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Remarks on the existence of a Kazantzis-Kravaris/Luenberger observer

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Abstract **We state sufficient conditions for the existence, on a given open set, of the extension, to non linear systems, of the Luenberger observer as it has been proposed by Kazantzis and Kravaris. To weaken these conditions, the observer is modified in a way which induces a time rescaling and which follows from a forward unboundedness observability property. Also, we state it is sufficient to choose the dimension of the dynamic system, giving the observer, less than or equal to 2 + twice the dimension of the state to be observed. Finally we show how approximation is allowed and we establish a link with high gain observers.**

I. INTRODUCTION

In a seminal paper [11], Kazantzis and Kravaris have proposed to extend to the nonlinear case the primary observer introduced by Luenberger in [17] for linear systems. Following this suggestion, for the system :

$$\dot{x} = f(x) \quad , \qquad y = h(x) \tag{1}$$

with state x in $\mathbb{R}^n$ and output y in $\mathbb{R}^p$, the estimate $\widehat{x}$ of x is obtained as the output of the dynamical system :

$$\left\{ \begin{array}{rcl} \dot{z} & = & A\,z + B(y) \;, \\ \widehat{x} & = & T^*(z) \;, \end{array} \right. \tag{2}$$

with state z in $\mathbb{C}^m$ and where A is a complex Hurwitz matrix and B and T^* are sufficiently smooth functions. In this context the main difficulty is in the choice of the function T^* which appears to depend very strongly on the other observer data A, B and m.

In the following we state sufficient conditions on f and h such that we can find A, B and m for which there exists T^* guaranteeing the convergence of $\widehat{x}$ to x.

This communication is an extended abstract of [3] where the reader can find all the technical details and the complete proofs of the results only claimed here.

II. SUFfICIENT CONDITION FOR THE EXISTENCE OF A KAZANTZIS- KRAVARIS/LUENBERGER OBSERVER

In [11], m, the dimension of z, is chosen equal to n and T^* is the inverse T^{-1} of a function T, solution of the following partial differential equation :

$$\frac{\partial T}{\partial x}(x)\, f(x) \;=\; A\,T(x) \;+\; B(h(x)) \;. \tag{3}$$

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The rationale for this equation, as more emphasized in [13] and [14] (see also [16] and [19]), is that, if T is a diffeomorphism satisfying (3), then the change of coordinates :

$$\zeta \;=\; T(x) \tag{4}$$

allows us to rewrite the dynamics (1) as :

$$\dot{\zeta} \;=\; A\,\zeta \;+\; B(h(T^{-1}(\zeta))) \;, \quad y \;=\; h(T^{-1}(\zeta)) \;. \tag{5}$$

We then have :

$$\dot{\overbrace{z-\zeta}} \;=\; A\,(z-\zeta) \;. \tag{6}$$

A being Hurwitz, z in (2) is an asymptotically convergent observer of $\zeta = T(x)$. Then, if the function $T^* = T^{-1}$ is uniformly continuous, $\widehat{x} = T^*(z)$ is an asymptotically convergent observer of :

$$x = T^*(\zeta) \;=\; T^*(T(x)) \;. \tag{7}$$

This equality says that the function T must be left invertible and therefore injective. This explains why, in general, m should be chosen larger or equal to n.

This way of finding the function T^* has motivated active research on the problem of existence of an analytic and invertible solution to (3) (see [11], [13], [14] for instance). These works establish a link between analyticity and non resonance conditions, and between invertibility and observability.

But, it turns out that having a (weak) solution to (3) which is only uniformly injective is already sufficient. This is made precise as follows :

We assume the functions f and h in (1) are C^1. So, for each x in $\mathbb{R}^n$ there exists a unique solution $X(x,t)$ to (1) with initial condition x. Given an open set $\mathcal{O}$ of $\mathbb{R}^n$, for each x in $\mathcal{O}$, we denote by $(\sigma^-_{\mathcal{O}}(x), \sigma^+_{\mathcal{O}}(x))$ the maximal interval of definition of the solution $X(x,t)$ conditioned to take values in $\mathcal{O}$. Also for a set S, we denote by $\mathtt{cl}(S)$ its closure and by $S+\delta$ the open set :

$$\begin{aligned} S+\delta &= \{x \in \mathbb{R}^n : \exists \chi \in S : |x-\chi| < \delta\} \;, && (8)\\ &= \bigcup_{\chi\in S} \mathcal{B}(\chi,\delta) \;. && (9) \end{aligned}$$

