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On the Asymptotic Properties of a System Arising in Non-equilibrium Theory of Output Regulation *

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Abstract

This paper provides a self-contained proof of the fact that certain systems arising in the non-equilibrium theory of output regulation, which possess a locally exponentially stable compact attractor, are input-to-state stable (with respect to the attractor, with restrictions) with a linear gain function.

Keywords: Lyapunov Functions, Input-to-State Stability, Regulation, Tracking, Nonlinear Control.

1 Terminology and Notations

Consider an *autonomous* ordinary differential equation

$$\dot{x} = f(x) \tag{1}$$

with $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, and let

$$\phi : \quad (t, x) \quad \mapsto \quad \phi(t, x)$$

define its flow. A set X is locally invariant under the flow of (1) if, for any $x \in X$, there exists an open interval I of 0 in $\mathbb{R}$ such that $\phi(t, x) \in X$ for all $t \in I$. A set X is forward invariant under the flow of (1) if, for any $x \in X$, $\phi(t, x)$ is defined for all for all $t \geq 0$ and $\phi(t, x) \in X$ for all $t \geq 0$. A set X is backward invariant under the flow of (1) if, for any

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$x \in X$, $\phi(t,x)$ is defined for all for all $t \leq 0$ and $\phi(t,x) \in X$ for all $t \leq 0$. A set X is *invariant* under the flow of (1) if it is backward and forward invariant.

Let B be a fixed subset of $\mathbb{R}^n$ and suppose that, for all $p \in B$, the map $t :\longrightarrow \phi(t,x)$ is defined for all $t \geq 0$. The *positive orbit* of B is the set

$$\mathcal{O}^+(B) := \bigcup_{x \in B} \bigcup_{t \geq 0} \phi(t,x)\,.$$

The ω-*limit set* of a subset $B \subset \mathbb{R}^n$, written $\omega(B)$, is the totality of all points $x \in \mathbb{R}^n$ for which there exists a sequence of pairs (x_k, t_k), with $x_k \in B$ and $t_k \to \infty$ as $k \to \infty$, such that

$$\lim_{k \to \infty} \phi(t_k, x_k) = x\,.$$

In case $B = \{x_0\}$ the set thus defined, $\omega(x_0)$, is precisely the ω-limit set, as defined by G.D.Birkhoff, of the point x_0. With a given set B, is it is also convenient to associate the set

$$\psi(B) = \bigcup_{x_0 \in B} \omega(x_0)$$

i.e. the union of the ω-limits set of all points of B. By definition $\psi(B) \subset \omega(B)$, but the equality may not hold.

Let $|x|$ denote the Euclidean norm of a vector $x \in \mathbb{R}^n$. Let A be a closed subset of $\mathbb{R}^n$ and, for any $x \in \mathbb{R}^n$ let

$$|x|_{\mathcal{A}} := \min_{y \in \mathcal{A}} |y - x|$$

denote the distance of x from $\mathcal{A}$. The A is said to *uniformly attract* a set B under the flow of (3) if for every $\varepsilon > 0$ there exists a time $\bar{t}$ such that

$$|\phi(t,x)|_{\mathcal{A}} \leq \varepsilon, \qquad \text{for all } t \geq \bar{t} \text{ and for all } x \in B.$$

Then the following holds (see [4] and, for the second property, [3] or [7]).

Lemma 1.1 *If B is a nonempty connected bounded set whose positive orbit is bounded, then $\omega(B)$ is a nonempty, connected, compact, invariant set which uniformly attracts B. Moreover, if $\omega(B) \in \mathrm{int}(B)$, then $\omega(B)$ is stable in the sense of Lyapunov.*

2 Preliminaries

The purpose of this paper is to analyze the consequence of certain asymptotic properties of a system of the form

$$\begin{array}{rcl} \dot{z} &=& f_0(z,w) \\ \dot{w} &=& s(w) \end{array} \qquad (2)$$

in which $z \in \mathbb{R}^n$, $w \in \mathbb{R}^r$.

The functions $f_0(z,w)$ and $s(w)$ in (2) are C^k (with k sufficiently large) functions. Initial conditions for w are allowed to range over a fixed compact set W. Moreover, the following assumptions are supposed to hold.

Assumption 0. *The set W is invariant for $\dot{w} = s(w)$ and $W = \psi(W)$.*

Note that, since W is invariant for $\dot{w} = s(w)$, the closed cylinder $\mathcal{C} = \mathbb{R}^n \times W$ is locally invariant for (2). Hence, it is natural regard (2) as a system defined on $\mathcal{C}$ and endow the latter with the subset topology. Let now Z be a fixed compact set of $\mathbb{R}^n$.

Assumption 1a. *The positive orbit of $Z \times W$ under the flow of (2) is bounded.*

This assumption implies that the set $\mathcal{A} := \omega(Z \times W)$ i.e the ω-limit set – under the flow of (2) – of the set $Z \times W$, is a nonempty, compact, invariant subset of $\mathcal{C}$ which uniformly attracts $Z \times W$ under the flow of (2). Moreover, Assumption 0 implies that for any $w \in W$ there is a $z \in Z$ such that $(z,w) \in \mathcal{A}$. In other words, the projection map $P : (z,w) \mapsto w$ carries $\mathcal{A}$ onto W(see [1]).

Assumption 1b. *There exists a number $d_0 > 0$ such that*

$$\mathcal{B}_0 := \{(z,w) \in \mathbb{R}^n \times W : |(z,w)|_{\mathcal{A}} \leq d_0\} \subset Z \times W\,.$$

This assumption implies that the set $\mathcal{A}$ is stable in the sense of Lyapunov, under the flow of (2).

