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Remarks on Equivalence between Full Order and Reduced Order Nonlinear Observers

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***Abstract*— Motivated by the fact that, for linear systems, existence conditions for a full order observer and for a reduced order observer are the same, we study relationship between full order and reduced order observers for general nonlinear systems. We employ coordinate transformations for dealing with reduced order observers. By restricting the change of coordinates to be linear, we obtain some equivalence between full order and reduced order observers and relationship between two corresponding error Lyapunov functions.**

I. INTRODUCTION

For linear systems, it is well-known that the existence of a full order (Luenberger) observer implies the existence of a reduced order observer, and vice versa. Motivated by this fact, we study relationship between full order and reduced order observers for nonlinear systems. Our approach is to investigate when the existence of a full order observer imply the existence of a reduced order observer and its converse, in view of error Lyapunov functions which are employed in order to analyze the convergence of their estimation error. This paper is to show that, if a reduced order observer exists for a general nonlinear system, a full order one is always constructed without any additional restrictions such as global Lipschitz or growth conditions that have frequently appeared in the literature (e.g., [1–4]). For the converse, we show that the existence of a full order observer implies the existence of a reduced order one if an error Lyapunov function for the full order observer satisfies a structural condition that will be presented in this paper.

We consider a system whose output is a partial state of the system, with which the whole state is divided into measurable and unmeasurable components. Then, a reduced order observer is constructed through a partial coordinate transformation of the system. The change of coordinates needs to be partial so that the measurable state remains unaltered and will be discarded after the coordinate change for constructing a reduced order observer. As an answer to the question how to construct a full order observer when we have the coordinate transformation for a reduced order observer, our idea is to design an additional estimator for the measurable state, which is robust to the estimation error for the unmeasurable state. In order to design this estimator, we have to incorporate nonlinear growth of the system equation. Then an error Lyapunov function will be constructed for the pair of the reduced order observer and the additional estimator. Finally, by changing the coordinate back to the original one, we obtain a full order observer with the corresponding error Lyapunov function. For the converse question, we show that the necessary coordinate transformation for a reduced order observer can be obtained from an error Lyapunov function for the full order observer. Then, the stability of the reduced order error dynamics is also ensured by manipulating the error Lyapunov function for the full order error dynamics.

In this paper, we denote the Euclidean norm of a vector by $|\cdot|$. For two column vectors x_1 and x_2, a simple notation $|(x_1,x_2)|$ implies $|[x_1^T,x_2^T]^T|$. When a function is said to be smooth, it means it is at least continuously differentiable. Finally, a system $\dot{z} = F_z(z,d)$ is said to be *incrementally uniformly globally asymptotically stable(UGAS)* with respect to z if there exists a class-$\mathcal{KL}$ function β such that

$$|z(t,\xi_1,d(t)) - z(t,\xi_2,d(t))| \le \beta(|\xi_1-\xi_2|,t),$$

for all ξ_1 and ξ_2 and for all admissible external input $d(\cdot)$, where $z(t,\xi,d(t))$ is the solution of the system corresponding the initial condition ξ and the input d.

II. EQUIVALENCE UNDER LINEAR CHANGE OF COORDINATES

We consider a system described by $\dot{x} = f(x)$ and $y = x_1$; that is,

$$\begin{aligned} \dot{x}_1 &= f_1(x_1,x_2), \qquad y = x_1, \\ \dot{x}_2 &= f_2(x_1,x_2), \end{aligned} \tag{1}$$

where $y = x_1 \in \mathbb{R}^p$, which is the measured output, and $x_2 \in \mathbb{R}^{n-p}$. Here, f_1 and f_2 are assumed to be locally Lipschitz. Since an *asymptotic* observer is of interest whose estimate converges to the true state of (1) as time goes to infinity, system (1) is assumed to be forward complete.

In order to formalize both full order and reduced order observers that will be used throughout the paper, we begin by their definitions.

Definition 1: System (1) is said to *admit a full order observer with a state-dependent error Lyapunov function (SDELF)* if there exist a locally Lipschitz function $F : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ and a smooth function $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that

$$\alpha_1(|e|) \le V(e,x) \le \alpha_2(|e|), \qquad \forall e,x \tag{2}$$

where α_1 and α_2 are class-$\mathcal{K}_\infty$ functions, and that

$$\frac{\partial V}{\partial e}(e,x)[F(e+x,x_1)-f(x)]+\frac{\partial V}{\partial x}(e,x)f(x)\le-\alpha_3(|e|), \tag{3}$$

for all e and x, where α_3 is a positive definite function.