We have :

Theorem 1: (Sufficient condition of existence of an observer) :

Assume :

1) The system[1] (1) is forward unboundedness observable conditioned to $\mathcal{O}$, i.e., there exists a proper and C^1 function $V_f : \mathbb{R}^n \to \mathbb{R}_+$ and a continuous function $\gamma_f : \mathbb{R}^p \to \mathbb{R}_+$ such that (see [2])[2] :

$$L_f V_f(x) \ \le \ V_f(x) \ + \ \gamma_f(h(x)) \qquad \forall x \in \mathcal{O} \ . \quad (10)$$

2) There exist an integer m, a complex Hurwitz matrix A in $\mathbb{C}^{m\times m}$ and functions $T : \mathtt{cl}(\mathcal{O}) \to \mathbb{C}^{m\times p}$, continuous, $B : \mathbb{R}^p \to \mathbb{C}^{m\times p}$, continuous, $\gamma : \mathbb{R}^p \to \mathbb{R}_+$, C^1, ρ, of class $\mathcal{K}_\infty$, satisfying :

$$L_f T(x) = \gamma(h(x))(AT(x) + B(h(x))) \quad \forall x \in \mathcal{O} \ , \ (11)$$

$$\gamma(h(x)) \ge 1 + \gamma_f(h(x)) \quad \forall x \in \mathtt{cl}(\mathcal{O}) \ , \qquad (12)$$

$$|x_1 - x_2| \le \rho(|T(x_1) - T(x_2)|) \quad \forall x_1, \, x_2 \in \mathtt{cl}(\mathcal{O}) \ (13)$$

Under these conditions, there exists a function $T^* : \mathbb{C}^{m\times p} \to \mathtt{cl}(\mathcal{O})$ such that, for each x in $\mathcal{O}$ and z in $\mathbb{C}^{m\times p}$ the (unique) solution $(X(x,t), Z(x,z,t))$ of :

$$\left\{ \begin{array}{rcl} \dot{x} &=& f(x) \ , \\ \dot{z} &=& \gamma(h(x)) \left(Az + B(h(x)) \right) \end{array} \right. \qquad (14)$$

is right maximally defined on $[0, \sigma^+_{\mathbb{R}^n}(x))$. Moreover, if we have[3] :

$$\sigma^+_{\mathcal{O}}(x) \ = \ \sigma^+_{\mathbb{R}^n}(x) \ , \qquad (15)$$

then we get :

$$\lim_{t \to \sigma^+_{\mathbb{R}^n}(x)} |T^*(Z(x,z,t)) - X(x,t)| \ = \ 0 \ . \qquad (16)$$

Remarks :

1) To be specific, the observer is :

$$\left\{ \begin{array}{rcl} \dot{z} &=& \gamma(h(x)) \left(Az + B(h(x)) \right) \ , \\ \widehat{x} &=& T^*(z) \ . \end{array} \right. \qquad (17)$$

The presence of γ is a key modification compared with the original Kazantzis-Kravaris/Luenberger observer (2). As written in (16), it allows us to get convergence to zero of the observation error within the domain of definition of the solution $X(x,t)$, even if it escapes to infinity, inside $\mathcal{O}$, in finite time. This modification of the time scale, as induced by γ, is one of the important contributions of [4]. For this modification to be possible, we need a forward unboundedness observability property. As already remarked in [4], this property is necessary for the existence of an observer satisfying (16).

2) Under the stringent assumption of existence of a function $C : \mathbb{R}^p \to \mathbb{C}^{m\times p\times q}$ such that we have the factorization :

$$L_g T(x) \ = \ \gamma(h(x)) C(h(x)) \quad \forall x \in \mathcal{O} \ , \qquad (18)$$

