Proceedings of the 42nd IEEE
Conference on Decision and Control
Maui, Hawaii USA, December 2003

On Global Output Feedback Stabilization of Uncertain Nonlinear Systems

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***Abstract*— This paper deals with the problem of global asymptotic stabilization of nonlinear systems by means of linear, high-gain, dynamic output feedback. The contribution of this paper is to show that linear high-gain control together with high-gain observer is enough to globally stabilize strongly nonlinear systems. The novelty is to introduce a (dynamic high) gain generated by an appropriate nonlinear filter. The key idea behind analysis and controller design is to apply ISS and nonlinear small-gain techniques.**

I. INTRODUCTION

Output feedback of nonlinear systems is a problem of paramount importance in control engineering. Some well-known challenging facts are the lack of a global "Separation Principle" and a systematic observer design for genuinely nonlinear systems. Our goals here are not to address any specific engineering applications nor to come up with a general solution in (applied) mathematics. Instead, this paper is addressed to engineers *at the conceptual level.* Specifically, we will show that linear control can still deal with nonlinear systems provided that a single dynamic gain is appropriately tuned. This has some resemblance with gain scheduling.

We approach our objective by considering a class of nonlinear systems whose dynamics are described by the trianguler form

$$\begin{aligned}
\dot z &= q(z,y)\\
\dot x_1 &= x_2 + \delta_1(z,x_1)\\
&\vdots\\
\dot x_i &= x_{i+1} + \delta_i(z,x_1,\ldots,x_i)\ .\\
&\vdots\\
\dot x_n &= u - u^\star + \delta_n(z,x_1,\ldots,x_n)\\
y &= x_1
\end{aligned} \tag{1}$$

where $u,y \in \mathbb{R}$ are the input and output, $u^\star$ is an unknown constant and $(z,x) \in \mathbb{R}^{n_0}\times\mathbb{R}^n$ is the state. Only the output y is available for feedback. The presence of $u^\star$ is motivated by the fact that, in some cases, the value of the control, related to a desired equilibrium point, may be unknown, such as in the case of set-point regulation or in the presence of sensor disturbance [2].

Throughout this paper, the following two hypotheses are imposed.

Hypothesis (H1):
(H1.1) The z-system in (1) is input-to-state stable (ISS) [14], [12]. Namely, there exist a positive definite and radially unbounded function V_z and a class $\mathcal{K}$ function γ satisfying

$$\frac{\partial V_z}{\partial z}(z)q(z,y) \le -V_z(z) + \gamma(|y|)\ . \tag{2}$$

(H1.2) There exist a Lipschitz continuous nonnegative function L and a class $\mathcal{K}$ function κ, satisfying

$$|\delta_i(z,x_1,\ldots,x_i)| \le L(|y|)\,(|x_1| + \cdots + |x_i|) + \kappa(V_z(z)) \tag{3}$$

$\forall i \in \{1,\ldots,n\}$.
(H1.3) There exist strictly positive real numbers k and s_0 such that

$$\kappa(2\gamma(s)) \ \le\ k\,s \quad \forall s \in [0,s_0]\ . \tag{4}$$

Hypothesis (H2):
(H2.1) There exist an integer m and a positive real number p satisfying[1]

$$L(s) \ +\ \kappa(2\gamma(s)) \ \le\ p \ +\ s^m \quad \forall s \ge 0\ . \tag{5}$$

(H2.2) The functions γ and κ are C^1 on $(0,+\infty)$ and there exists a real number $\ell \ge 1$ such that

$$\ell\, s\,\kappa'(s) \ \ge\ \kappa(s) \quad \forall s > 0\ . \tag{6}$$

Under the above hypotheses, the origin $(z,x) = (0,0)$ is an equilibrium point for the open-loop system (1) with $u = u^\star$. The control problem of interest is to design a *linear* output feedback controller to render this solution globally Lagrange stable and attractive.

Comments

1) The low-triangularity condition appears in numerous papers previously published on the topic of robust

*This work was done in part while Laurent Praly was visiting the Institut Mittag-Leffler from The Royal Swedish Academy of Sciences.
**The work of the second author has been supported partly by the Othmer Institute for Interdisciplinary Studies of Polytechnic University, and the National Science Foundation under grants ECS-0093176 and INT-9987317.

