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Nonlinear output feedback design via domination and generalized weighted homogeneity

V. Andrieu, L. Praly and A. Astolfi

Abstract— **We consider the problem of output feedback stabilization for a class of nonlinear systems without zero dynamics. We address this problem by means of a combination of domination and homogeneity methods. The novelty is twofold: a) the resulting controller is weighted homogeneous at infinity but with a linear growth close to the origin; b) a new recursive observer design method is proposed.**

I. INTRODUCTION

The problem of global asymptotic stabilization by output feedback for nonlinear systems has been addressed by several authors following different routes. In this paper we focus on the approach which exploits domination in combination with homogeneity. This approach has been exploited for state feedback design in [3], [8], [1] and for output feedback design in [9] and more recently in [6]. The main idea of this approach is to design a stabilizing feedback for a chain of integrators and to handle other, possibly nonlinear, terms by domination. Domination is possible, from a technical point of view, by invoking weighted homogeneity which is a way to formalize how nonlinear terms with polynomial growth can be considered negligible. However, nominal weighted homogeneity imposes the same growth limit on the whole state space, with the consequence that linearly bounded terms may not be *tolerated* – they are dominant close to the origin. To solve this problem we exploit weighted homogeneity at infinity and linear tools close to the origin. This allows us to obtain the following result.

Theorem 1 (Main result): Consider a nonlinear single-input and single-output system described by $\dot{x} = f(x,u)$ and $y = h(x)$. Suppose the following two assumptions are satisfied.

ASSUMPTION 1 (Structure of the system) : The system can be written in the following form :

$$\left\{\begin{array}{lll} \dot{x}_1 & = x_2 + & \delta_1(x_1) \\ & \vdots & \\ \dot{x}_i & = x_{i+1} + & \delta_i(x_1,\dots,x_i) \\ & \vdots & \\ \dot{x}_n & = u + & \delta_n(x_1,\dots,x_n) \end{array}\right. \qquad y = x_1 \qquad (1)$$

where y is the output in $\mathbb{R}$ and u is the control input in $\mathbb{R}$.

ASSUMPTION 2 (Non linearity bound) : There exists a real number τ in $\left(-\frac{1}{n-1}, 0\right]$ and a positive real number d such that the functions δ_i satisfy the bound :

$$|\delta_i(x_1,\dots,x_i)| \leq d\left(\sum_{j=1}^{i} |x_j| + |x_j|^{\frac{1+(n-i-1)\tau}{1+(n-j)\tau}}\right) . \qquad (2)$$

Then we can construct an output feedback which renders the origin of the closed loop system a globally asymptotically stable equilibrium.

A. Discussion on assumptions

There is nothing new in Assumption 1. Note moreover that we restrict our interest to systems without inverse dynamics. On the other hand, as for any other approach exploiting domination, the unstructured terms δ_i may depend on many other things as long as inequality (2) holds.

The novelty of our result is in Assumption 2 and more specifically in the presence of both $|x_j|$ and $|x_j|^{\frac{1+(n-i-1)\tau}{1+(n-j)\tau}}$. If only the terms $|x_j|$ are considered then the problem has been studied, for example, in [2], whereas if only the terms $|x_j|^{\frac{1+(n-i-1)\tau}{1+(n-j)\tau}}$ are allowed the problem has been dealt with in [6]. To understand the way in which our assumption generalizes existing tools, consider the system :

$$\left\{\begin{array}{lll} \dot{x}_1 & = & x_2 \ , \\ \dot{x}_2 & = & u + \delta_2(x_2) \end{array}\right. \quad , \qquad y = x_1 \ ,$$

with $\delta_2(x_2) = x_2 + |x_2|^{\frac{3}{2}}$. Assumption 2 holds with $\tau = -\frac{1}{2}$, hence the system is globally asymptotically stabilizable by the output feedback result given in this paper, whereas none of the existing results can be applied. In particular, the presence of the term x_2 in δ_2 impedes the use of nominal weighted homogeneity. On the contrary, in our approach

1) when $|x_2| << 1$, we have $|\delta_2(x_2)| \leq 2|x_2|$ and linear tools can be used;
2) when $|x_2| >> 1$, we have $|\delta_2(x_2)| \leq 2|x_2|^{\frac{3}{2}}$, hence we rely on weighted homogeneity tools.

To deal with the linear terms we introduce, implicitly, a generalization of weighted homogeneity which is straightforward and is in line with the ideas given, for example, in [6]. However, this generalization relies on a novel recursive observer design tool, which constitutes a much more significant contribution. This tool extends to the nonlinear case the recursive procedure proposed in [5, Lemmas 1 and 2] (see also [9]) and it is in some sense dual of backstepping. Without the linear part, this observer would be truly homogeneous; this differs from the one obtained in [9]. Also our observer works in open loop (with no feedback)

V. Andrieu is with the Electrical and Electronic Engineering Dept, Imperial College, London, SW7 2AZ, UK (vincent.andrieu@gmail.com)
L. Praly is with CAS – Ecole des Mines de Paris, 35, rue St-Honore, 77305 Fontainebleau CEDEX, France, (laurent.praly@ensmp.fr)
A. Astolfi is with the Electrical and Electronic Engineering Dept, Imperial College, London, SW7 2AZ, UK (a.astolfi@ic.ac.uk) and with Dipartimento di Informatica Sistemi e Produzione, University of Rome Tor Vergata, Via del Politecnico 1, 00133 Roma, Italy
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for a chain of integrators; this differs from the one proposed in [6]).