A straightforward consequence of the definition is that a dynamic system given by

$$\dot{\hat{x}}=F(\hat{x},y), \qquad \hat{x}\in\mathbb{R}^n, \tag{4}$$

guarantees that the estimate $\hat{x}(t)$ converges to the state $x(t)$ (that is, $e(t):=\hat{x}(t)-x(t)$ converges to zero) uniformly with respect to the initial condition $x(0)$, and thus, system (4) is said to be a full order observer for (1). Note that we call an observer 'full order' when its state $\hat{x}$ is an estimate of x (not just when the order of the observer is n).

Definition 2: System (1) is said to *admit a reduced order observer with an SDELF under a linear change of coordinates* if there exists a partial linear coordinate transformation

$$z_2=x_2+K_1x_1, \tag{5}$$

by which the reduced order system

$$\dot{z}_2=f_2(x_1,z_2-K_1x_1)+K_1f_1(x_1,z_2-K_1x_1)=:F_z(z_2,x_1)$$

is incrementally UGAS with respect to z_2, and there exists a smooth function $V_z:\mathbb{R}^{n-p}\times\mathbb{R}^n\to\mathbb{R}$ such that

$$\alpha_a(|e_z|)\le V_z(e_z,x)\le\alpha_b(|e_z|), \qquad \forall e_z,x \tag{6}$$

where α_a and α_b are class-$\mathcal{K}_\infty$ functions, and that

$$\frac{\partial V_z}{\partial e_z}(e_z,x)[F_z(e_z+z_2,x_1)-F_z(z_2,x_1)] + \frac{\partial V_z}{\partial x}(e_z,x)f(x)\le-\alpha_c(|e_z|) \tag{7}$$

for all e_z, x with $z_2=x_2+K_1x_1$, where α_c is a positive definite function.

Under this definition, a reduced order observer for system (1) is simply given by

$$\dot{\hat{z}}_2=F_z(\hat{z}_2,y), \tag{8}$$

$$\hat{x}_2=\hat{z}_2-K_1y. \tag{9}$$

Indeed, since system (8) equals $\dot{\hat{z}}_2=F_z(\hat{z}_2,x_1)$, inequalities (6) and (7), with $e_z=\hat{z}_2-z_2$, guarantees that $e_z(t)\to0$. Therefore, with (9), system (8) is a reduced order observer.

Now we state the main result of the paper assuming $e_1\in\mathbb{R}^p$ and $e_2\in\mathbb{R}^{n-p}$.

Theorem 1: The following two statements are equivalent:

1) System (1) admits a full order observer with an SDELF that satisfies

$$\frac{\partial V}{\partial e_1}(0,e_2,x)=\frac{\partial V}{\partial e_2}(0,e_2,x)K, \qquad \forall e_2,x \tag{10}$$

with some matrix $K\in\mathbb{R}^{(n-p)\times p}$.

2) System (1) admits a reduced order observer with an SDELF under a linear change of coordinates.

Remark 1: Regarding the implication of $(1\Rightarrow2)$, what is already known in the literature [1], [2], [5] is that, if a full order observer exists with a *quadratic* error Lyapunov function, then a reduced order observer can be constructed. By *quadratic* error Lyapunov function, we mean a positive definite function

$$V(e_1,e_2)=\frac{1}{2}\begin{bmatrix}e_1\\e_2\end{bmatrix}^T\begin{bmatrix}P_1 & P_2\\P_2^T & P_3\end{bmatrix}\begin{bmatrix}e_1\\e_2\end{bmatrix},$$

where $e_1=\hat{x}_1-x_1$ and $e_2=\hat{x}_2-x_2$. It is obvious that this function satisfies (10) with $K=P_3^{-1}P_2^T$ (and it will be seen that a linear change of coordinates $z_2=x_2+P_3^{-1}P_2^Tx_1$ will yield a reduced order observer). We comment here that many nonlinear (full order) observer design methods in the literature (e.g., [2–4], [6], [7]) actually leads to a quadratic error Lyapunov function, and thus, the design of reduced order observer easily follows from their construction of full order observer.

