \\ \forall\, \mathbf{z} \in \mathcal{A}_0\,,\ \exists\, i \in I\ :\ |\mathbf{z}_i - \mathbf{z}|\ \le\ \nu \end{array} \tag{18}$$

for a positive number ν which is a design parameter. The second instrumental step is to approximate the function τ solution of (7) on the grid $\{\mathbf{z}_i\}_{i\in I}$. As $\mathbf{z}_i$ not necessarily belongs to $\mathcal{A}_0$ on which τ is defined, we let $\bar{\mathbf{z}}_i$ be the projection of $\mathbf{z}_i$ on $\mathcal{A}_0$ and define a set of points T_i supposed to approximate $\tau(\mathbf{z}_i)$, for all $i \in I$, as

$$\sup_{i\in I} |T_i - \tau(\bar{\mathbf{z}}_i)|\ \le\ \nu \qquad \bar{\mathbf{z}}_i := \texttt{argmin}_{\mathbf{z}\in\mathcal{A}_0} |\mathbf{z} - \mathbf{z}_i|\,. \tag{19}$$

We refer the interested reader to the expanded journal version of this paper for the presentation of efficient algorithms to compute the previous sets. Here we limit ourself to present the following proposition which presents an approximated expression of γ on the basis of the grids $\{\mathbf{z}_i\}_{i\in I}$ and $\{T_i\}_{i\in I}$.

Proposition 5: For a positive ν let $\{\mathbf{z}_i\}_{i\in I}$ and $\{T_i\}_{i\in I}$ be two finite family of points which satisfy (18) and (19). Furthermore let γ be the function defined in (14). For any $\epsilon > 0$ there exist $\nu^\star > 0$ such that for any positive $\nu \le \nu^\star$ the function

$$\gamma_\epsilon(x) = \min_{i\in I}\ -\mathbf{q}_0(\mathbf{z}_i) + \varrho(|\tau_\nu(\mathbf{z}_i) - x|) \tag{20}$$

satisfies

$$|\gamma_\epsilon(x) - \gamma(x)| \le \epsilon \qquad \forall\, x \in I\!\!R^m\,.$$

V. Conclusions

We have presented possible explicit expressions of the nonlinear regulator proposed in [7] in the context of nonlinear output regulation in absence of immersion assumptions. An approximated expression of the regulator, suitable for real design purposes, has been also presented yielding practical convergence to zero of the regulation error uniformly in the local gain of the stabilizer and in the dimension of the internal model.

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Appendix

Consider system

$$\begin{array}{rcl} \dot{w} & = & s(w) \\ \dot{p} & = & F(w,p) \end{array} \tag{21}$$

with initial conditions in $W \times I\!\!R^q$, $W \subset I\!\!R^r$ invariant for $\dot{w} = s(w)$, with $F(\cdot,\cdot)$ a continuous function. As W is an invariant set for $\dot{w} = s(w)$ the closed cylinder $\mathcal{C} := W \times I\!\!R^n$ is locally invariant and it is natural to regard (21) as a system defined on $\mathcal{C}$ and endow the latter with the subset topology. This indeed is assumed in the following. Let $\mathcal{B} \subset I\!\!R^r \times I\!\!R^q$ be an asymptotically stable compact set for (21) with domain of attraction $W \times \mathcal{D}$, $\mathcal{D}$ an open subset of $I\!\!R^q$. Then the following holds.

Proposition 6: Consider the perturbed system

$$\begin{array}{rcl} \dot{w} & = & s(w) \\ \dot{p} & = & F(w,p) + \Delta(w,p) \end{array} \tag{22}$$

in which $\Delta(\cdot,\cdot)$ is a continuous function. For any compact set $P \subset \mathcal{D}$ there exists a $\hat{P} \supset P$ and, for any $\epsilon > 0$, there exists a $\delta_\epsilon > 0$, such that if

$$|\Delta(w,p)| \leq \delta_\epsilon \qquad \forall\,(w,p) \in W \times \hat{P}$$

the trajectories of (22) originating from $W \times P$ are bounded and

$$\lim_{t\to\infty} \sup |((w(t),p(t))|_{\mathcal{B}} \leq \epsilon\,.$$

Proof: Let $\bar{p} := \mathrm{col}(w,p)$ and rewrite system (21) in the compact form

$$\dot{\bar{p}} = f(\bar{p}) + \Delta(\bar{p})$$

in which $f(\bar{p}) := \mathrm{col}(s(w),\,F(w,p))$, $\delta(\bar{p}) := \mathrm{col}(0,\,\Delta(w,p))$. Let

$$|\bar{p}|_{\mathcal{B}\setminus\mathcal{D}} := \left(1 + \frac{1}{|p|_{\partial \mathrm{cl}\mathcal{D}}}\right)|\bar{p}|_{\mathcal{B}}\,. \tag{23}$$