Notations:

- c is used to denote a generic positive real number. Therefore the following holds $c + c = c$, $c * c = c$.
- For any real number r, we define the function $w \mapsto w^r = \text{sign}(w)|w|^r \; \forall \; w \in \mathbb{R}$. According to this definition, we have for instance when $r \geq 0$:

$$\frac{dw^r}{dw} = r|w|^{r-1} \,,\; w^2 = w|w| \,,\; w_1 > w_2 \Rightarrow w_1^r > w_2^r \;. \quad (3)$$

II. Output feedback design

The output feedback is designed by domination on a chain of integrators, and then it is shown to be adequate for the stabilization of the original system by means of robustness arguments. Such a domination approach can be found, in the linear context, in [2] (see also [7]). The nonlinear context has been studied in [4] and it has been widely exploited, in particular in combination with weighted homogeneity, by Lin, Qian and coworkers (see [9], [6] and references therein). Following this approach, we begin by considering the system on $\mathbb{R}^n$ described by :

$$\dot{x}_1 = x_2 \,,\; \ldots \,,\; \dot{x}_n = u \,. \quad (4)$$

This system can be made homogeneous, using feedback, with any arbitrary degree τ in $\left(-\frac{1}{n-1}, 0\right]$ and weights :

$$r_n = 1 \,, \quad r_i = r_{i+1} + \tau = 1 + \tau\,(n-i) \,.$$

A. Observer design

The observer we propose is described by :

$$\left\{\begin{array}{lcl} \dot{\hat{x}}_1 & = & \hat{x}_2 - k_1 K_1(\hat{x}_1 - y) \\ & \vdots & \\ \dot{\hat{x}}_i & = & \hat{x}_{i+1} - \left(\prod_{j=1}^i k_j\right) K_i(\hat{x}_1 - y) \\ & \vdots & \\ \dot{\hat{x}}_n & = & u - \left(\prod_{j=1}^n k_j\right) K_n(\hat{x}_1 - y) \end{array}\right. \quad (5)$$

where the K_i's are C^1 functions defined recursively as :

$$K_1(e_1) = e_1 + e_1^{\frac{r_2}{r_1}} \,, \quad (6)$$

$$K_i(e_1) = K_{i-1}(e_1) + K_{i-1}(e_1)^{\frac{r_{i+1}}{r_i}} \quad (7)$$

and the $k_i's$ are positive real numbers defined recursively starting from k_n.

1) Selection of k_n: Consider the system $\dot{x}_n = u$ with $y = x_n$. An observer of the form (5) is :

$$\dot{\hat{x}}_n = u - k_n \left(e_n + e_n^{\frac{r_{n+1}}{r_n}}\right) \,, \quad e_n = \hat{x}_n - x_n \,,$$

and this yields the error equation

$$\dot{e}_n = -k_n \left(e_n + e_n^{\frac{r_{n+1}}{r_n}}\right) \,. \quad (8)$$

Picking $k_n = 1$, the function $W_n(e_n) = \frac{1}{2}|e_n|^2 + \frac{r_n}{2}|e_n|^{\frac{2}{r_n}}$ is such that, along the trajectories of (8), one has :

$$\dot{\overline{W_n(e_n)}} \leq -\left(|e_n|^2 + |e_n|^{\frac{2-\tau}{r_n}}\right) \,. \quad (9)$$

Finally, to be consistent with the following, let $s_n(w) = 0$.

2) Selection of k_i: Consider the system with state $\mathcal{E}_{i+1} = (\varepsilon_{i+1}, \ldots, \varepsilon_n)$ in $\mathbb{R}^{n-i}$ described by :

$$\left\{\begin{array}{ll} \dot{\varepsilon}_{i+1} & = k_{i+1}(\varepsilon_{i+2} - P_{i+1,i+1}(\varepsilon_{i+1})) \\ \dot{\varepsilon}_{i+2} & = k_{i+2}(\varepsilon_{i+3} - P_{i+2,i+1}(\varepsilon_{i+1})) \\ \vdots & \\ \dot{\varepsilon}_n & = -k_n\, P_{n,i+1}(\varepsilon_{i+1}) \end{array}\right. \quad (10)$$

where the k_j's for j in $[i+1, n]$ are positive real numbers and the functions $P_{j,i+1}$ are obtained from the recursion :

$$P_{j,i+1}(\varepsilon_{i+1}) = P_{j-1,i+1}(\varepsilon_{i+1}) + P_{j-1,i+1}(\varepsilon_{i+1})^{\frac{r_{j+1}}{r_j}} \quad (11)$$

starting from

$$P_{i+1,i+1}(\varepsilon_{i+1}) = \varepsilon_{i+1} + \varepsilon_{i+1}^{\frac{r_{i+2}}{r_{i+1}}} \,. \quad (12)$$

The change of coordinates, for $j = i+1, \ldots, n$, defined as :

$$e_{i+1} = \varepsilon_{i+1} \,, \quad e_j = \left(\prod_{l=i+1}^{j-1} k_l\right) \varepsilon_j \,, \quad (13)$$

allows us to rewrite the system (10) as :

$$\dot{e}_j = e_{j+1} + \left(\prod_{l=i+1}^{j} k_l\right) P_{j,i+1}(e_{i+1}) \,, \quad (14)$$

which corresponds to the dynamics of the estimation error associated with an observer of the type (5).