Proof: $(1\Rightarrow2)$. Full order observer (4) is rewritten here for convenience

$$\begin{aligned}\dot{\hat{x}}_1&=F_1(\hat{x}_1,\hat{x}_2,x_1)\\ \dot{\hat{x}}_2&=F_2(\hat{x}_1,\hat{x}_2,x_1).\end{aligned} \tag{11}$$

By virtue of (2) and (3), it is seen that, once the estimate $\hat{x}(t)$ of (11) gets equal to the true state $x(t)$ (i.e., $e(t)=0$), they must remain the same afterwards. This necessarily implies that $F_i(x_1,x_2,x_1)=f_i(x_1,x_2)$, $i=1,2$, for all x_1 and x_2. Therefore, if we let $e_1=\hat{x}_1-x_1=0$, inequality (3) becomes

$$\begin{aligned}&\frac{\partial V}{\partial e_1}(0,e_2,x)[f_1(x_1,e_2+x_2)-f_1(x_1,x_2)]\\ &\quad+\frac{\partial V}{\partial e_2}(0,e_2,x)[f_2(x_1,e_2+x_2)-f_2(x_1,x_2)]\\ &\quad\quad+\frac{\partial V}{\partial x}(0,e_2,x)f(x)\ \le\ -\alpha_3(|e_2|)\end{aligned}$$

for all e_2 and x, since $F_i(x_1,\hat{x}_2,x_1)=f_i(x_1,\hat{x}_2)=f_i(x_1,e_2+x_2)$ for $i=1,2$. Under the condition (10), we have

$$\begin{aligned}&\frac{\partial V}{\partial e_2}(0,e_2,x)[(f_2(x_1,e_2+x_2)-f_2(x_1,x_2))\\ &\quad+K(f_1(x_1,e_2+x_2)-f_1(x_1,x_2))]\\ &\quad\quad+\frac{\partial V}{\partial x}(0,e_2,x)f(x)\ \le\ -\alpha_3(|e_2|).\end{aligned} \tag{12}$$

On the other hand, by taking $z_2=x_2+Kx_1$, we obtain from (1)

$$\dot{z}_2=f_2(x_1,z_2-Kx_1)+Kf_1(x_1,z_2-Kx_1).$$

Then, it can be shown that this system is incrementally UGAS with respect to z_2 by a corresponding Lyapunov function $V_z(e_z,x)=V(0,e_z,x)$. In fact, with this $V_z(e_z,x)$ and $K_1=K$, the inequality (7) is nothing but (12). The proof completes with $\alpha_a(\cdot)=\alpha_1(\cdot)$, $\alpha_b(\cdot)=\alpha_2(\cdot)$ and $\alpha_c(\cdot)=\alpha_3(\cdot)$

since $\alpha_a(|e_z|) = \alpha_1(|(0,e_z)|) = \alpha_1(|e_z|)$ and so on. We finally obtain a reduced order observer

$$\dot{\hat{z}}_2 = f_2(y,\hat{z}_2 - Ky) + K f_1(y,\hat{z}_2 - Ky), \qquad \hat{x}_2 = \hat{z}_2 - Ky.$$

($2 \Rightarrow 1$). Outline of the proof is as follows. We first show a dynamic system

$$\begin{aligned} \dot{\hat{x}}_1 &= f_1(y,\hat{z}_2 - K_1 y) - L_1(y,\hat{x}_1,\hat{z}_2), \\ \dot{\hat{z}}_2 &= f_2(y,\hat{z}_2 - K_1 y) + K_1 f_1(y,\hat{z}_2 - K_1 y), \end{aligned} \tag{13}$$

where L_1 will be constructed, is an observer for the system (1) in (x_1,z_2)-coordinates, that is,

$$\begin{aligned} \dot{x}_1 &= f_1(x_1,z_2 - K_1 x_1), \qquad y = x_1, \\ \dot{z}_2 &= f_2(x_1,z_2 - K_1 x_1) + K_1 f_1(x_1,z_2 - K_1 x_1). \end{aligned} \tag{14}$$