Theorem 1 extends readily to the case where the system is :

$$\dot{x} \ = \ f(x) + g(x)u \quad , \qquad y \ = \ h(x) \, , \qquad (19)$$

where $u : \mathbb{R} \to \mathcal{U} \subset \mathbb{R}^q$ is in L^∞_{loc}. Moreover, in this case the unboundedness observability property has to be modified in :

$$L_{f+gu} V_f(x) \ \le \ V_f(x) \ + \ \gamma_f(h(x)) \ + \ \omega(u) \quad \forall x \in \mathcal{O} \qquad (20)$$

where ω is a continuous function satisfying, for some real number k,

$$\omega(u) \le k(1 + V_f(x) + \gamma(h(x))), \quad \forall (x,u) \in \mathcal{O} \times \mathcal{U} \ . \qquad (21)$$

Assuming we have a continuous function T satisfying (11), to implement the observer, we have to find a uniformly continuous function T^* satisfying (see (7)) :

$$T^*(T(x)) \ = \ x \quad \forall x \in \mathcal{O}. \qquad (22)$$

If the restriction that $\mathcal{O}$ be bounded is acceptable, i.e. we are happy with a local result, then the existence of such a function T^* is guaranteed as soon as T is injective. But, if we need a global result, a necessary condition for the existence of T^* is that T be uniformly injective as prescribed by (13). This uniform injectivity of T is the corner stone of the contribution of Kreisselmeier and Engel in [12].

In conclusion, a Kazantzis-Kravaris/Luenberger observer exists mainly if we can find a continuous function T solving (11) and uniformly injective in the sense of (13).

III. EXISTENCE OF T SOLVING (11)

To exhibit conditions guaranteeing the existence of a function T solution of (11), we abandon the interpretation above of a change of coordinates (see (4)) and come back to the original idea in [17] (see also [11] and [4]) of dynamic extension. Namely, we consider the augmented system (14). Because of its triangular structure and the fact that A is Hurwitz, we may expect this system to have, at least may be only locally, an exponentially attractive invariant manifold in the augmented (x,z) space which could even be described as the graph of a function as :

$$\{(x,z) \in \mathbb{R}^n \times \mathbb{R}^m \, : \, z = T(x)\} \, .$$

In this case, the function T must satisfy the following identity, for all t in the domain of definition of the solution $(X(x,t), Z((x,z),t))$ of (14) issued from (x,z) (compare with [19, Definition 5]),

$$T(X(x,t)) \ = \ Z((x,T(x)),t) \ . \qquad (23)$$

But, with such an identity, we get readily :

$$\lim_{t\to 0} \frac{T(X(x,t)) - T(x)}{t} \ = \ \lim_{t\to 0} \frac{Z((x,T(x)),t) - T(x)}{t} \ . \qquad (24)$$

[1] When $\mathcal{O}$ is bounded, forward unboundedness observability conditioned to $\mathcal{O}$ is necessarily satisfied with $\gamma_f = 0$.

[2] Here $L_f V$ denotes the Lie derivative of V along f, i.e. $L_f V(x) = \lim_{h\to 0} \frac{V(X(x,h)) - V(x)}{h}$.

[3] This is a compact way for expressing that the only way the solution $X(x,t)$ can exit $\mathcal{O}$ is by escaping at infinity.

It follows that T has a Lie derivative $L_f T$ satisfying (11) which can be usefully rewritten :

$$\frac{d}{\gamma\, dt}T \;=\; A T \;+\; B(y)\ . \tag{25}$$

Moreover, since we need (11) to hold only on $\mathcal{O}$, from (25), it is sufficient that T satisfies :

$$\begin{aligned} T(x) \;=\; & \exp(-At)\, T(\breve{X}(x,t)) \\ & - \int_0^t \exp(-As) B(h(\breve{X}(x,s)))ds \end{aligned} \tag{26}$$

where $\breve{X}(x,s)$ is a solution of :

$$\dot{x} \;=\; \frac{f(x)\chi(x)}{\gamma(h(x))} \tag{27}$$

where $\chi : \mathbb{R}^n \to \mathbb{R}$ is an arbitrary locally Lipschitz function satisfying :

$$\left\{ \begin{array}{rcll} \chi(x) & = & 1 & \text{if} \quad x \in \mathcal{O}\ , \\ & = & 0 & \text{if} \quad x \notin \mathcal{O} + \delta_u\ , \end{array} \right. \tag{28}$$

for some positive real number δ_u. So, as standard in the literature on invariant manifold (see [6, (2.3.4)] for instance), by letting t go to $-\infty$, we get the following candidate expression for T (compare with [12] and [15]) :

$$T(x) \;=\; \int_{-\infty}^0 \exp(-As) B(h(\breve{X}(x,s)))ds\ . \tag{29}$$

The above non rigorous reasoning can be made correct as follows :