Let q_i be the function defined as :

$$q_i(w) = w + w^{\frac{r_{i+1}}{r_i}} \,. \quad (15)$$

It is strictly increasing, onto, and with a non zero derivative (see (3)). So it admits an inverse, denoted s_i and satisfying :

$$s_i(q_i(\varepsilon_i)) = \varepsilon_i \quad , \quad \varepsilon_{i+1} = q_i(s_i(\varepsilon_{i+1})) \,. \quad (16)$$

We observe that the functions $P_{j,i}$ satisfying (11) (and also (7)) satisfy also :

$$P_{i,i}(\varepsilon_i) = q_i(\varepsilon_i) \,,$$
$$P_{j,i}(\varepsilon_i) = P_{j,i+1}(q_i(\varepsilon_i)) \,,\; \forall j \in \{i+1, \ldots, n\} \,. \quad (17)$$

Let now W_{i+1} be the C^1 positive definite function defined as :

$$W_{i+1}(\mathcal{E}_{i+1}) = \sum_{j=i+1}^{n} \frac{1}{2}\left|\varepsilon_j - s_j(\varepsilon_{j+1})\right|^2 + \sum_{j=i+1}^{n} \int_{s_j(\varepsilon_{j+1})}^{\varepsilon_j} \left(h^{\frac{2-r_j}{r_j}} - s_j(\varepsilon_{j+1})^{\frac{2-r_j}{r_j}}\right) dh \,. \quad (18)$$

Inductive Assumption : *There exist $k_n, \ldots, k_{i+1}$ such that the function W_{i+1} satisfies, along the trajectories of (10), the inequality :*

$$\dot{\overline{W_{i+1}(\mathcal{E}_{i+1})}} \leq -c\left(\sum_{j=i+1}^{n} |\eta_j|^2 + |\eta_j|^{\frac{2-\tau}{r_j}}\right)$$

where $\eta_j = \varepsilon_j - s_j(\varepsilon_{j+1})$.

From (9), this inductive assumption holds for $i + 1 = n$. To show that this property holds also for step i, we write the dynamics of the system with state $\mathcal{E}_i = (\varepsilon_i, \dots, \varepsilon_n)$ in $\mathbb{R}^{n-i+1}$:

$$\begin{cases} \dot{\varepsilon}_i &= k_i\,(\varepsilon_{i+1} - P_{i,i}(\varepsilon_i)) \\ \dot{\varepsilon}_{i+1} &= k_{i+1}(\varepsilon_{i+2} - P_{i+1,i}(\varepsilon_i)) \\ \vdots & \\ \dot{\varepsilon}_n &= -k_n\, P_{n,i}(\varepsilon_i) \end{cases} \tag{19}$$

Consider the Lyapunov function :

$$W_i(\mathcal{E}_i) \;=\; W_{i+1}(\mathcal{E}_{i+1}) + \frac{1}{2}\left|\varepsilon_i - s_i(\varepsilon_{i+1})\right|^2 + \int_{s_i(\varepsilon_{i+1})}^{\varepsilon_i} \left(h^{\frac{2-r_i}{r_i}} - s_i(\varepsilon_{i+1})^{\frac{2-r_i}{r_i}}\right) dh\,, \tag{20}$$

Proposition 1 (Inductive Proposition): There exists a positive real number k_i such that, along the trajectories of (19), the Lyapunov function W_i satisfies :

$$\overbrace{W_i(\mathcal{E}_i)}^{\cdot} \;\le\; -c\left(\sum_{j=i}^{n} |\eta_j|^2 + |\eta_j|^{\frac{2-\tau}{r_j}}\right) . \tag{21}$$

Proof : With the inductive assumption, we obtain, along the trajectories of (19),

$$\overbrace{W_i(\mathcal{E}_i)}^{\cdot} \le -c\left(\sum_{j=i+1}^{n} |\eta_j|^2 + |\eta_j|^{\frac{2-\tau}{r_j}}\right) + T_1 + T_2 + T_3 \tag{22}$$

where the T_i's are defined as :

$$\begin{aligned} T_1 &= k_i\left(\eta_i + \varepsilon_i^{\frac{2-r_i}{r_i}} - s_i(\varepsilon_{i+1})^{\frac{2-r_i}{r_i}}\right)(\varepsilon_{i+1} - q_i(\varepsilon_i)) \\ T_2 &= -k_{i+1}\, s_i'(\varepsilon_{i+1})\left(1 + \frac{2-r_i}{r_i}|s_i(\varepsilon_{i+1})|^{\frac{2-2r_i}{r_i}}\right) \\ &\qquad \times \eta_i\,(\varepsilon_{i+2} - P_{i+1,i}(\varepsilon_i))\,, \\ T_3 &= -\sum_{j=i+1}^{n} k_j \frac{\partial W_{i+1}}{\partial \varepsilon_j}(\mathcal{E}_i)\,(P_{j,i}(\varepsilon_i) - P_{j,i+1}(\varepsilon_{i+1}))\,. \end{aligned}$$

Bounding $\mathbf{T_1}$: Using (3), (15) and (16), we obtain :

$$\text{sign}(\eta_i)\;(q_i(\varepsilon_i) - \varepsilon_{i+1}) \;=\; |\eta_i| + \left|\varepsilon_i^{\frac{r_{i+1}}{r_i}} - s_i(\varepsilon_{i+1})^{\frac{r_{i+1}}{r_i}}\right| .$$

Then point 2) in Lemma 1 in the Appendix yields two positive real numbers c such that :

$$\text{sign}(\eta_i)\left(\eta_i + \varepsilon_i^{\frac{2-r_i}{r_i}} - s_i(\varepsilon_{i+1})^{\frac{2-r_i}{r_i}}\right) \ge c\left(|\eta_i| + |\eta_i|^{\frac{2-r_i}{r_i}}\right) ,$$

$$\text{sign}(\eta_i)\;(q_i(\varepsilon_i) - \varepsilon_{i+1}) \;\ge\; c\left(|\eta_i| + |\eta_i|^{\frac{r_{i+1}}{r_i}}\right) .$$