In particular, we prove uniform convergence of $e_1(t) = \hat{x}_1(t) - x_1(t)$ and $e_z(t) = \hat{z}_2(t) - z_2(t)$ to zero by an error Lyapunov function $\bar{V}(e_1,e_z,x)$ such that $\bar{\alpha}_1(|(e_1,e_z)|) \le \bar{V}(e_1,e_z,x) \le \bar{\alpha}_2(|(e_1,e_z)|)$ and $\frac{d}{dt}\bar{V} \le -\bar{\alpha}_3(|(e_1,e_z)|)$, where $\bar{\alpha}_1$ and $\bar{\alpha}_2$ are class-$\mathscr{K}_\infty$ functions and $\bar{\alpha}_3$ is a positive definite function. Then, by virtue of the coordinate transformation (5) and a similar transformation of the estimate:

$$\hat{x}_2 := \hat{z}_2 - K_1 \hat{x}_1, \tag{15}$$

we obtain a full order observer[1]

$$\begin{aligned} \dot{\hat{x}}_1 &= f_1(y,\hat{x}_2 + K_1(\hat{x}_1 - y)) - L_1(y,\hat{x}_1,\hat{x}_2 + K_1(\hat{x}_1 - y)) \\ \dot{\hat{x}}_2 &= f_2(y,\hat{x}_2 + K_1(\hat{x}_1 - y)) + K_1 \cdot L_1(y,\hat{x}_1,\hat{x}_2 + K_1(\hat{x}_1 - y)) \end{aligned}$$

and an SDELF $V(e_1,e_2,x) = \bar{V}(e_1,e_2 + K_1 e_1,x)$ which satisfies (2) and (3), where $e_2 = \hat{x}_2 - x_2$, with $\alpha_i(|(e_1,e_2)|) = \bar{\alpha}_i(|(e_1,e_2 + K_1 e_1)|)$, $i = 1,2,3$. Therefore, the rest of proof will concentrate on the construction of such L_1 and $\bar{V}$. Finally, we will show the condition (10) holds with the obtained SDELF V.

From the local Lipschitz property of f_1, there exists a continuous nonnegative function l_f such that

$$|f_1(y,\hat{z}_2 - K_1 y) - f_1(y,(\hat{z}_2 - e_z) - K_1 y)| \le l_f(y,\hat{z}_2,e_z)|e_z|. \tag{16}$$

Then, we have

$$l_f(y,\hat{z}_2,e_z) \le \beta_{f,1}(|y| + |\hat{z}_2|) + \beta_{f,2}(|e_z|) \tag{17}$$

where $\beta_{f,1}$ and $\beta_{f,2}$ are locally Lipschitz, nondecreasing functions satisfying

$$\beta_{f,1}(s) \ge \sup\{l_f(y,\hat{z}_2,e_z) : |e_z| \le |y| + |\hat{z}_2| \le s\}, \tag{18}$$

$$\beta_{f,2}(s) \ge \sup\{l_f(y,\hat{z}_2,e_z) : |y| + |\hat{z}_2| \le |e_z| \le s\}. \tag{19}$$

Without loss of generality, we also assume that $\beta_{f,1}(0) > 0$. Here, for convenience we define a function δ_f as

$$\delta_f(|e_z|) := 1 + \beta_{f,2}(|e_z|)$$

[1]This form of observer has error injection terms inside the function arguments, which recalls the contribution of [7].

and a class-$\mathscr{K}_\infty$ function ψ as

$$\psi(s) := \begin{cases} \frac{\eta(s)}{\delta_f(1)}, & \text{if} \quad 0 \le s \le 1, \\ \frac{\eta(1)}{\delta_f(1)} + (s-1), & \text{if} \quad 1 < s \end{cases} \tag{20}$$

where η is a class-$\mathscr{K}$ function such that $\alpha_c(s) \ge \eta(s)$ for all $s \in [0,1]$ whose existence is guaranteed since α_c is positive definite [8, Thm. 5.7.1].

Claim 1: There exist a locally Lipschitz function λ_f and a class-$\mathscr{K}_\infty$ function σ satisfying

$$\begin{aligned} &\frac{\sigma(|e_1|)}{|e_1|} e_1^T [f_1(y,\hat{z}_2 - K_1 y) - f_1(y,(\hat{z}_2 - e_z) - K_1 y)] \\ &\qquad \le |e_1|\sigma(|e_1|)\lambda_f(y,\hat{z}_2,e_1) + |e_z|\psi(|e_z|)\delta_f(|e_z|). \end{aligned} \tag{21}$$