For convenience, in what follows we rewrite (2) in the form of a single autonomous system

$$\dot{p} = f(p) \tag{3}$$

in which $p := (z,w)$, and we let $\phi(t,p)$ denote its flow. Consistently, we set $\mathcal{P} := Z \times W$ (and note that $\mathcal{A} = \omega(\mathcal{P})$).

As observed above, a consequence of Assumptions 1a and 1b is that $\mathcal{A}$ is stable in the sense of Lyapunov and uniformly attracts $\mathcal{P}$, under the flow of (3). Hence, there exist a strictly increasing function $\delta(\cdot)$, carrying $\mathbb{R}_{\geq 0}$ into $\mathbb{R}_{\geq 0}$ and vanishing at zero, such that

$$|p|_{\mathcal{A}} \leq \delta(\varepsilon) \qquad \Rightarrow \qquad |\phi(t,p)|_{\mathcal{A}} \leq \varepsilon \qquad \forall t \geq 0\,,\ \ \forall p \in \mathcal{P}$$

and a continuous and strictly decreasing function $T(\cdot)$, carrying $\mathbb{R}_{>0}$ onto itself, such that

$$|\phi(t,p)|_{\mathcal{A}} \leq \varepsilon \qquad \forall t \geq T(\varepsilon),\ \ \forall p \in \mathcal{P}\,.$$

We define the domain of attraction of $\mathcal{A}$ as the set $\mathcal{D}$ of all points $p \in \mathcal{C}$ such that $\lim_{t\to\infty} |\phi(t,p)|_{\mathcal{A}} = 0$. The set $\mathcal{D}$, open in the subset topology of $\mathcal{C}$, is forward invariant for (3) and, obviously, $\mathcal{P} \subset \mathcal{D}$. In what follows we let $\bar{\mathcal{D}}$ denote the complement of $\mathcal{D}$ in $\mathcal{C}$ and let $\partial\mathcal{D}$ denote the boundary of $\mathcal{D}$ (in the subset topology).

Appropriate adaptations of the arguments of [9] and [6] can be used to show the existence, for system (3), of a Lyapunov function. In the present note, we consider a "perturbed" version of (3), namely a system of the form

$$\dot{p} = f(p) + r(p,u)u$$

in which $u \in \mathbb{R}$ is an external input, and we are interested in determining its input-to-state stability properties (with restrictions) with respect to the compact set $\mathcal{A}$ (see [7]), with an input-to-state gain function which is linear at the origin. To this end, it is convenient to assume that the set $\mathcal{A}$ is locally exponentially stable.

Assumption 2. *There exists numbers $M \geq 1$ and $\lambda > 0$ such that, for all $p \in \mathcal{B}_0$,*

$$|\phi(t,p)|_{\mathcal{A}} \leq Me^{-\lambda t}|p|_{\mathcal{A}}\,, \qquad \forall t \geq 0\,.$$

Note that, in this case, there is no loss of generality in assuming that the function $\delta(\cdot)$ is linear at the origin, in particular that $\delta(\varepsilon) = (1/M)\varepsilon$ for all $\varepsilon \in [0, Md_0]$.

3 Lyapunov functions for (3)

3.1 The rescaled-time system

System (3) is not necessarily (backward and forward) complete. Since completeness plays an important role in the construction of Lyapunov functions, as in [6] we construct a complete system as follows. Let $a_f : \mathbb{R}^{n+r} \to \mathbb{R}$ be a smooth function satisfying

$$\begin{array}{rcll} a_f(p) & = & 1\,, & \text{for all } p \text{ such that } |p|_{\mathcal{A}} \leq d_0 \\ a_f(p) & \geq & 1 + |f(p)|\,, & \text{for all } p \text{ such that } |p|_{\mathcal{A}} \geq 2d_0. \end{array}$$

Indeed, the system

$$\dot{p} = \frac{1}{a_f(p)} f(p) \qquad (4)$$

is complete. In what follows, we denote by $\psi(t,p)$ its flow.

Proposition 3.1 *The sets $\mathcal{C}$ and $\mathcal{A}$ are invariant for (4).*

Proof. The two sets are locally invariant for (3) and hence, since $f(p)$ and $f(p)/a_f(p)$ only differ by a scalar factor, these sets are also locally invariant for (4). To prove that $\mathcal{C}$ is forward invariant, take $p \in \mathcal{C}$, observe that $\psi(t,p)$ is defined for all $t \in \mathbb{R}$, let $\overline{\mathcal{C}}$ denote the complement of $\mathcal{C}$ in $\mathbb{R}^n \times \mathbb{R}^r$ and suppose, by contradiction, that the set

$$S = \{t > 0 : \psi(t,p) \in \overline{\mathcal{C}}\}$$

is not empty. Let t^* denote the lower bound of S. Note that S is open, because $\overline{\mathcal{C}}$ is open and $\psi(t,p)$ is continuous in t. Thus, $t^* \notin S$ and $\psi(t^*,p) \in \mathcal{C}$. But, as $\mathcal{C}$ is locally invariant, $\psi(t,p) \in \mathcal{C}$ for all t in a neighborhhod of t^*. This contradicts the fact that t^* is a lower bound of S. An identical argument shows that $\mathcal{C}$ is backward invariant. The same proof shows also that $\mathcal{A}$, a closed locally invariant set, is invariant. ◁