Then point 3) in Lemma 1 yields another positive real number c satisfying :

$$T_1 \;\le\; -k_i\, c\left(|\eta_i|^2 + |\eta_i|^{\frac{2-\tau}{r_i}}\right) .$$

Bounding $\mathbf{T_2}$: Using the bounds on $P_{j,i}$ given in Lemma 2 (with $w_2 = 0$) we get a positive real number c such that :

$$|P_{i+1,i}(\varepsilon_i) - \varepsilon_{i+2}| \;\le\; c\left(|\varepsilon_i| + |\varepsilon_i|^{\frac{r_{i+2}}{r_i}}\right) + |\varepsilon_{i+2}| .$$

On the other hand, by differentiating (16) and using (15), we obtain :

$$s_i'(\varepsilon_{i+1})\left(1 + \frac{r_{i+1}}{r_i}|s_i(\varepsilon_{i+1})|^{\frac{r_{i+1}-r_i}{r_i}}\right) \;=\; 1\,. \tag{24}$$

Using the bounds on s_i given in Lemma 3, we obtain a positive real number c such that :

$$s_i'(\varepsilon_{i+1})\left(1 + \frac{2-r_i}{r_i}|s_i(\varepsilon_{i+1})|^{\frac{2-2r_i}{r_i}}\right) \le c\left(1 + |\varepsilon_{i+1}|^{\frac{2-r_i-r_{i+1}}{r_{i+1}}}\right) . \tag{25}$$

Then, with the bounds (25), Young's inequality, the bounds on ε_i given in Lemma 4 and point 3) in Lemma 1, we can prove the existence of a continuous function $b_2 : \mathbb{R}_+ \to \mathbb{R}_+$ such that, for each strictly positive real number ρ_2, we have :

$$T_2 \;\le\; \rho_2\left(\sum_{l=i+1}^{n} |\eta_l|^2 + |\eta_l|^{\frac{2-\tau}{r_l}}\right) + b_2(\rho_2)\left(|\eta_i|^2 + |\eta_i|^{\frac{2-\tau}{r_i}}\right) . \tag{26}$$

Bounding $\mathbf{T_3}$: From (16), (17) Lemma 2, we get $\forall j \ge i+1$:

$$\begin{aligned} |P_{j,i}(\varepsilon_i) - P_{j,i+1}(\varepsilon_{i+1})| &= |P_{j,i}(\varepsilon_i) - P_{j,i}(s_i(\varepsilon_{i+1}))| \\ &\le c\,|\eta_i|\left(1 + |\eta_i|^{\frac{r_{j+1}-r_i}{r_i}} + |\varepsilon_i|^{\frac{r_{j+1}-r_i}{r_i}}\right) . \end{aligned} \tag{27}$$

Note now that the definition of W_{i+1} and (24) give :

$$\begin{aligned} \left|\frac{\partial W_{i+1}}{\partial \varepsilon_j}(\mathcal{E}_{i+1})\right| &\le |\eta_j| + |\varepsilon_j|^{\frac{2-r_j}{r_j}} + |s_j(\varepsilon_{j+1})|^{\frac{2-r_j}{r_j}} \\ &\quad + c\left(1 + |s_{j-1}(\varepsilon_j)|^{\frac{2-r_{j-1}-r_j}{r_{j-1}}}\right)|\eta_{j-1}|\,. \end{aligned}$$

Then, the bound on s_i given in Lemma 3, Young's inequality, the bound on ε_i and point 3) in Lemma 1, imply that there exists a positive real number c such that, for each j in $[i+1, n]$, we have :

$$\left|\frac{\partial W_{i+1}}{\partial \varepsilon_j}(\mathcal{E}_{i+1})\right| \;\le\; c\left(\sum_{l=j}^{n} |\eta_l| + |\eta_l|^{\frac{2-r_j}{r_l}}\right) . \tag{28}$$

Finally, as for the term T_2, by (27), (28) and Young's inequality, we can prove the existence of a continuous function $b_3 : \mathbb{R}_+ \to \mathbb{R}_+$ such that, for any strictly positive real number ρ_3, we have :

$$T_3 \;\le\; \rho_3\left(\sum_{l=i+1}^{n} |\eta_l|^2 + |\eta_l|^{\frac{2-\tau}{r_l}}\right) + b_3(\rho_3)\left(|\eta_i|^2 + |\eta_i|^{\frac{2-\tau}{r_i}}\right) . \tag{29}$$

Conclusion : With the bounds obtained for T_1, T_2 and T_3 and with the inductive assumption (21), the inequality (22) becomes :

$$\overbrace{W_i(\mathcal{E}_i)}^{\cdot} \le (\rho_2 + \rho_3 - c)\left(\sum_{j=i+1}^{n} |\eta_j|^2 + |\eta_j|^{\frac{2-\tau}{r_j}}\right) + (b_2(\rho_2) + b_3(\rho_3) - k_i c)\left(|\eta_i|^2 + |\eta_i|^{\frac{2-\tau}{r_i}}\right) . \tag{30}$$

Hence, by picking ρ_2 and ρ_3 sufficiently small and k_i sufficiently large, the claim is established. □