(*Proof of Claim 1*): From (16) and (17), it can be seen[2] that

$$\begin{aligned} &\frac{\sigma(|e_1|)}{|e_1|} e_1^T [f_1(y,\hat{z}_2 - K_1 y) - f_1(y,(\hat{z}_2 - e_z) - K_1 y)] \\ &\quad \le \sigma(|e_1|)\,|e_z|\,[\beta_{f,1}(|y| + |\hat{z}_2|) + \beta_{f,2}(|e_z|)] \\ &\quad \le \sigma(|e_1|)\beta_{f,1}(*)\psi^{-1}(\sigma(|e_1|)\beta_{f,1}(*)) \\ &\qquad + \sigma(|e_1|)\psi^{-1}(\sigma(|e_1|))\beta_{f,2}(\psi^{-1}(\sigma(|e_1|))) \\ &\qquad + |e_z|\psi(|e_z|)[1 + \beta_{f,2}(|e_z|)] \end{aligned}$$

where $(*)$ implies $(|y| + |\hat{z}_2|)$. On the other hand, since ψ^{-1} is a class-$\mathscr{K}_\infty$ function, there exists[3] a class-$\mathscr{K}_\infty$ function $\bar{\psi}$ such that

$$\psi^{-1}(ab) \le \bar{\psi}(a)\bar{\psi}(b), \qquad \psi^{-1}(a) \le \bar{\psi}(a), \qquad \forall a,b \ge 0,$$

and $\bar{\psi}(\cdot)$ is locally Lipschitz on $(0,\infty)$. Then, the above inequality can proceed as

$$\begin{aligned} \ldots &\le \sigma(|e_1|)\beta_{f,1}(*)\bar{\psi}(\sigma(|e_1|))\bar{\psi}(\beta_{f,1}(*)) \\ &\quad + \sigma(|e_1|)\bar{\psi}(\sigma(|e_1|))\beta_{f,2}(\bar{\psi}(\sigma(|e_1|))) \\ &\quad + |e_z|\psi(|e_z|)[1 + \beta_{f,2}(|e_z|)]. \end{aligned}$$

Let us now pick $\sigma(\cdot) = \bar{\psi}^{-1}(\cdot)$ and take

$$\begin{aligned} \lambda_f(y,\hat{z}_2,e_1) &= \beta_{f,1}(|y| + |\hat{z}_2|)\bar{\psi}(\beta_{f,1}(|y| + |\hat{z}_2|)) \\ &\quad + \beta_{f,2}(|e_1|), \end{aligned}$$

which leads to the inequality (21). Here, λ_f is locally Lipschitz because the nondecreasing function $\beta_{f,1}(\cdot)$ is such that $\beta_{f,1}(0) > 0$ and $\bar{\psi}$ is locally Lipschitz away from zero. This completes the proof.

[2]Recall the inequalities

$$ab \le a\psi^{-1}(a) + b\psi(b), \qquad \forall a,b \ge 0$$

$$cd\beta_{f,2}(d) \le d\psi(d)\beta_{f,2}(d) + c\psi^{-1}(c)\beta_{f,2}(\psi^{-1}(c)), \qquad \forall c,d \ge 0$$

for any class-$\mathscr{K}_\infty$ function ψ and nondecreasing function $\beta_{f,2}$. We can proceed the proof by putting $a = \sigma(|e_1|)\beta_{f,1}(*)$, $c = \sigma(|e_1|)$ and $b = d = |e_z|$.

[3]Refer to [9, Corollary 10] for details.

Now we pick

$$L_1(y,\hat{x}_1,\hat{z}_2) = e_1 + \lambda_f(y,\hat{z}_2,e_1)e_1 \tag{22}$$

for the observer (13) and construct $\bar{V}(e_1,e_z,x)$ as follows.

Claim 2: There exists a class-$\mathscr{K}_\infty$ function $\rho(\cdot)$ such that the function

$$\bar{V}(e_1,e_z,x) := \int_0^{|e_1|} \sigma(s)ds + 2\int_0^{V_z(e_z,x)} \rho(s)ds \tag{23}$$

satisfies

$$\frac{d}{dt}\bar{V} \le -\sigma(|e_1|)|e_1| - \rho(\alpha_a(|e_z|))\alpha_c(|e_z|), \tag{24}$$

where σ is given in Claim 1.
(*Proof of Claim 2*): The time derivative along the trajectories of (13) and (14) is obtained from (21) as

$$\begin{aligned}\frac{d}{dt}\bar{V} &\le |e_1|\sigma(|e_1|)\lambda_f(y,\hat{z}_2,e_1) + |e_z|\psi(|e_z|)\delta_f(|e_z|)\\ &\quad - \frac{\sigma(|e_1|)}{|e_1|}e_1^T L_1(y,\hat{x}_1,\hat{z}_2) - 2\rho(V_z(e_z,x))\alpha_c(|e_z|)\\ &= -\sigma(|e_1|)\,|e_1| + |e_z|\psi(|e_z|)\delta_f(|e_z|)\\ &\quad - 2\rho(V_z(e_z,x))\alpha_c(|e_z|).\end{aligned}$$