Theorem 2 (Existence of T): Assume the existence of a strictly positive real number δ_u such that the system (1) is backward unboundedness observable conditioned to $\mathcal{O}+\delta_u$, i.e., there exists a proper and C^1 function $V_b : \mathbb{R}^n \to \mathbb{R}_+$ and a continuous function $\gamma_b : \mathbb{R}^p \to \mathbb{R}_+$ such that :

$$L_f V_b(x) \;\geq\; -V_b(x) \,-\, \gamma_b(h(x)) \qquad \forall x \,\in\, \mathcal{O}+\delta_u\ . \tag{30}$$

Then, for each complex Hurwitz matrix A in $\mathbb{C}^{m\times p}$ and for each C^1 function $\gamma : \mathbb{R}^p \to \mathbb{R}$ satisfying :

$$\gamma(h(x)) \;\geq\; 1 + \gamma_b(h(x)) \qquad \forall x \in \mathcal{O} + \delta_u\ , \tag{31}$$

we can find a C^1 function $B : \mathbb{R}^p \to \mathbb{R}^{m\times p}$ such that the function $T : \mathtt{cl}(\mathcal{O}) \to \mathbb{C}^{m\times p}$, given by (29), satisfies (11).

Remark : For the next Theorem to come, it is important to stress here that the function $B : \mathbb{R}^p \to \mathbb{R}^{m\times p}$, given by the proof of this statement, is injective and such that the function $t \mapsto |\exp(-At)B(h(\breve{X}(x,t)))|$ is exponentially decaying with t going to $-\infty$. So in particular when $\mathtt{cl}(\mathcal{O})$ is bounded, B can be chosen simply as a linear function.

Approaching the problem from another perspective, Kreisselmeier and Engel have introduced in [12] this same expression (29) (but with X instead of $\breve{X}$ and B the identity function) and interpreted each of the m components of $T(x)$ as a coefficient of a decomposition of the past output path $t \mapsto h(X(x,t))$ on a particular time functions basis. Namely, (29) is compressing the whole past output path into m real numbers. Another link between [11] and [12] has been established in [15].

IV. T INJECTIVE

Assuming now we have at our disposal the continuous function T, we need to make sure that it is injective, if not uniformly injective as specified by (13). Here is where observability enters the game. Following [17], in [11], [13], [14], observability of the first order approximation at an equilibrium together with an appropriate choice of A and B is shown to imply injectivity of the solution T of the PDE (3) in a neighborhood of this equilibrium when $m = n$. In [12], uniform injectivity of T is obtained under the following two assumptions:

1) The past output path $t \mapsto h(X(x,t))$ is injective in x in an L_2 sense, i.e. for some negative real number ℓ and class $\mathcal{K}_\infty$ function ρ, we have :

$$\begin{aligned} \rho\left(\int_{-\infty}^0 \exp(-2\ell s)|h(X(x_1,s)) - h(X(x_2,s))|^2 ds\right) \\ \geq\; |x_1 - x_2| \quad \forall x_1, x_2 \in \mathcal{O}\ . \end{aligned} \tag{32}$$

2) The system (1) has finite complexity, i.e. there exists a finite number M of piecewise continuous function ϕ_i in $L_2(\mathbb{R}_-;\mathbb{R}^p)$ and a strictly positive real number δ such that we have :

$$\begin{aligned} \sum_{i=1}^M \left[\int_{-\infty}^0 \exp(-\ell s)\phi_i(s)^T [h(X(x_1,s)) - h(X(x_2,s))]ds\right]^2 \\ \geq\; \delta \int_{-\infty}^0 \exp(-2\ell s)|h(X(x_1,s)) - h(X(x_2,s))|^2 ds \\ \forall x_1, x_2 \in \mathcal{O}\ . \end{aligned} \tag{33}$$

We state below, that, with the only assumption that the past output path $t \mapsto h(X(x,t))$ is (may be non uniformly in t) injective in x, it is sufficient to choose $m = n+1$ generic complex eigen values for A to get T injective. However this guarantees only injectivity, not uniform injectivity. As already mentioned, if $\mathcal{O}$ is bounded, the former implies the latter. But, to get a global result, it is not clear at this point what are the minimum conditions we need to obtain this stronger injectivity property.