We can iterate the procedure to obtain $k_1, \ldots, k_n$ and the Lyapunov function :

$$W_1(\mathcal{E}_1) \;=\; \sum_{j=1}^{n} \frac{1}{2}\left|\varepsilon_j - s_j(\varepsilon_{j+1})\right|^2 + \int_{s_j(\varepsilon_{j+1})}^{\varepsilon_j} \left(h^{\frac{2-r_j}{r_j}} \;-\; s_j(\varepsilon_{j+1})^{\frac{2-r_j}{r_j}}\right) dh \tag{31}$$

which satisfies :

$$\overbrace{W_1(\mathcal{E}_1)}^{\cdot} \;\le\; -c\left(\sum_{j=1}^{n} |\eta_j|^2 \;+\; |\eta_j|^{\frac{2-\tau}{r_j}}\right) . \tag{32}$$

where $\mathcal{E}_1 \;=\; (\varepsilon_1, \ldots, \varepsilon_n)$ and $\eta_j \;=\; \varepsilon_j - s_j(\varepsilon_{j+1})$. (33)

B. State Feedback design

In the same spirit as for the observer design, the state feedback is initially derived recursively for a chain of integrators. We apply a classical homogeneous backstepping design (see for instance [3], [8], [1]) but with an additional linear part. The controller obtained from the proposed procedure is $u = \phi_n(x_1, \ldots, x_n)$, with ϕ_n defined by the recursion :

$$\phi_i(\mathcal{X}_i) = -\ell_i\Big(x_i - \phi_{i-1}(\mathcal{X}_{i-1}) + (x_i - \phi_{i-1}(\mathcal{X}_{i-1}))^{\frac{r_{i+1}}{r_i}}\Big) \tag{34}$$

starting from[1] $$\phi_1(x_1) \;=\; -\ell_1\left(x_1 \;+\; x_1^{\frac{r_2}{r_1}}\right) . \tag{35}$$

To choose the ℓ_i's we apply a recursive design starting from ℓ_1.

1) Initial Step: Consider the Lyapunov function $V_1(x_1) = \frac{r_1}{2}|x_1|^{\frac{2}{r_1}} + \frac{1}{2}|x_1|^2$. With (35) and when $\ell_1 = 1$, we get, along the solutions of the system $\dot{x}_1 \;=\; \phi_1(x_1)$,

$$\overbrace{V_1(x_1)}^{\cdot} \;\le\; -|x_1|^2 \;-\; |x_1|^{\frac{2-\tau}{r_1}} . \tag{36}$$

2) Induction step: Consider a chain of integrators of order i with state $\mathcal{X}_i \;=\; (x_1, \ldots, x_i)$:

$$\dot{x}_1 \;=\; x_2 \;, \quad \ldots \;, \quad \dot{x}_i \;=\; u \; . \tag{37}$$

Consider the control law $u = \phi_i(\mathcal{X}_i)$, defined recursively by (34) and (35), and the Lyapunov function :

$$V_i(\mathcal{X}_i) \;\;=\;\; \sum_{j=1}^{i} \frac{1}{2}\,|z_j|^2 \;+\; \frac{r_j}{2}\,|z_j|^{\frac{2}{r_j}} \;,$$

where $z_j \;=\; x_j - \phi_{j-1}(\mathcal{X}_{j-1}) \quad , \qquad j = 1 \;\ldots, i$. (38)

Inductive Assumption : *The ℓ_j's are such that the function V_i satisfies, along the trajectories of (37) :*

$$\overbrace{V_i(\mathcal{X}_i)}^{\cdot} \;\le\; -c\left(\sum_{j=1}^{i} |z_j|^2 \;+\; |z_j|^{\frac{2-\tau}{r_j}}\right) .$$

[1]Compare with (11)-(12).

From (36), this inductive assumption holds for $i \;=\; 1$. Consider now a chain of integrators of order $i+1$ with state $\mathcal{X}_{i+1} \;=\; (x_1, \ldots, x_{i+1})$:

$$\dot{x}_1 \;=\; x_2 \;, \quad \ldots \;, \quad \dot{x}_{i+1} \;=\; u \tag{39}$$

and the Lyapunov function V_{i+1} defined as :

$$V_{i+1}(\mathcal{X}_{i+1}) \;=\; V_i(\mathcal{X}_i) + \frac{1}{2}\,|z_{i+1}|^2 + \frac{r_{i+1}}{2}|z_{i+1}|^{\frac{2}{r_{i+1}}} \; .$$

Proposition 2 (Inductive Proposition): There exists ℓ_{i+1} such that, by taking $u = \phi_{i+1}(\mathcal{X}_{i+1})$, with :

$$\phi_{i+1}(\mathcal{X}_{i+1}) \;=\; -\ell_{i+1}\left(z_{i+1} \;+\; z_{i+1}^{\frac{r_{i+2}}{r_{i+1}}}\right) \;, \tag{40}$$

we have that V_{i+1} satisfies, along the trajectories of (39),

$$\overbrace{V_{i+1}(\mathcal{X}_{i+1})}^{\cdot} \;\le\; -c\left(\sum_{j=1}^{i+1} |z_j|^2 \;+\; |z_j|^{\frac{2-\tau}{r_i}}\right) . \tag{41}$$

Proof : With the inductive assumption, we obtain, along the trajectories of (39),

$$\overbrace{V_{i+1}(\mathcal{X}_{i+1})}^{\cdot} \le -c\left(\sum_{j=1}^{i} |z_j|^2 \;+\; |z_j|^{\frac{2-\tau}{r_j}}\right) + T_4 + T_5 + T_6 \tag{42}$$

where the T_i's are defined as :

$$\begin{aligned} T_4 &= \frac{\partial V_i}{\partial x_i}(\mathcal{X}_i)\, z_{i+1} \;, \\ T_5 &= -\ell_{i+1}\left(z_{i+1} \;+\; z_{i+1}^{\frac{2-r_{i+1}}{r_{i+1}}}\right)\left(z_{i+1} \;+\; z_{i+1}^{\frac{r_{i+2}}{r_{i+1}}}\right) \;, \\ T_6 &= -\left(z_{i+1} \;+\; z_{i+1}^{\frac{2-r_{i+1}}{r_{i+1}}}\right)\sum_{j=1}^{i} \frac{\partial \phi_i}{\partial x_j}(\mathcal{X}_i)\, x_{j+1} \; . \end{aligned}$$