Proposition 3.2 *The set $\mathcal{A}$ uniformly attracts $\mathcal{P}$ under the flow of (4).*

Proof. Pick any $p_0 \in \mathbb{R}^{n+r}$. Since $a_f(\psi(t,p_0))$ takes values in $[1,+\infty)$ and is locally Lipschitz in the argument t, there exists a unique solution $\tau_0(t)$ of the initial value problem

$$\dot{\tau} = a_f(\psi(\tau,p_0)), \quad \tau(0) = 0\,, \tag{5}$$

maximally defined over an open interval (t_0,t_1) of 0 in $\mathbb{R}$. In particular $\lim_{t\to t_1}\tau_0(t) = +\infty$. Indeed this follows from the fact that if t_1 is finite, then $\tau_0(t)$ goes to ∞ by the maximality of (t_0,t_1), while if t_1 is infinite, the result follows from $\tau_0(t) \geq t$. Similar arguments show that $\lim_{t\to t_0}\tau_0(t) = -\infty$.

It is easy to check that

$$\psi(\tau_0(t),p_0) = \phi(t,p_0)\,, \qquad \forall t \in (t_0,t_1)\,. \tag{6}$$

In fact

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi(\tau_0(t),p_0) = \frac{1}{a_f(\psi(\tau_0(t),p_0))}f(\psi(\tau_0(t),p_0))\frac{\mathrm{d}}{\mathrm{d}t}\tau_0(t) = f(\psi(\tau_0(t),p_0))\,,$$

and, by uniqueness, (6) follows. The function $\tau_0(t)$ is continuously differentiable and strictly increasing. Therefore, there exists a function $\tau_0^{-1}(t)$ defined on $(-\infty,+\infty)$ such that

$$\tau_0^{-1}\circ\tau_0(t) = t, \quad \forall t \in (t_0,t_1)$$

and

$$\tau_0^{-1}(t) \leq t\,, \quad \tau_0\circ\tau_0^{-1}(t) = t, \quad \forall t \in (-\infty,+\infty)\,.$$

Clearly

$$\psi(t,p_0) = \phi(\tau_0^{-1}(t),p_0) \qquad \forall t \in (-\infty,+\infty)\,. \tag{7}$$

By Assumption 1a, there exists a number K such that $|\phi(t,p_0)| \leq K$ for all $t \geq 0$ and all $p_0 \in \mathcal{P}$. Hence, from (7) we obtain

$$|\psi(t,p_0)| \leq K\,, \qquad \forall t \in [0,\infty)\,,\ \forall p_0 \in \mathcal{P}\,.$$

Let now

$$N = \max_{|p|\leq K} a_f(p)\,.$$

Thus, for all $p_0 \in \mathcal{P}$, we have $\tau_0(t) \leq Nt$ for all $t \in [0,t_1)$, which in turn implies $t_1 = \infty$. Set now $\tilde{T}(\epsilon) = NT(\epsilon)$ and note that $t \geq \tilde{T}(\epsilon)$ implies $t \geq \tau_0(T(\epsilon))$, i.e. $\tau_0^{-1}(t) \geq T(\epsilon)$. Therefore

$$t \geq \tilde{T}(\epsilon) \qquad \Rightarrow \qquad |\psi(t,p_0)|_{\mathcal{A}} = |\phi(\tau_0^{-1}(t),p_0)|_{\mathcal{A}} \leq \epsilon$$

for all $p_0 \in \mathcal{P}$ and this proves that $\mathcal{A}$ uniformly attracts $\mathcal{P}$ under the flow of (4). $\triangleleft$

Proposition 3.3 *The set $\mathcal{D}$ is the set of all points $p \in \mathcal{C}$ such that* $\lim_{t\to\infty} |\psi(t,p)|_{\mathcal{A}} = 0$.

Proof. First of all, we observe that, if $p_0 \in \mathcal{D}$, there exists a number K_0 such that $|\phi(t,p_0)| \le K_0$ for all $t \ge 0$. Thus, the same argument used in the proof of Proposition 3.2, setting $N_0 = \max_{|p|\le K_0} a_f(p)$, shows that $\tau_0(t) \le N_0 t$ so long as $\tau_0(t)$ is defined, which in turn implies that the function in question is defined for all $t \ge 0$ (and $\lim_{t\to\infty} \tau_0^{-1}(t) = \infty$). Thus

$$\lim_{t\to\infty} |\psi(t,p_0)|_{\mathcal{A}} = \lim_{t\to\infty} |\phi(\tau_0^{-1}(t),p_0)|_{\mathcal{A}} = 0\,,$$

i.e. all points in $\mathcal{D}$ are such that $\lim_{t\to\infty} |\psi(t,p)|_{\mathcal{A}} = 0$. It remains to show that no other point of $\mathcal{C}$ has this property. To this end, it suffices to show that $\mathcal{D}$ is invariant under the flow of (4). Pick any $p \in \mathcal{D}$ and any $s \in \mathbb{R}$, and set $p_1 := \psi(s,p)$. Clearly,

$$\lim_{t\to\infty} |\psi(t,p_1)|_{\mathcal{A}} = 0\,.$$

As $\psi(t,p_1)$ is bounded on $[0,\infty)$, arguments identical to those used above show that the solution $\tau_1(t)$ of

$$\dot\tau = a_f(\psi(\tau,p_1)), \quad \tau(0) = 0$$

is defined for all $t \ge 0$ (and $\lim_{t\to\infty} \tau_1(t) = \infty$). Thus

$$\lim_{t\to\infty} |\phi(t,p_1)|_{\mathcal{A}} = \lim_{t\to\infty} |\psi(\tau_1(t),p_1)|_{\mathcal{A}} = 0\,.$$