Therefore, the claim is proved if there exists a class-$\mathscr{K}_\infty$ function ρ satisfying

$$\begin{aligned}\rho(V_z(e_z,x))\alpha_c(|e_z|) &\ge \rho(\alpha_a(|e_z|))\alpha_c(|e_z|)\\ &\ge |e_z|\psi(|e_z|)\delta_f(|e_z|). \end{aligned}\tag{25}$$

Indeed, we can choose a class-$\mathscr{K}_\infty$ function ρ as

$$\rho(\tau) = \begin{cases} \alpha_a^{-1}(\tau) & \text{if } 0 \le \alpha_a^{-1}(\tau) \le 1,\\ \alpha_a^{-1}(\tau) + \frac{\alpha_a^{-1}(\tau)\psi(\alpha_a^{-1}(\tau))\delta_f(\alpha_a^{-1}(\tau))}{\alpha_c(\alpha_a^{-1}(\tau))} - \frac{\psi(1)\delta_f(1)}{\alpha_c(1)} & \\ & \text{if } 1 < \alpha_a^{-1}(\tau).\end{cases}$$

By the choice of ρ, it follows from (20) that, for $0 \le s \le 1$,

$$\rho(\alpha_a(s))\alpha_c(s) \ge s\alpha_c(s) \ge s\eta(s) \ge s\psi(s)\delta_f(s),$$

and for $1 < s$,

$$\begin{aligned}\rho(\alpha_a(s))\alpha_c(s) &= \left[s + \frac{s\psi(s)\delta_f(s)}{\alpha_c(s)} - \frac{\psi(1)\delta_f(1)}{\alpha_c(1)}\right]\alpha_c(s)\\ &= s\psi(s)\delta_f(s) + \left[s - \frac{\psi(1)\delta_f(1)}{\alpha_c(1)}\right]\alpha_c(s)\\ &\ge s\psi(s)\delta_f(s).\end{aligned}$$

(Note that $\psi(1)\delta_f(1) \le \alpha_c(1)$ by the choice of ψ in (20).) This proves the inequality (25).

From (23) and (24), it is seen that there exist $\bar{\alpha}_i$ $(i = 1,2,3)$ such that $\bar{\alpha}_1(|(e_1,e_z)|) \le \bar{V}(e_1,e_z,x) \le \bar{\alpha}_2(|(e_1,e_z)|)$ and that

$$\frac{d}{dt}\bar{V} \le -\sigma(|e_1|)|e_1| - \rho(\alpha_a(|e_z|))\alpha_c(|e_z|) \le -\bar{\alpha}_3(|(e_1,e_z)|).$$

Finally, it is not difficult to show from (23) that the condition (10) holds with $K = K_1$ for $V(e_1,e_2,x) = \bar{V}(e_1,e_2 + K_1e_1,x)$ since

$$\begin{aligned}\frac{\partial V}{\partial e_1}(e_1,e_2) &= \sigma(|e_1|)\frac{\partial |e_1|}{\partial e_1}\\ &\quad + 2\rho(V_z(e_1,e_2+K_1e_1,x))\frac{\partial V_z}{\partial e_z}(e_2+K_1e_1)K_1\end{aligned}$$

and

$$\frac{\partial V}{\partial e_2}(e_1,e_2) = 2\rho(V_z(e_1,e_2+K_1e_1,x))\frac{\partial V_z}{\partial e_z}(e_2+K_1e_1).$$

■

III. CONCLUSION

In this paper we have presented a result of equivalence between full order and reduced order nonlinear observers. A straightforward corollary is that, if one obtains a full order observer with an error Lyapunov function that satisfies the proposed condition (10) (e.g., a quadratic error Lyapunov function), a reduced order observer is always constructed. And, if one obtains a reduced order observer (which may be more easily done since the order of the system is reduced), a full order observer can also be constructed through the proposed procedure.

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