Bounding $\mathbf{T_4}$: We have :

$$\frac{\partial V_i}{\partial x_i}(\mathcal{X}_i) \;=\; z_i \;+\; z_i^{\frac{2-r_i}{r_i}} \; .$$

Hence, with Young's inequality, we can prove the existence of a continuous function $b_4 \;: \mathbb{R}_+ \to \mathbb{R}_+$ such that, for each strictly positive real number ρ_4, we have :

$$\frac{\partial V_i}{\partial x_i}(\mathcal{X}_i)\, z_{i+1} \;\le\; \rho_4\left(|z_i|^2 + |z_i|^{\frac{2-\tau}{r_i}}\right) + b_4(\rho_4)\left(|z_{i+1}|^2 + |z_{i+1}|^{\frac{2-\tau}{r_{i+1}}}\right) .$$

Bounding $\mathbf{T_5}$: As ℓ_{i+1} is a positive real number, we have readily :

$$T_5 \;\le\; -\ell_{i+1}\left(|z_{i+1}|^2 \;+\; |z_{i+1}|^{\frac{2-\tau}{r_{i+1}}}\right) .$$

Bounding $\mathbf{T_6}$: The definitions (38) and (40) of z_{j+1} and ϕ_{j+1} give, for $j = 1 \ldots, i$:

$$|x_{j+1}| \quad\le\quad |z_{j+1}| \;+\; \ell_j|z_j| \;+\; \ell_j|z_j|^{\frac{r_{j+1}}{r_j}} \; .$$

Therefore, by Lemma 5, and Young's inequality, we can prove the existence of a continuous function $b_6 \;: \mathbb{R}_+ \to \mathbb{R}_+$

such that, for each strictly positive real number ρ_6, we have :

$$T_6 \;\leq\; \rho_6 \left(\sum_{j=1}^{i} |z_j|^2 + |z_j|^{\frac{2-\tau}{r_j}}\right) + b_6(\rho_6)\left(|z_{i+1}|^2 + |z_{i+1}|^{\frac{2-\tau}{r_{i+1}}}\right) \;.$$

Conclusion : With the bounds obtained for T_4, T_5 and T_6, the inequality on the derivative of the Lyapunov function in (42) becomes :

$$\overbrace{V_{i+1}(\mathcal{X}_{i+1})}^{\cdot} \leq (\rho_4 + \rho_6 - c)\sum_{j=1}^{i} |z_j|^2 + |z_j|^{\frac{2-\tau}{r_j}} + \;(b_4(\rho_4) + b_6(\rho_6) - \ell_{i+1})\left(|z_{i+1}|^2 + |z_{i+1}|^{\frac{2-\tau}{r_{i+1}}}\right) \;.$$

Hence, by picking ρ_4 and ρ_6 sufficiently small and ℓ_{i+1} sufficiently large, the claim is established. □

We can iterate the procedure to obtain $\ell_1, \ldots, \ell_n$ and the Lyapunov function :

$$V_n(\mathcal{X}_n) \;=\; \sum_{j=1}^{n} \frac{1}{2}|z_j|^2 + \frac{r_j}{2}|z_j|^{\frac{2}{r_j}} \;, \qquad (43)$$

where $\mathcal{X}_n = (x_1, \ldots, x_n)$ and which satisfies :

$$\overbrace{V_n(\mathcal{X}_n)}^{\cdot} \;\leq\; -c\left(\sum_{j=1}^{n} |z_j|^2 \;+\; |z_j|^{\frac{2-\tau}{r_j}}\right) \qquad (44)$$

where $z_j = x_j - \phi_{j-1}(\mathcal{X}_{j-1})$. (45)

III. Proof of the main result

Following the procedure introduced in [6] (see also [7]) the output feedback is obtained from the observer and the state feedback previously defined by introducing an additional high gain parameter L. More specifically the output feedback is given by :

$$\left\{\begin{array}{rcl} \dot{\hat{x}}_1 &=& L\,\hat{x}_2 - L\,k_1\,K_1(\hat{x}_1 - y) \\ &\vdots& \\ \dot{\hat{x}}_i &=& L\,\hat{x}_{i+1} - L\left(\prod_{j=1}^{i} k_j\right)\,K_i(\hat{x}_1 - y)) \\ &\vdots& \\ \dot{\hat{x}}_n &=& \frac{u}{L^{n-1}} - L\,\left(\prod_{j=1}^{n} k_j\right)\,K_n(\hat{x}_1 - y) \\ u &=& L^n\,\phi_n(\hat{x}_1, \ldots, \hat{x}_n). \end{array}\right.$$

Here the ℓ_i's and k_i's are given. It remains to choose L. In order to do so, we write the dynamics of the closed loop system with the coordinates $\hat{\mathcal{X}}_n = (\hat{x}_i)$ and $\mathcal{E}_1 = (\varepsilon_i)$ where :

$$\varepsilon_1 \;=\; \hat{x}_1 - x_1 \;, \quad \varepsilon_i \;=\; \frac{1}{\left(\prod_{l=1}^{i-1} k_l\right)}\left[\hat{x}_i - \frac{x_i}{L^{i-1}}\right] \;. \qquad (46)$$