This shows that $\psi(s,p) \in \mathcal{D}$, i.e. that $\mathcal{D}$ is invariant under the flow of (4). $\lhd$

Finally, set $d_1 := d_0/M$ and

$$\mathcal{B}_1 = \{p \in \mathcal{B}_0 : |p|_{\mathcal{A}} \le d_1\}\,.$$

As system (3) and system (4) agree on $\mathcal{B}_0$, it is seen from Assumption 2 that for all $p \in \mathcal{B}_1$

$$|\psi(t,p)|_{\mathcal{A}} \le M e^{-\lambda t} |p|_{\mathcal{A}}\,, \qquad \forall t \ge 0\,. \tag{8}$$

In particular, the function $\delta : [0,d_0] \to \mathbb{R}$ defined as $\delta(\varepsilon) = (1/M)\varepsilon$ is such that, for all $p \in \mathcal{B}_1$,

$$|p|_{\mathcal{A}} \le \delta(\varepsilon) \qquad \Rightarrow \qquad |\psi(t,p)|_{\mathcal{A}} \le \varepsilon \quad \forall t \ge 0\,.$$

3.2 Wilson's Lyapunov function for (4)

We follow Wilson's construction [9]. First of all, define $g : \mathcal{B}_1 \to \mathbb{R}$ by

$$g(p) = \inf_{t\le 0}\{|\psi(t,p)|_{\mathcal{A}}\}\,.$$

Lemma 3.1 *The function g has the following properties:*

1. $g(p) \geq g(\psi(t,p))$ *for all* $t \geq 0$*.*

2. $\delta(|p|_{\mathcal{A}}) \leq g(p) \leq |p|_{\mathcal{A}}$*.*

3. There is a time $T > 0$ *such that, for all* $p \in \mathcal{B}_1$*,* $g(p) = \min_{t\in[-T,0]}\{|\psi(t,p)|_{\mathcal{A}}\}$*.*

4. The function g *is Lipschitz on* $\mathcal{B}_1$*.*

Proof. Property 1 is a direct consequence of the definition. In Property 2, the inequality on the right is a direct consequence of the definition. The inequality on the left is proven by contradiction. Suppose it is not true. Then there exists $t_0 \leq 0$ such that $|\psi(t_0,p)|_{\mathcal{A}} < \delta(|p|_{\mathcal{A}})$. As $\delta(\cdot)$ is strictly increasing, it is always possible to find $0 < \varepsilon < |p|_{\mathcal{A}}$ such that

$$|\psi(t_0,p)|_{\mathcal{A}} < \delta(\varepsilon) < \delta(|p|_{\mathcal{A}})\,.$$

This implies

$$|p|_{\mathcal{A}} = |\psi(-t_0,\psi(t_0,p))|_{\mathcal{A}} \leq \varepsilon < |p|_{\mathcal{A}}\,,$$

which is a contradiction.

To prove Property 3, recall (8) and set

$$T := \frac{1}{\lambda}\log M\,.$$

We claim that

$$t < -T \qquad \Rightarrow \qquad |\psi(t,p)|_{\mathcal{A}} \geq |p|_{\mathcal{A}}\,. \tag{9}$$

The property is indeed true for all $t < 0$ such that $|\psi(t,p)|_{\mathcal{A}} > d_1$, because $d_1 \geq |p|_{\mathcal{A}}$ for all $p \in \mathcal{B}_1$. Consider now a $t < -T$ such that $|\psi(t,p)|_{\mathcal{A}} \leq d_1$ and, by contradiction, suppose $|\psi(t,p)|_{\mathcal{A}} < |p|_{\mathcal{A}}$. Set $t = -T - \epsilon$, with $\epsilon > 0$, and use (8) to obtain

$$|p|_{\mathcal{A}} = |\psi(-t,\psi(t,p))|_{\mathcal{A}} = |\psi(T+\epsilon,\psi(t,p))|_{\mathcal{A}} \leq Me^{-\lambda(T+\epsilon)}|\psi(t,p)|_{\mathcal{A}} \leq e^{-\lambda\epsilon}|p|_{\mathcal{A}}\,.$$

This is a contradiction, and hence (9) is true. This property, since $g(p) \leq |p|_{\mathcal{A}}$, shows that

$$\inf_{t\leq 0}\{|\psi(t,p)|_{\mathcal{A}}\} = \min_{t\in[-T,0]}\{|\psi(t,p)|_{\mathcal{A}}\}\,. \tag{10}$$

Finally, to see that Property 4 holds, pick any two points η and ζ in $\mathcal{B}_1$ and, in view of Property 3, let $\bar{t} \in [-T,0]$ be a time at which the minimum in (10) is reached, i.e. such that $g(\zeta) = |\psi(\bar{t},\zeta)|_{\mathcal{A}}$. Observe also that $g(\eta) \leq |\psi(\bar{t},\eta)|_{\mathcal{A}}$. Thus

$$g(\eta) - g(\zeta) \leq |\psi(\bar{t},\eta)|_{\mathcal{A}} - |\psi(\bar{t},\zeta)|_{\mathcal{A}} \leq |\psi(\bar{t},\eta) - \psi(\bar{t},\zeta)|\,.$$