[1] With the hypothesis (H1.3), p could be replaced by ks. But we want to stress here that the constraint is for large s only.

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and adaptive nonlinear control via state and output feedback; see [6], [7], [15] and references therein. A common feature of earlier work is that the functions δ_i's and q are output nonlinearities, i.e., the unmeasured states occur linearly. This restriction is completely removed for the state z of the inverse dynamics when they are ISS, as expressed by (H1.1) (see [11]).

2) It should be emphasized that our design of a desired (linear!) dynamic high-gain output feedback law does *not* require the nonlinearities to be globally Lipschitz [4], or linearly bounded [13]. In addition, the controller structure is simpler than [13], and is linear as opposed to the nonlinear controller [5], which uses the idea of dynamic gain introduced in [10]. Certainly, the use of a nonlinear controller would allow us theoretically to handle a larger class of nonlinear systems.
3) Hypothesis (H1) is reminiscent of previous work involving ISS and nonlinear small-gain techniques for partial-state and output feedback; see, for instance, [14], [11], [2], [3], [1].

II. OUTPUT FEEDBACK DESIGN

The output feedback we propose is made of a linear high gain (partial) observer and a linear high gain controller.

For the design of these two blocks, we select a real number a and compute a set of data (d_0, d_1, Q, P, K, F) according to the following Lemma which was already announced in [9] and whose proof is given in the Appendix.

Let I_i be the identity matrix of order i, and set

$$\begin{array}{rcl} A_i & = & \begin{pmatrix} 0 & \\ \vdots & I_{i-1} \\ 0 & 0\cdots 0 \end{pmatrix} , \quad B = \mathrm{col}(0,\ldots,0,1) \in \mathbb{R}^n , \\ C & = & \mathrm{col}(1,0,\ldots,0) \in \mathbb{R}^{n+1} , \\ D_i & = & \mathrm{diag}(0,1,\ldots,i-1) . \end{array} \tag{7}$$

We have

Lemma 1: For any strictly positive real number a, there exist real numbers d_0 and d_1, symmetric matrices P and Q, and column and row vectors K and F satisfying the following set of inequalities :

$$\begin{array}{c} 0 < d_0 , \quad 0 \leq d_1 , \quad 0 < P , \quad 0 < Q , \\ P(A_{n+1} - KC^T) + (A_{n+1} - KC^T)^T P \leq -d_0 P , \\ Q(A_n - BF) + (A_n - BF)^T Q \leq -d_0 Q , \\ -aP \leq PD_{n+1} + D_{n+1}P \leq d_1 P , \\ -aQ \leq QD_n + D_nQ \leq d_1 Q . \end{array} \tag{8}$$

Remark 1: As a consequence of Lemma 1, the set (d_0, d_1, Q, P, K, F) is dependent on the design parameter a, as well as all the real numbers d_i's to be introduced later on.

A. Observer design

We adopt here the high gain observer of [10].

For notational convenience, we denote $x_{n+1} = -u^\star$. Let the k_i's be the entries of the vector K given by Lemma 1, i.e.,

$$K = \mathrm{col}(k_1, \ldots, k_{n+1}) . \tag{9}$$

We introduce the following $(n+1)$-th order observer

$$\begin{array}{rcl} \dot{\widehat{x}}_i & = & \widehat{x}_{i+1} + k_i r^i (x_1 - \widehat{x}_1) , \quad 1 \leq i \leq n-1 \\ \dot{\widehat{x}}_n & = & \widehat{x}_{n+1} + u + k_n r^n (x_1 - \widehat{x}_1) \\ \dot{\widehat{x}}_{n+1} & = & k_{n+1} r^{n+1} (x_1 - \widehat{x}_1) \end{array} \tag{10}$$

Following [10], the gain r, involved in this observer, is obtained as a solution of the system

$$\dot{r} = -r\,(br - \sigma(y,r)) , \tag{11}$$

where the strictly positive real number b and the function σ are other design parameters to be made precise later on. However, we note at this time that, by imposing

$$\sigma(y,r) \geq b \quad , \qquad r(0) \geq 1 , \tag{12}$$

we get that $r(t)$ is larger than or equal to 1, for all positive times t and for any solution.