The precise injectivity condition on the past output path we need is :

Definition 1 (Backward $\mathcal{O}$-distinguishability): There exists a strictly positive real number δ_d such that, for each pair of distinct points x_1 and x_2 in $\mathcal{O}$, there exists a negative time t, satisfying :

$$\max\{\sigma^-_{\mathcal{O}+\delta_d}(x_1), \sigma^-_{\mathcal{O}+\delta_d}(x_2)\} \;<\; t\ , \tag{34}$$

such that we have :

$$h(X(x_1,t)) \;\neq\; h(X(x_2,t))\ . \tag{35}$$

This distinguishability assumption says that the present state x can be distinguished from other states by looking at the past output path restricted to the negative time interval where the solution $X(x,t)$ is in $\mathcal{O}+\delta_d$.

Theorem 3: Assume :

1) The system (1) is backward unboundedness observable conditioned to $\mathcal{O}+\delta_u$.
2) The system (1) is backward $\mathcal{O}$-distinguishable with some positive real number δ_d in $(0,\delta_u)$.
3) There exist an injective C^1 function $b:\mathbb{R}^p\to\mathbb{R}^p$, a C^1 function $\gamma:\mathbb{R}^p\to\mathbb{R}_+$, a continuous function $M:\mathtt{cl}(\mathcal{O})\to\mathbb{R}^+$, and a negative real number ℓ such that, for each x in $\mathtt{cl}(\mathcal{O})$, the two functions $t\mapsto \exp(-\ell t)b(h(\breve{X}(x,t)))$ and $t\mapsto \exp(-\ell t)\frac{\partial b\circ h\circ \breve{X}}{\partial x}(x,t)$ satisfy, for all t in $(\breve{\sigma}^-_{\mathcal{O}+\delta_u}(x),0]$,
$$|\exp(-\ell t)b(h(\breve{X}(x,t)))| \quad\leq\quad M(x) \qquad (36)$$
$$|\exp(-\ell t)\frac{\partial b\circ h\circ \breve{X}}{\partial x}(x,t)| \quad\leq\quad M(x) \qquad (37)$$
where as above $\breve{X}$ is a solution of (27), but this time with the function χ satisfying :
$$\left\{\begin{array}{lcll}\chi(x) &=& 1 & \text{if}\quad x\in\mathcal{O}+\delta_d\ ,\\ &=& 0 & \text{if}\quad x\notin\mathcal{O}+\delta_u\ .\end{array}\right. \qquad (38)$$

Under these conditions, there exists a subset S of $\mathbb{C}^{n+1}$ of zero Lebesgue measure such that the function $T:\mathtt{cl}(\mathcal{O})\to\mathbb{C}^{(n+1)\times p}$ defined by :
$$T(x) \;=\; \int_{-\infty}^0 \exp(-As)\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix} b(h(\breve{X}(x,s)))ds\ , \qquad (39)$$
is injective provided A is the diagonal matrix :
$$A \;=\; \mathtt{diag}(\lambda_1,\ldots,\lambda_{n+1}) \qquad (40)$$
where the $n+1$ complex numbers λ_i are arbitrarily chosen in $\mathbb{C}^{n+1}\setminus S$ but with real part strictly smaller than ℓ.

This theorem states that, if we choose $n+1$ complex generic eigenvalues for the matrix A, then the function T given by (39) (or equivalently (29)) is injective. This says that the (real) dimension of z is $m=2n+2$. It is a well known fact in observer theory that it is generically sufficient to extract $m=2n+1$ pieces of information from the output path to observe a state of dimension n (see for instance [1], [21], [8], [7], [20]). It can be understood from the adage that, the relation $T(x_1)=T(x_2)$ between the two states x_1 and x_2 in $\mathbb{R}^n$, i.e. for $2n$ unknowns, has generically the unique trivial solution $x_1=x_2$ if we have strictly more than $2n$ equations, i.e. $T(x)$ has strictly more than $2n$ components.