It is known (see e.g. [2]) that there is a constant C, which only depends on T and $\mathcal{B}_1$, such that

$$|\psi(t,\eta) - \psi(t,\zeta)| \leq C|\eta - \zeta|$$

for all $|t| \le T$ and all $\eta, \zeta \in \mathcal{B}_1$. Thus, the previous inequality yields

$$g(\eta) - g(\zeta) \le C|\eta - \zeta|\,.$$

Reversing the roles of η and ζ yields $g(\zeta) - g(\eta) \le C|\eta - \zeta|$ and this proves the desired property. $\triangleleft$

Set now

$$\mathcal{B}_2 := \{p \in \mathcal{B}_1 : |p|_{\mathcal{A}} \le d_1/M\}$$

and define $U : \mathcal{B}_2 \to \mathbb{R}$ by

$$U(p) = \sup_{t \ge 0}\{g(\psi(t,p))k(t)\}$$

in which

$$k(t) = \frac{1+2t}{1+t}\,.$$

Lemma 3.2 *The function U has the following properties:*

1. *For all $p \in \mathcal{B}_2$*
$$\frac{1}{M}|p|_{\mathcal{A}} \le U(p) \le 2|p|_{\mathcal{A}}\,. \tag{11}$$
2. *There is a time $T^* > 0$ such that, for all $p \in \mathcal{B}_2$, $U(p) = \max_{t\in[0,T^*]}\{g(\psi(t,p))k(t)\}$.*
3. *The function U is Lipschitz on $\mathcal{B}_2$.*
4. *There is a number $\kappa > 0$ such that, for all $p \in \mathrm{int}(\mathcal{B}_2)$,*
$$\limsup_{h\to 0^+} \frac{U(\psi(h,p)) - U(p)}{h} \le -\kappa U(p)\,. \tag{12}$$

Proof. Since $U(p) \ge g(p)$ by definition, and $\delta(\varepsilon) = \varepsilon/M$, the estimate on the left in Property 2 of g yields the estimate on the left in (11). Likewise, as Property 1 of g implies $g(\psi(t,p))k(t) \le g(\psi(t,p))2 \le g(p)2$, the estimate on the right in Property 2 of g yields the estimate on the right in (11).

To prove Property 2, set

$$T^* = \frac{1}{\lambda}\log(2M^2)\,.$$

Suppose the property is false. Then, there exists a time $\bar{t} > T^*$ such that

$$\frac{1}{M}|p|_{\mathcal{A}} \le g(p) \le g(\psi(\bar{t},p)) \le |\psi(\bar{t},p)|_{\mathcal{A}} \le Me^{-\lambda\bar{t}}|p|_{\mathcal{A}}\,,$$

having used the property (8). This yields

$$|p|_{\mathcal{A}} \le M^2 e^{-\lambda\bar{t}}|p|_{\mathcal{A}} < M^2 e^{-\lambda T^*}|p|_{\mathcal{A}} \le (1/2)|p|_{\mathcal{A}}\,,$$

which is a contradiction.

To prove Property 3, pick any two points η and ζ in $\mathcal{B}_2$ and, in view of Property 2, let $\bar{t} \in [0, T^*]$ be a time such that $U(\zeta) = g(\psi(\bar{t}, \zeta))k(\bar{t})$. Observe also that $U(\eta) \geq g(\psi(\bar{t}, \eta))k(\bar{t})$. Thus

$$U(\zeta) - U(\eta) \leq g(\psi(\bar{t}, \zeta))k(\bar{t}) - g(\psi(\bar{t}, \eta))k(\bar{t}) \leq 2|g(\psi(\bar{t}, \zeta)) - g(\psi(\bar{t}, \eta))|\,.$$

Both points $\psi(\bar{t}, \zeta)$ and $\psi(\bar{t}, \eta)$ are in $\mathcal{B}_1$. Hence, by Property 4 of g,

$$|g(\psi(\bar{t}, \zeta)) - g(\psi(\bar{t}, \eta))| \leq C|\psi(\bar{t}, \zeta) - \psi(\bar{t}, \eta)|\,.$$

Using again the properties of $\psi(t, p)$ as in the proof of the previous Proposition, we know that there exist a constant C^*, which only depends of T^* and $\mathcal{B}_2$ such that $|\psi(\bar{t}, \zeta) - \psi(\bar{t}, \eta)| \leq C^*|\zeta - \eta|$ and this yields

$$U(\zeta) - U(\eta) \leq D|\zeta - \eta|$$

with $D = 2CC^*$. Reversing the roles of ζ and η, we finally obtain

$$|U(\zeta) - U(\eta)| \leq D|\zeta - \eta|\,.$$

To prove Property 4, it is shown first that for $p \in \mathrm{int}(\mathcal{B}_2)$ and sufficiently small $h > 0$

$$\frac{U(\psi(h, p)) - U(p)}{h} \leq -\frac{1}{(1 + 2T^* + 2h)^2} U(p)\,. \tag{13}$$

In fact, set $p' = \psi(h, p)$ and let t' be a point such that $U(p') = g(\psi(t', p'))k(t')$ (recall that $0 \leq t' \leq T^*$). Then, setting $t = h + t'$, we have

$$\begin{aligned} U(p') &= g(\psi(t', p'))k(t') = g(\psi(t', \psi(h, p)))k(t') = g(\psi(t, p))k(t)\frac{k(t')}{k(t)} \\ &\leq U(p)\frac{k(t')}{k(t)} = U(p)\Big[1 - \frac{k(t) - k(t')}{k(t)}\Big] \leq U(p)\Big[1 - \frac{h}{(1 + t')(1 + 2t' + 2h)}\Big]\,. \end{aligned}$$

As $t' \leq T^*$,

$$1 - \frac{h}{(1 + t')(1 + 2t' + 2h)} \leq 1 - \frac{h}{(1 + 2T^* + 2h)^2}\,,$$

and therefore

$$U(p') \leq U(p)\Big[1 - \frac{h}{(1 + 2T^* + 2h)^2}\Big]\,.$$

This yields (13), which in turn yields (12). $\triangleleft$

As in [9], the function U can be extended to $\mathcal{D}$ in the following way.