For every $1 \leq i \leq n+1$, let

$$e_i = x_i - \widehat{x}_i . \tag{13}$$

Then, with (1) and (10), it holds

$$\begin{array}{rcl} \dot{e}_i & = & e_{i+1} - k_i r^i e_1 + \delta_i , \quad 1 \leq i \leq n \\ \dot{e}_{n+1} & = & -k_{n+1} r^{n+1} e_1 \end{array} \tag{14}$$

We introduce the scaled estimation error ε in $\mathbb{R}^{n+1}$

$$\varepsilon = \mathrm{col}(\varepsilon_1, \ldots, \varepsilon_{n+1}) , \tag{15}$$

as follows

$$\varepsilon_i = \frac{e_i}{r^{i-1+a}} , \quad \forall 1 \leq i \leq n+1 . \tag{16}$$

We have

$$\dot{\varepsilon} = r(A_{n+1} - KC^T)\varepsilon - (aI_{n+1} + D_{n+1})\frac{\dot{r}}{r}\varepsilon + \Delta_1 \tag{17}$$

where

$$\Delta_1 = \mathrm{col}\left(\frac{\delta_1}{r^a}, \ldots, \frac{\delta_n}{r^{n-1+a}}, 0\right) . \tag{18}$$

The main property of interest to us of the observer can be expressed with the help of the quadratic function

$$V_e = \varepsilon^T P \varepsilon . \tag{19}$$

By means of (8) and (11), we can see that, along the solutions of (17), the time derivative of V_e satisfies

$$\dot{V}_e \leq -([d_0 - (2a + d_1)\,b]r + a\sigma)\,V_e + 2\varepsilon^T P \Delta_1 \tag{20}$$

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B. Controller design

Here, we follow the suggestion of [5]. But we restrict the controller to be linear in $(y, \widehat{x}_2, \dots, \widehat{x}_{n+1})$. For this, as usual, we consider the following auxiliary system

$$\begin{aligned} \dot{y} &= \widehat{x}_2 + e_2 + \delta_1 \\ \dot{\widehat{x}}_2 &= \widehat{x}_3 + k_2 r^2 e_1 \\ &\vdots \\ \dot{\widehat{x}}_n &= \widehat{x}_{n+1} + u + k_n r^n e_1 \end{aligned} \tag{21}$$

We introduce the scaled state and input variables

$$\overline{x} = \mathrm{col}\left(\tfrac{y}{r^a}, \tfrac{\widehat{x}_2}{r^{1+a}}, \dots, \tfrac{\widehat{x}_n}{r^{n-1+a}}\right) \tag{22}$$

$$\overline{u} = \frac{\widehat{x}_{n+1} + u}{r^n} \tag{23}$$

Using the notation introduced previously, we have

$$\dot{\overline{x}} = rA_{n-1}\overline{x} + rB\overline{u} - (aI_n + D_n)\frac{\dot{r}}{r}\overline{x} + r\,\Delta_2 \tag{24}$$

where

$$\Delta_2 = \mathrm{col}\left(\frac{\delta_1}{r^{1+a}} + \varepsilon_2, k_2\varepsilon_1, \dots, k_n\varepsilon_1\right) . \tag{25}$$

We compute the scaled input $\overline{u}$ as

$$\overline{u} = -F\overline{x} \tag{26}$$

with F given by Lemma 1.

The main property of interest to us of the controller can be expressed with the help of the quadratic function

$$V_c = \overline{x}^T Q \overline{x} . \tag{27}$$

By means of (8) and (11), we can see that, along the solutions of (24), the time derivative of V_c satisfies

$$\dot{V}_c \le -([d_0 - (2a + d_1)\, b]r + a\sigma)\, V_c + 2r\overline{x}^T Q \Delta_2 . \tag{28}$$