To be able to prove Theorem 3 we require the condition (37) in order to guarantee that T is a C^1 function. A simple case where this condition holds is when the data f, h, γ and b have bounded derivative on $\mathtt{cl}(\mathcal{O})$ (see [15] for instance). But this is a severe restriction. Actually, we conjecture that, with the help of techniques of maximal monotone operators, such as those presented in [5], we should be able to prove that a weaker sufficient condition on f and γ for instance is :
$$\frac{\partial f_\gamma}{\partial x}(x)Q \;+\; \frac{\partial f_\gamma}{\partial x}(x)^TQ \;\geq\; -q\,Q \qquad \forall x\in\mathtt{cl}(\mathcal{O}) \quad (41)$$
for some positive definite matrix Q and real number q and with the notation :
$$f_\gamma(x) \;=\; \frac{f(x)}{\gamma(h(x))}\ . \qquad (42)$$

V. Injectivity in the case of complete observability

Another setup where injectivity can be obtained is when we have complete observability. Namely we can find a dimension m, a function $b:y\in\mathbb{R}^p\mapsto b(y)=(b_1(y),\ldots,b_p(y))\in\mathbb{R}^p$, and a strictly positive function $\gamma:\mathbb{R}^p\to\mathbb{R}_{+*}$, so that the following function $H:\mathbb{R}^n\to\mathbb{R}^{m\times p}$ is injective :
$$H(x)=\begin{pmatrix} b_1(h(x)) & \ldots & b_p(h(x))\\ L_{f_\gamma}b_1(h(x)) & \ldots & L_{f_\gamma}b_p(h(x))\\ \ldots & \ldots & \ldots\\ L_{f_\gamma}^{m-1}b_1(h(x)) & \ldots & L_{f_\gamma}^{m-1}b_p(h(x))\end{pmatrix} \qquad (43)$$
for each x in $\mathbb{R}^n$ and where L_f^ih denotes the ith iterate Lie derivative, i.e. $L_f^{i+1}h=L_f(L_f^ih)$. Of course, for this to make sense, the functions b, f, h and γ must be sufficiently smooth. This setup has been popularized and studied in deep details by Gauthier and his coworkers (see [10] and the references therein, see also [19]). In particular, again it is known (see [8] for instance) that, generically we have $m=2n+1$.

With a Taylor expansion of the output path at $t=0$, we see that the injectivity of H implies that the function which associates the initial condition x to the output path, restricted to a very small time interval, is injective. This property is nicely exploited by observers with very fast dynamics as high gain observers (see [9]). Specifically, we have :

Theorem 4: (Injectivity in the case of complete observability) : Assume there exist functions $b:\mathbb{R}^p\to\mathbb{R}^p$ and $\gamma:\mathbb{R}^p\to\mathbb{R}_+$ such that :

1) there exists a positive real number L such that, for each x_1 and x_2 in $\mathtt{cl}(\mathcal{O})$, we have :
$$|L_{f_\gamma}^m b(h(x_1)-L_{f_\gamma}^m b(h(x_2))|\leq L|H(x_1)-H(x_2)| \quad (44)$$
2) there exists a class $\mathcal{K}_\infty$ function ρ such that, for each x_1 and x_2 in $\mathtt{cl}(\mathcal{O})$, the function H satisfies :
$$\rho(|H(x_1)-H(x_2)|)\geq|x_1-x_2|\ . \qquad (45)$$

Then, for any diagonal complex Hurwitz matrix A in $\mathbb{C}^{m\times m}$, there exists a real number k^* such that, for any k larger than k^*, there exists a function $T:\mathcal{O}\to\mathbb{C}^{m\times p}$

which is uniformly injective and satisfies :

$$L_f T(x) = \gamma(h(x)) \left[kAT(x) + \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} b(h(x)) \right] \qquad (46)$$

$$\forall x \in \mathcal{O} \ .$$

VI. APPROXIMATION

Fortunately the problem of finding an expression for T can be simplified. Indeed a function T satisfying (11) only approximately is allowed. But, in this case, we have to modify the observer dynamics.