As a result, we obtain :

$$\left\{\begin{array}{rcl} \dot{\varepsilon}_i &=& Lk_i\,(\varepsilon_{i+1} - \;K_i(e_1)) - \dfrac{\delta_i}{L^{i-1}\prod_{j=1}^{i-1} k_j} \\ \dot{\hat{x}}_i &=& L\,\hat{x}_{i+1} - L\left(\prod_{j=1}^{i} k_j\right)\,K_i(\varepsilon_1)) \\ \dot{\hat{x}}_n &=& L\,\phi_n(\hat{\mathcal{X}}_n) - L\,\left(\prod_{j=1}^{n} k_j\right)\,K_n(\varepsilon_1) \end{array}\right. \qquad (47)$$

To study the stability of this system, we consider the function

$$U(\hat{\mathcal{X}}_n, \mathcal{E}_1) \;=\; V_n(\hat{\mathcal{X}}_n) \;+\; \alpha\,W_1(\mathcal{E}_1)\;, \qquad (48)$$

where α is a positive real number and V_n and W_1 are given in (43) and (31), respectively. Using (44) and (32), we have, along the trajectories of (47),

$$\overbrace{U(\hat{\mathcal{X}}_n, \mathcal{E}_1)}^{\cdot} \leq -L\,c\left(\textstyle\sum_{j=1}^{n} |\hat{z}_j| + |\hat{z}_j|^{\frac{2-\tau}{r_j}}\right) + L\,T_7 - \alpha\,L\,c\left(\textstyle\sum_{j=1}^{n} |\eta_j| \;+\; |\eta_j|^{\frac{2-\tau}{r_j}}\right) + L\,T_8 \qquad (49)$$

where the T_i's are defined as :

$$T_7 \;=\; -\textstyle\sum_{j=1}^{n} \frac{\partial V_n}{\partial \hat{x}_j}(\hat{\mathcal{X}}_n)\left(\prod_{l=1}^{j} k_l\right)\;K_j(e_1)\;,$$

$$T_8 \;=\; -\alpha\,\textstyle\sum_{i=1}^{n} \frac{\partial W_1}{\partial \varepsilon_i}(\mathcal{E}_1)\dfrac{\delta_i}{L^i\prod_{j=1}^{i-1} k_j} \;.$$

Bounding $\mathbf{T_7}$: From (43) and (45) and the bound on $\frac{\partial \phi_\ell}{\partial \hat{x}_j}$ given in Lemma 5 we obtain a positive real number c such that :

$$\left|\frac{\partial V_n}{\partial \hat{x}_j}(\hat{\mathcal{X}}_n)\right| \;\leq\; |\hat{z}_j| + |\hat{z}_j|^{\frac{2-r_j}{r_j}} + \;c\sum_{l=j+1}^{n}\left(1 + \sum_{m=1}^{l-1}|\hat{z}_m|^{\frac{r_l - r_j}{r_m}}\right)\left(|\hat{z}_l| + |\hat{z}_l|^{\frac{2-r_l}{r_l}}\right) \;.$$

Then, by Young's inequality, we can prove the existence of a positive real number c satisfying :

$$\left|\frac{\partial V_n}{\partial \hat{x}_j}(\hat{\mathcal{X}}_n)\right| \;\leq\; c\left(\sum_{l=1}^{n} |\hat{z}_l| + |\hat{z}_l|^{\frac{2-r_j}{r_l}}\right) \;. \qquad (50)$$

On the other hand, by definition, we have $K_j = P_{j,1}$. It follows from Lemma 2 (with $w_2 = 0$) that there exists a positive real number c such that :

$$|K_j(e_1)| \;\leq\; c\left(|e_1| + |e_1|^{\frac{r_{j+1}}{r_1}}\right) \;.$$

Once again, using Young's inequality, Lemma 4, and point 1) in Lemma 1, we can prove the existence of a continuous function $b_7\;:\mathbb{R}_+ \to \mathbb{R}_+$ such that, for each strictly positive real number ρ_7, we have :

$$T_7 \leq \rho_7\left(\sum_{l=1}^{n} |\hat{z}_l|^2 + |\hat{z}_l|^{\frac{2-\tau}{r_l}}\right) + \;b_7(\rho_7)\left(\sum_{l=1}^{n} |\eta_j|^2 + |\eta_j|^{\frac{2-\tau}{r_j}}\right) \;. \qquad (51)$$

Bounding $\mathbf{T_8}$: To begin with we recall (2) in Assumption 2, namely :

$$|\delta_i| \leq d\left(\sum_{j=1}^{i} |x_j| + |x_j|^{\frac{r_{i+1}}{r_j}}\right) \;.$$

From (45) and (46), we get :

$$x_j \;=\; L^{j-1}\left[\hat{z}_j + \phi_{j-1}(\hat{\mathcal{X}}_{j-1}) \;-\; \left(\prod_{l=1}^{j-1} k_l\right)\varepsilon_j\right] \;.$$

We remark that, for $1 \leq j \leq i$ we have $(j-1)\frac{r_{i+1}}{r_j} - i \leq -r_1$. Since $r_1 > 0$ and we can impose $L \geq 1$, by point 1) of Lemma 1, we get a positive real number c such that :

$$\frac{|x_j| + |x_j|^{\frac{r_{i+1}}{r_j}}}{L^i} \leq c\, L^{-r_1}\Big(|\hat{z}_j| + |\phi_{j-1}(\hat{\mathcal{X}}_{j-1})| \;+\; |\hat{z}_j|^{\frac{r_{i+1}}{r_j}} + |\phi_{j-1}(\hat{\mathcal{X}}_{j-1})|^{\frac{r_{i+1}}{r_j}} \;+\; |\varepsilon_j| + |\varepsilon_j|^{\frac{r_{i+1}}{r_j}}\Big) \;.$$