By Theorem 4 in [8] there exists a continuous function $V : W \times \mathcal{D} \to I\!R$ with the following properties:

(a) there exist class $\mathcal{K}_\infty$ functions $\underline{a}(\cdot)$, $\overline{a}(\cdot)$ such that

$$\underline{a}(|\bar{p}|_{\mathcal{B}/\mathcal{D}}) \leq V(\bar{p}) \leq \overline{a}(|\bar{p}|_{\mathcal{B}/\mathcal{D}}) \qquad \forall\,\bar{p} \in W \times \mathcal{D}\,;$$

(b) there exists $c > 0$ such that

$$\begin{array}{rcl} D^+V(\bar{p}_1) & = & \displaystyle\lim_{h\to 0^+} \sup \frac{1}{h}[V(\bar{p}(h,\bar{p}_1)) - V(\bar{p}_1)] \\ & \leq & -cV(\bar{p}_1) \qquad \forall\,\bar{p}_1 \in \mathcal{D}\,, \end{array}$$

having denoted by $\bar{p}(h,\bar{p}_1)$ the trajectories of (21) at time h passing through $\bar{p}_1$ at $h = 0$;

(c) for all $\alpha > 0$ there exists $L_\alpha > 0$ such that for all $\bar{p}_1$, $\bar{p}_2 \in W \times \mathcal{D}$ such that $|\bar{p}_1|_{\mathcal{B}/\mathcal{D}} \leq \alpha$, $|\bar{p}_2|_{\mathcal{B}/\mathcal{D}} \leq \alpha$ the following holds

$$|V(\bar{p}_1) - V(\bar{p}_2)| \leq L_\alpha|\bar{p}_1 - \bar{p}_2|\,.$$

Now, given $\epsilon > 0$, let a, b be positive numbers (with $a > b$) such that

$$V^{-1}([0,b]) \subset \mathrm{int}B_\epsilon \quad B_\epsilon := \{\bar{p} \in W \times I\!R^n \;:\; |\bar{p}|_{\mathcal{B}} \leq \epsilon\}$$

and $P \subset \mathrm{int}\,\mathrm{Proj}_w(V^{-1}([0,a]))$, $\mathrm{Proj}_w(\cdot)$ being the projection operator along the w direction. These numbers exist by property (a) above and by definition (23). Furthermore let $\alpha > 0$ be a number such that $\bar{p} \in V^{-1}([0,a]) \Rightarrow |\bar{p}|_{\mathcal{B}\setminus\mathcal{D}} < \alpha$.

By denoting with $\bar{p}(h,\bar{p}_1)$ the solution of (22) with initial condition $\bar{p}_1$ and by bearing in mind properties (b) and (c) above, it turns out that, so long as $\bar{p}_1 \in V^{-1}([0,a])$,

$$\begin{array}{l} D^+V(\bar{p}_1) = \displaystyle\lim_{h\to 0^+}\sup \frac{1}{h}[V(\bar{p}(h,\bar{p}_1)) - V(\bar{p}_1)] \\ = \displaystyle\lim_{h\to 0^+}\sup \frac{1}{h}[V(\bar{p}_1 + hf(\bar{p}_1) + h\delta(\bar{p}_1)) - V(\bar{p}_1)] \\ \leq \displaystyle\lim_{h\to 0^+}\sup \frac{1}{h}[V(\bar{p}_1 + hf(\bar{p}_1) + h\delta(\bar{p}_1)) - V(\bar{p}_1 + \\ hf(\bar{p}_1))] + \displaystyle\lim_{h\to 0^+}\sup \frac{1}{h}[V(\bar{p}_1 + hf(\bar{p}_1)) - V(\bar{p}_1)] \\ \leq \displaystyle\lim_{h\to 0^+}\sup \frac{1}{h}L_\alpha h|\delta(\bar{p}_1)| - cV(\bar{p}_1) \\ = L_\alpha|\delta(\bar{p}_1)| - cV(\bar{p}_1)\,. \end{array} \tag{24}$$

From this, by applying the appropriate comparison lemma and by standard arguments, it turns out that if

$$|\delta(\bar{p}_1)| \leq \delta_\epsilon := \frac{cb}{2L_\alpha} \qquad \forall\,\bar{p}_1 \in V^{-1}([0,a])$$

the set $V^{-1}([0,a])$ is forward invariant for (22) and the set $V^{-1}([0,b])$ is reached in finite time. This, in turn, yields the desired result with $\hat{P} = \mathrm{Proj}_w(V^{-1}([0,a]))$ ■

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