Theorem 5 (Approximation): Assume the system (1) is forward unboundedness observable conditioned to $\mathcal{O}$, i.e. (10) holds. Assume also the existence of an integer m, a complex Hurwitz matrix A in $\mathbb{C}^{m\times m}$ and functions $T_a : \mathtt{cl}(\mathcal{O}) \to \mathbb{C}^{m\times p}$, $B : \mathbb{R}^p \to \mathbb{C}^{m\times p}$ continuous, $\gamma : \mathbb{R}^p \to \mathbb{R}_+$, C^1, and ρ of class $\mathcal{K}_\infty$, such that :

1) we have :

$$\gamma(h(x)) \ \geq \ 1 + \gamma_f(h(x)) \qquad \forall x \in \mathtt{cl}(\mathcal{O}) \ , \qquad (47)$$

$$|x_1 - x_2| \ \leq \ \rho(|T_a(x_1) - T_a(x_2)|) \qquad (48)$$

$$\forall x_1,\, x_2 \in \mathtt{cl}(\mathcal{O}) \ .$$

2) the function $L_{f_\gamma} T_a$ is well defined on $\mathcal{O}$ and the function $\mathfrak{E} : \mathtt{cl}(\mathcal{O}) \to \mathbb{C}^{m\times p}$ de ned by :

$$\mathfrak{E}(x) = L_{f_\gamma} T_a(x) - [AT_a(x) + B(h(x))] \ \forall x \in \mathcal{O} \quad (49)$$

satis es :

$$|\mathfrak{E}(x_1) - \mathfrak{E}(x_2)| \ \leq \ L\left(|T_a(x_1) - T_a(x_2)|\right) \qquad (50)$$

$$\forall x_1,\, x_2 \in \mathtt{cl}(\mathcal{O}) \ ,$$

where L is a positive real number satisfying :

$$2L\lambda_{\max}(P) \ < \ 1 \ , \qquad (51)$$

with $\lambda_{\max}(P)$ the largest eigenvalue of the Hermitian matrix P solution of :

$$\overline{A}^\top P + PA = -I \ . \qquad (52)$$

Under these conditions, there exists a function $T_a^* : \mathbb{C}^{m\times p} \to \mathtt{cl}(\mathcal{O})$ and a locally Lipschitz function $\mathfrak{F} : \mathbb{C}^{m\times p} \to \mathbb{C}^{m\times p}$ such that, for each x in $\mathcal{O}$ and z in $\mathbb{C}^{m\times p}$ each solution $(X(x,t), Z(x,z,t))$ of :

$$\left\{ \begin{array}{rcl} \dot{x} & = & f(x) \ , \\ \dot{z} & = & \gamma(h(x))\left(Az + \mathfrak{F}(z) + B(h(x))\right) \end{array} \right. \qquad (53)$$

is right maximally defined on $[0, \sigma^+_{\mathbb{R}^n}(x))$. Moreover, if we have :

$$\sigma^+_{\mathcal{O}}(x) \ = \ \sigma^+_{\mathbb{R}^n}(x) \ , \qquad (54)$$

then we get :

$$\lim_{t\to\sigma^+_{\mathbb{R}^n}(x)} |T_a^*(Z(x,z,t)) - X(x,t)| \ = \ 0 \ . \qquad (55)$$

Remarks :

1) In (49), $\mathfrak{E}$ represents the error given by the approximation T_a of T. This error should not be too large in an incremental sense as specified by (50) and (51). This indicates that one way to approximate T is to look for T_a in a set of functions minimizing the L_∞ norm on $\mathtt{cl}(\mathcal{O})$ of the gradient of the associated error $\mathfrak{E}$.

2) The function $\mathfrak{F}$ in the observer (53) is in fact a Lipschitz extension of $\mathfrak{E}(T_a^*)$ outside $T_a(\mathtt{cl}(\mathcal{O}))$. This is very similar to what is done in [19] where a constructive procedure for this extension is proposed. Fortunately, this Lipschitz extension is not needed in the case where the function $\mathfrak{E}$ satisfies :

$$|\mathfrak{E}(x_1) - \mathfrak{E}(x_2)| \leq \frac{L}{4}\,\rho^{-1}(|x_1 - x_2|) \quad \forall x_1,\, x_2 \in \mathtt{cl}(\mathcal{O}) \ . \qquad (56)$$

In this case we take simply :

$$\mathfrak{F}(z) = \mathfrak{E}(T_a^*(z)) \quad \forall z \in \mathbb{C}^{m\times p}. \qquad (57)$$