By Property 1, it follows that there is a $d > 0$ such that $U^{-1}(d) \subset \mathrm{int}(\mathcal{B}_2)$. It easy to prove that every trajectory of (3) with initial condition in $\mathcal{D} \setminus \mathcal{A}$ intersects $U^{-1}(d)$ in a single point. In fact, consider the set

$$\mathcal{U}_d = U^{-1}([0, d])\,.$$

For any $p \in \mathcal{U}_d \setminus \mathcal{A}$, there must exist some time $t < 0$ at which $\psi(t,p) \in \mathcal{D} \setminus \mathcal{U}_d$ because, otherwise, p would be a point in $\omega(\mathcal{U}_d)$, which contradicts the fact that $\omega(\mathcal{U}_d) \subset \omega(\mathcal{P}) = \mathcal{A}$. On the other hand, for any $p \in \mathcal{D} \setminus \mathcal{U}_d$, the integral curve $\psi(t,p)$ must intersect $U^{-1}(d)$ at some time $t_p > 0$, as $|\psi(t,p)|_{\mathcal{A}} \to 0$ as $t \to \infty$. Property 4 implies that, if $\psi(t,p) \in \mathrm{int}(\mathcal{B}_2)$ for all $t \in (t_0, t_1)$,

$$D^+U(\psi(t,p)) < 0\,, \qquad \forall t \in (t_0, t_1)$$

and hence $U(\psi(t,p))$ is strictly decreasing on (t_0,t_1). Form this, and the fact that $U^{-1}(d) \subset \mathrm{int}(\mathcal{B}_2)$, it is concluded that the time t_p is unique.

Define now a function $V : \mathcal{D} \to \mathbb{R}$ as

$$V(p) = \begin{cases} U(p) & \text{if } p \in \mathcal{U}_d \\ d + t_p & \text{if } p \in \mathcal{D} \setminus \mathcal{U}_d\,. \end{cases}$$

Lemma 3.3 *The function V has the following properties.*

1. *It is continuous.*
2. *It is locally Lipschitz on $\mathcal{D} \setminus \mathcal{U}_d$.*
3. *$V^{-1}(a) = \psi(d-a, U^{-1}(d))$ for all $a \geq d$, and $\lim_{n\to\infty} V(p_n) = \infty$ for any sequence p_n in $\mathcal{D} \setminus \mathcal{U}_d$ which has its limit on $\partial\mathcal{D}$, or whose distance from $\mathcal{A}$ becomes infinite.*
4. *For all $p \in \mathcal{D} \setminus \mathcal{U}_d$,*

$$\limsup_{h \to 0^+} \frac{V(\psi(h,p)) - V(p)}{h} = -1\,. \tag{14}$$

Proof. For Properties 1,3,4, see [9]. To prove Property 2, pick a point $p \in \mathcal{D} \setminus \mathcal{U}_d$, a neighborhood $N \subset \mathcal{D} \setminus \mathcal{U}_d$ of p and two points $p_1, p_2 \in N$. Let t_1 and t_2 be the (unique) times such that

$$d = U(\psi(t_1, p_1)) = U(\psi(t_2, p_2))$$

and, without loss of generality, assume $t_2 \geq t_1$. Note that, by definition

$$|t_2 - t_1| = |V(p_2) - V(p_1)|\,.$$

Thus, since V is continuous and $\psi(t,p)$ is continuous in the argument t, if N is small enough, $\psi(t_1 - t_2, \psi(t_2, p_2)) = \psi(t_1, p_2) \in \mathrm{int}(\mathcal{B}_2)$. Using the fact that U is Lipschitz and that $\psi(t_1, p)$ is Lipschitz in the argument p, it is seen that for some M_0

$$|U(\psi(t_1,p_1)) - U(\psi(t_1,p_2))| \leq M_0 |p_1 - p_2|\,. \tag{15}$$

Next, we show that

$$U(\psi(t_2,p_2)) \leq U(\psi(t_1,p_2)) - \kappa c|t_2 - t_1|\,. \tag{16}$$

In fact, using Property 4 of U (which is legitimate because $\psi(t_1, p_2) \in \text{int}(\mathcal{B}_2)$), note that

$$D^+U(\psi(t, p_2)) \leq -\kappa U(\psi(t, p_2)) \leq -\kappa c\,, \qquad \forall t \in [t_1, t_2]$$

from which, by the Comparison Lemma (see [8]), (16) follows. Using the latter and (15) we obtain

$$\begin{aligned} c\kappa|V(p_2) - V(p_1)| = \kappa c|t_2 - t_1| &\leq U(\psi(t_1, p_2)) - U(\psi(t_2, p_2)) \\ &= U(\psi(t_1, p_2)) - U(\psi(t_1, p_1)) + U(\psi(t_1, p_1)) - U(\psi(t_2, p_2)) \\ &\leq M_0|p_1 - p_2| + d - d\,, \end{aligned}$$

which proves Property 2. $\triangleleft$

Remark. Note that, since $U^{-1}(d) \subset \text{int}(\mathcal{B}_2)$, it is possible to find a number $b > d$ such that $U^{-1}(b) \subset \text{int}(\mathcal{B}_2)$. Moreover, it is possible to find numbers number $c_1 > c_2 > d$ such that

$$V^{-1}(c_i) \subset \{p \in \mathcal{B}_2 : d < U(p) < b\} \qquad i = 1, 2. \quad \triangleleft$$