By (40) and points 1) and 3) of Lemma 1, we obtain a positive real number c such that,

$$\frac{|\delta_i|}{L^i \prod_{j=1}^{i-1} k_j} \;\leq\; c\, L^{-r_1} \sum_{j=1}^{i} \left(|\hat{z}_j| + |\hat{z}_j|^{\frac{r_{i+1}}{r_j}} + |\varepsilon_j| + |\varepsilon_j|^{\frac{r_{i+1}}{r_j}}\right) \;.$$

Hence by (28), Young's inequality, Lemma 4 and point 1) of Lemma 1, we infer the existence of a positive real number c such that :

$$T_8 \;\leq\; \alpha\, c\, L^{-r_1}\left(\sum_{j=1}^{n} |\hat{z}_j|^2 \;+\; |\eta_j|^2 \;+\; |\eta_j|^{\frac{2-\tau}{r_j}} \;+\; |\hat{z}_j|^{\frac{2-\tau}{r_j}}\right) \;.$$

Conclusion : With the obtained bounds on T_7 and T_8, we have :

$$\overbrace{U(\hat{\mathcal{X}}_n, \mathcal{E}_1)}^{\cdot} \leq -\left(\frac{\alpha}{\beta}\right)^{\frac{1}{r_1}} (\alpha\, c - b_7(\rho_7) - \beta\, c) \left(\sum_{j=1}^{n} |\eta_j| + |\eta_j|^{\frac{2-\tau}{r_j}}\right) - \left(\frac{\alpha}{\beta}\right)^{\frac{1}{r_1}} (c - \rho_7 - \beta\, c) \left(\sum_{j=1}^{n} |\hat{z}_j| + |\hat{z}_j|^{\frac{2-\tau}{r_j}}\right)$$

where $\beta = \alpha\, L^{-r_1}$. Finally, picking ρ_7 and β sufficiently small, and α sufficiently large, the left hand side of the above equation is strictly negative for all $(\eta_j, \hat{z}_j) \neq 0$. This shows global asymptotic stability of the closed loop system, which completes the proof of Theorem 1.

IV. Conclusion

We have extended the class of nonlinear systems which can be stabilized by output feedback by allowing terms with growth both of low and high polynomial degrees. This has been made possible by following a domination approach where a linear part is dominant close to the origin and a weighted homogeneous part is dominant close to infinity. This domination approach allows to design the feedback for a simple chain of integrators and to accommodate with the other terms involved in the dynamics by adjusting a high gain parameter, a technique which has been introduced in [6].

Our result has been obtained by means of a novel recursive observer design methodology which is somewhat dual to backstepping. We believe that this design is interesting *per se*, and can be used in other frameworks.

Appendix

In this appendix we present some inequalities without proof, due to space limitation.

Lemma 1: Let $a_i \geq 0$ and $p \geq 1$ be real numbers.

1) We have :

$$\left(\sum_{i=1}^{m} a_i\right)^p \;\leq\; m^{p-1}\sum_{i=1}^{m} a_i^p \quad , \quad \left(\sum_{i=1}^{m} a_i\right)^{\frac{1}{p}} \;\leq\; \sum_{i=1}^{m} a_i^{\frac{1}{p}} \;.$$

2) There exists a positive number c such that for all real numbers a and b, we have :

$$\Big| a^p - b^p \Big| \;\geq\; c\,|a-b|^p \;.$$

3) Let $0 \leq d_1 \leq \ldots \leq d_m$ be real numbers. We have :

$$a_1|w|^{d_1} + a_m|w|^{d_m} \;\leq\; \textstyle\sum_{j=1}^{m} a_j\,|w|^{d_j} \;\leq\; \leq\; \left(\sum_{j=1}^{m} a_j\right)\left(|w|^{d_1} + |w|^{d_m}\right) \quad \forall w \in \mathbb{R} \;.$$

Lemma 2 (Bound on $\Delta P_{j,i}$): There exists a positive real number c such that, for each real numbers w_1 and w_2, $P_{j,i}$ satisfies :

$$|P_{j,i}(w_1) - P_{j,i}(w_2)| \;\leq\; c\,|w_1 - w_2|\Big(1 + |w_1 - w_2|^{\frac{r_{j+1}-r_i}{r_i}} + |w_2|^{\frac{r_{j+1}-r_i}{r_i}}\Big) \;.$$

This gives a bound on $P_{j,i}(w_1)$ by noting that $P_{j,i}(0) = 0$.

Lemma 3 (Bound on s_j): The function s_i satisfies :

$$|s_i(w)| \;\leq\; \min\{|w|, |w|^{\frac{r_i}{r_{i+1}}}\} \qquad \forall w \in \mathbb{R} \;. \qquad (52)$$

Lemma 4 (Bound on ε_i): We have :

$$|\varepsilon_i| \;\leq\; \min\left\{\sum_{l=n}^{i} |\eta_l|,\; \sum_{l=n}^{i} |\eta_l|^{\frac{r_i}{r_l}}\right\} \;.$$

Lemma 5 (Bound on $\frac{\partial \phi_i}{\partial x_j}$): There exists a positive real number c such that, for all i and j,

$$\left|\frac{\partial \phi_i}{\partial x_j}(\mathcal{X}_i)\right| \;\leq\; c\,\left(1 + \sum_{l=1}^{i} |z_l|^{\frac{r_{i+1}-r_j}{r_l}}\right) \;.$$

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