3) As for Theorem 1, we can extend Theorem 5 to the case with input. Then the function $\mathfrak{E}$ should satisfy :

$$\mathfrak{E}(x,u) = L_{f_\gamma + g_\gamma u} T_a(x) \qquad (58)$$

$$-[AT_a(x) + B(h(x)) + C(h(x))u]$$

where $g_\gamma(x) = \frac{g(x)}{\gamma(x)}$. In this expression, the function $C : \mathbb{R}^p \to \mathbb{C}^{m\times p\times q}$, and the set $\mathcal{U} \subset \mathbb{R}^q$ of admissible u's are to be chosen, if possible, so that we have (20), (21) and :

$$|\mathfrak{E}(x_1,u) - \mathfrak{E}(x_2,u)| \ \leq \ L(|T_a(x_1) - T_a(x_2)|) \qquad (59)$$

$$\forall x_1,\, x_2 \in \mathtt{cl}(\mathcal{O}) \ , \ \forall u \in \mathcal{U} \ .$$

Theorem 5 gives us a new insight in the classical high gain observer of order m as studied in [8] or [19] for instance.

Corollary 1 (Classical high gain Observer): Assume :

1) The system (1) is forward unboundedness observable conditioned to $\mathcal{O}$, i.e. (10) holds.
2) With the function γ satisfying (12), there exist a sufficiently smooth function $b : \mathbb{R}^p \to \mathbb{R}^p$, a class $\mathcal{K}_\infty$ function ρ and a positive real number L such that (44) and (45) hold.

Under these conditions, for any diagonal complex Hurwitz matrix A in $\mathbb{C}^{m\times m}$, there exist a matrix B and a real number k^* such that, for any k larger than k^*, there exists a function $T_a^* : \mathbb{R}^{m\times p} \to \mathtt{cl}(\mathcal{O})$ and a function $\mathfrak{F} : \mathbb{R}^{m\times p} \to \mathbb{R}^{m\times p}$ such that, for each x in $\mathcal{O}$ and z in $\mathbb{R}^{m\times p}$ each solution $(X(x,t), Z(x,z,t))$ of :

$$\left\{ \begin{array}{rcl} \dot{x} & = & f(x) \ , \\ \dot{z} & = & \gamma(h(x))\left(kAz + \mathfrak{F}(z) + Bb(h(x))\right) \end{array} \right. \qquad (60)$$

is right maximally defined on $[0, \sigma_{\mathbb{R}^n}^+(x))$. Moreover, if we have :

$$\sigma_{\mathcal{O}}^+(x) \ = \ \sigma_{\mathbb{R}^n}^+(x) \ , \tag{61}$$

then we get :

$$\lim_{t \to \sigma_{\mathbb{R}^n}^+(x)} |T_a^*(Z(x,z,t)) - X(x,t)| \ = \ 0 \ . \tag{62}$$

Remark : When $\mathcal{O}$ is bounded and H is injective, uniform injectivity (45) and forward unboundedness observability conditioned to $\mathcal{O}$ hold necessarily. Thus, in this case, we recover [19, Lemma 1].

VII. CONCLUSION

We have stated sufficient conditions under which the extension to non linear systems of the Luenberger observer, as it has been proposed by Kazantzis and Kravaris in [11], can be used as long as the state to be observed remains in a given open set. In doing so, we have exploited the fact, already mentioned in [15], that the observer proposed by Kreisselmeier and Engel in [12] is a possible way of implementing the Kazantzis-Kravaris/Luenberger observer.

To get as less restrictive sufficient conditions as possible we have found useful to modify the observer in a way which induces a time rescaling as already suggested in [4].

We have also claimed that a sufficient dimension of the dynamic system giving the observer is 2 + twice the dimension of the state to be observed. This is in agreement with many other results known on the generic number of pieces of information to be extracted from the output paths to be able to reconstruct the state.

Finally, we have shown that it is sufficient to know only an approximation of a solution of a partial differential equation which we need to solve to implement the observer. In this way, we have been able to make a connection with high gains observers.

At this stage, our results are mainly of theoretical nature. They are concerned with existence. Several problems of prime importance for practice remain to be addressed like type and speed of convergence. In these regards, the contribution of Rapaport and Maloum [19] is an important starting point.

Even, about existence, we have to note that the conditions we have given can be strongly relaxed if an estimation of the norm of the state is available. This idea has been exploited in [4] where a truly global observer has been proposed under the assumption of global complete observability and unboundedness observability.

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