3.3 Back to system (3)

We recall a result from [10]. Consider a system

$$\dot{p} = f(t, p)\,,$$

in which $f : \mathbb{R} \times D \to \mathbb{R}^n$, with D an open set in $\mathbb{R}^n$, is continuous. Let $p : I \to \mathbb{R}^n$, with I an open interval in $\mathbb{R}$, be a solution. Let $V : D \to \mathbb{R}$ be a locally Lipschitz function. Then

$$\limsup_{h\to 0^+} \frac{1}{h}[V(p(t+h)) - V(p(t))] = \limsup_{h\to 0^+} \frac{1}{h}[V(p(t) + hf(t, p(t))) - V(p(t))]\,. \tag{17}$$

Using this result for systems (3) and (4), we get

$$\begin{aligned} \limsup_{h\to 0^+} \frac{1}{h}[V(\phi(h, p)) - V(p)] &= \limsup_{h\to 0^+} \frac{a_f(p)}{a_f(p)h}[V(p + a_f(p)h\frac{f(p)}{a_f(p)}) - V(p)] \\ &= a_f(p) \limsup_{\ell\to 0^+} \frac{1}{\ell}[V(p + \ell\frac{f(p)}{a_f(p)}) - V(p)] \\ &= a_f(p) \limsup_{\ell\to 0^+} \frac{1}{\ell}[V(\psi(\ell, p)) - V(p)]\,. \end{aligned}$$

As a consequence, since $a_f(p) \geq 1$, the function V constructed in the previous sub-section is such that

$$\limsup_{h\to 0^+} \frac{1}{h}[V(\phi(h, p)) - V(p)] \leq -1 \quad \text{for all } p \in \mathcal{D} \setminus \mathcal{U}_d\,, \tag{18}$$

in which now $\phi(t, p)$ is the flow of (3). Likewise, the function U satisfies

$$\limsup_{h\to 0^+} \frac{1}{h}[U(\phi(h, p)) - U(p)] \leq -\kappa U(p) \quad \text{for all } p \in \text{int}(\mathcal{B}_2)\,. \tag{19}$$

4 Asymptotic Regulation

In the non-equilibrium theory of nonlinear output regulation (see [1]), one is interested in the asymptotic behavior of systems of the form

$$\begin{aligned} \dot{z} &= f_0(z,w) + f_1(z,w,e)e \\ \dot{w} &= s(w) \\ \dot{e} &= h_0(z,w) + h_1(z,w,e)e - ke\,, \end{aligned} \tag{20}$$

with $z \in \mathbb{R}^n$, $w \in \mathbb{R}^r$, $e \in \mathbb{R}$, in which $f_0(z,w)$, $f_1(z,w,e)$, $s(w)$, $h_0(z,w)$, $h_1(z,w,e)$ are C^k functions, $s(w)$ and $f_0(z,w)$ are such that Assumptions 0, 1a, 1b, and 2 hold for some fixed pair of compact sets $W \subset \mathbb{R}^r$ and $Z \subset \mathbb{R}^n$, and $h_0(z,w)$ vanishes on $\mathcal{A}$. Let this system be rewritten in the form

$$\begin{aligned} \dot{p} &= f(p) + r(p,e)e \\ \dot{e} &= h(p) + [q(p,e) - k]e\,. \end{aligned} \tag{21}$$

Let E be a closed interval of $\mathbb{R}$. In what follows we prove the following result.

Proposition 4.1 *Suppose $h(p) = 0$ for all $p \in \mathcal{A}$. There exists a number k^* such that, if $k \geq k^*$, the positive orbit of $\mathcal{P} \times E$ under the flow of (21) is bounded and* $\lim_{t\to\infty} e(t) = 0$.

Let $E = [e_0, e_1]$ and set $E_1 = [e_0 - 1, e_1 + 1]$. Pick a number $a > 0$ such that $\mathcal{P} \subset V^{-1}([0,a])$, which is possible because V is proper on $\mathcal{D}$, and define

$$h_0 = \max_{p \in V^{-1}([0,a+1])} |h(p)|, \qquad q_0 = \max_{p \in V^{-1}([0,a+1])\,, e \in E_1} |q(p,e)|$$

Fix $0 < \delta < 1$, pick μ such that $\mu h_0^2 = \delta^2$ and pick k so that

$$\lambda_k := k - q_0 - 1/8\mu > 1\,.$$

Then, standard arguments show that, *so long as* $p(t) \in V^{-1}([0,a+1])$,[1]

$$|e(t)| \leq \exp(-\lambda_k t)|e(0)| + \delta\,. \tag{22}$$

This shows, in particular, that for any $\varepsilon > 0$ and any $T > 0$, there is a number $k^*_{\varepsilon,T}$ such that if, $k \geq k^*_{\varepsilon,T}$, then $|e(t)| \leq \varepsilon$ for all $t \geq T$, *provided that* $p(t) \in V^{-1}([0,a+1])$ *for all* $t \geq 0$.

Pick numbers b and c_1, c_2 as in specified in the Remark at the end of the sub-section 3.2 and note that, since the function V is locally Lipschitz on $\mathcal{D} \setminus \mathcal{U}_d$ and the set

$$\mathcal{S} = \{p \in \mathcal{D} : c_2 \leq V(p) \leq a+1\}$$

is compact, there is a number $\bar{L}$ such that

$$|V(p) - V(q)| \leq \bar{L}|p - q| \qquad \text{for all } p, q \in \mathcal{S}.$$

[1] See e.g. [5].

Moreover, from (18), it is seen that

$$\limsup_{h\to 0^+} \frac{1}{h}[V(\phi(h,p)) - V(p)] \leq -1 \qquad \text{for all } p \in \mathcal{S},$$

in which $\phi(h,p)$ is the flow of (3). Finally, let

$$r_0 = \max_{p\in V^{-1}([0,a+1]),\, e\in E_1} |r(p,e)|\,.$$

Note that, as $p(0)$ ranges on a compact set, contained in $V^{-1}([0,a])$, there is a time T (independent of $(p(0), e(0))$) such that, for all $t \in [0,T]$, $p(t)$ exists and satisfies $p(t) \in V^{-1}([0, a+1/2])$ for all $t \in [0,T]$. Thus, on this time interval (22) holds.

Note that (see (17) and (18)), so long as $p(t) \in \mathcal{S}$ and $e(t) \in E_1$,

$$\begin{aligned}
D^+V(p(t)) &= \limsup_{h\to 0^+} \frac{1}{h}[V(p(t+h)) - V(p(t))] \\
&= \limsup_{h\to 0^+} \frac{1}{h}[V(p(t) + hf(p(t)) + hg(p(t),e(t))e(t)) - V(p(t))] \\
&= \limsup_{h\to 0^+} \frac{1}{h}[V(p(t) + hf(p(t)) + hg(p(t),e(t))e(t)) - V(p(t) + hf(p(t))] \\
&\quad + \limsup_{h\to 0^+} \frac{1}{h}[V(p(t) + hf(p(t))) - V(p(t))] \\
&\leq \limsup_{h\to 0^+} \frac{1}{h}\bar{L}|hg(p(t),e(t))e(t))| + \limsup_{h\to 0^+} \frac{1}{h}[V(\phi(h,p(t))) - V(p(t))] \\
&\leq \bar{L}r_0|e(t)| - 1\,.
\end{aligned}$$

Pick $\varepsilon > 0$ such that $\bar{L}r_0\varepsilon \leq 1/2$, and let $k \geq k^*_{\varepsilon,T}$. Then, it is seen that, so long as $p(t) \in \mathcal{S}$,

$$D^+V(p(t)) \leq -1/2\,. \tag{23}$$

This proves that $V(p(t))$ is strictly decreasing for $t > T$ and hence $p(t) \in V^{-1}([0, a+1])$ for all $t \geq 0$. Moreover, it also proves that in finite time $p(t)$ intersects $V^{-1}(c_1)$. In fact, (23) implies

$$V(t) \leq V(T) - \frac{1}{2}a_0 t \leq a + \frac{1}{2} - \frac{1}{2}(t-T)$$

for $t \geq T$ and therefore $p(t)$ must enter the set $V^{-1}([0,c_1])$ at some time $\bar{t} \leq 2(a-c_1)+T+1$. Inequality (23) also proves that the set $V^{-1}([0,c_1]) \times \{e \in \mathbb{R} : |e| \leq \varepsilon\}$ is invariant in positive time under the flow of (21).

It remains to show that $e(t) \to 0$ as $t \to \infty$. This is a direct consequence of the small-gain theorem for input-to-state stable systems. First of all note that a calculation identical to the one above leading to (23) leads (using this time (19), which is legitimate because $p(t) \in \mathrm{int}(\mathcal{B}_2)$) to

$$D^+U(p(t)) \leq -\kappa U(p(t)) + \bar{L}r_0|e(t)|\,. \tag{24}$$

Thus, by the Comparison Lemma (see [8]),

$$U(p(t)) \leq e^{-\kappa(t-t_0)}U(p(t_0)) + \frac{\bar{L}r_0}{\kappa} \max_{t\in[t_0,t]} |e(t)|$$

for all $t \geq t_0$. This, in view of the estimates (11), yields

$$|p(t)|_{\mathcal{A}} \leq 2Me^{-\kappa(t-t_0)}|p(t_0)|_{\mathcal{A}} + \gamma \max_{t\in[t_0,t]} |e(t)| \tag{25}$$

in which $\gamma = M\bar{L}r_0/\kappa$

On the other hand, assuming $k > q_0$ it is easily seen that

$$|e(t)| \leq e^{-(k-q_0)(t-t_0)}|e(t_0)| + \frac{1}{(k-q_0)} \max_{t\in[t_0,t]} |h(p(t))| \,.$$

Recall now that $h(p)$, a smooth function, vanishes on $\mathcal{A}$. Thus, there exists a number β such that $|h(p)| \leq \beta|p|_{\mathcal{A}}$ for all $p \in \mathcal{B}_2$. Thus,

$$|e(t)| \leq e^{-(k-q_0)(t-t_0)}|p(t_0)| + \frac{\beta}{(k-q_0)} \max_{t\in[t_0,t]} |p(t)|_{\mathcal{A}} \,. \tag{26}$$

At this point, comparing (25) and (26), the classical arguments of the small-gain theorem for input-to-state stable systems prove that, if

$$k > \gamma\beta + q_0$$

then $e(t) \to 0$ and $|p(t)|_{\mathcal{A}} \to 0$ as $t \to \infty$. This proves the Proposition.

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