III. Main result

Theorem 1: Consider the system (1). Under the hypotheses (H1) and (H2), the following output feedback makes the solutions of the closed loop system bounded and their components z and x_i, $1 \le i \le n$, to converge to the origin :

$$\begin{aligned} \dot{\widehat{x}}_i &= \widehat{x}_{i+1} + k_i r^i (x_1 - \widehat{x}_1) , \quad 1 \le i \le n-1 \\ \dot{\widehat{x}}_n &= \widehat{x}_{n+1} + u + k_n r^n (x_1 - \widehat{x}_1) \\ \dot{\widehat{x}}_{n+1} &= k_{n+1} r^{n+1} (x_1 - \widehat{x}_1) \\ u &= -\widehat{x}_{n+1} - r^n F \,\mathrm{col}\left(\tfrac{y}{r^a}, \tfrac{\widehat{x}_2}{r^{1+a}}, \dots, \tfrac{\widehat{x}_n}{r^{n-1+a}}\right) \\ \dot{r} &= -r\,(br - \sigma(y,r)) \end{aligned} \tag{29}$$

where the scalars k_i's, a and b, the matrix F and the Lipschitz continuous function σ are the appropriately chosen feedback parameters.[2]

[2]See eqs. (8), (36) and (57).

Proof of Theorem 1

From the hypothesis (H1.2) and the fact that r can be imposed to stay larger than 1, we get

$$\begin{aligned} \frac{|\delta_i|}{r^{i-1+a}} \le\ & L(|y|)\,[(|\overline{x}_1| + \dots + |\overline{x}_i|) + (|\varepsilon_2| + \dots + |\varepsilon_i|)] \\ & + \frac{\kappa(V_z)}{r^{i-1+a}} . \end{aligned} \tag{30}$$

By completing the squares, this yields

$$2\varepsilon^T P \Delta_1 \le L(|y|)\, V_c + d_2\, L(|y|)\, V_e + d_3 \sqrt{V_e}\frac{\kappa(V_z)}{r^a} , \tag{31}$$

with some nonnegative real numbers d_2, d_3, depending on a. So, it holds

$$\begin{aligned} \dot{V}_e \le\ & -([d_0 - (2a + d_1)\, b]r + a\sigma - d_2 L(|y|))\, V_e \\ & + L(|y|)\, V_c + d_3 \sqrt{V_e}\frac{\kappa(V_z)}{r^a} . \end{aligned} \tag{32}$$

Similarly, we get

$$2r\overline{x}^T Q \Delta_2 \le \left(d_4 L(y) + \frac{d_0}{2} r\right) V_c + r\, d_5\, V_e + d_6 \sqrt{V_c}\frac{\kappa(V_z)}{r^a} \tag{33}$$

with some strictly positive real numbers d_4 to d_6, depending on a. This, in turn, implies

$$\begin{aligned} \dot{V}_c \le\ & -\left(\left[\frac{d_0}{2} - (2a + d_1)\, b\right] r + a\sigma - d_4 L(|y|)\right) V_c \\ & + r\, d_5\, V_e + d_6 \sqrt{V_c}\frac{\kappa(V_z)}{r^a} . \end{aligned} \tag{34}$$

Now, consider the function

$$V_{ec} = \frac{2d_5}{d_0} V_e + V_c . \tag{35}$$

Its time derivative satisfies

$$\begin{aligned} \dot{V}_{ec} \le\ & -\frac{2d_5}{d_0}\left(\left[\frac{d_0}{2} - (2a + d_1)\, b\right] r + a\sigma - d_2 L(|y|)\right) V_e \\ & - \left(\left[\frac{d_0}{2} - (2a + d_1)\, b\right] r + a\sigma - \left[d_4 + \frac{2d_5}{d_0}\right] L(|y|)\right) V_c \\ & + d_7 \sqrt{V_{ec}}\frac{\kappa(V_z)}{r^a} \end{aligned}$$

with some nonnegative real number d_7, depending on a. So let us select our design parameters b and σ to satisfy

$$\frac{d_0(a)}{4(2a + d_1(a))} \ge b > 0 , \tag{36}$$

$$\sigma(y,r) = \sigma_1(y) + \sigma_2(y,r) , \tag{37}$$

$$\sigma_1(y) = \frac{\max\left\{d_2(a) , \left[d_4(a) + \frac{2d_5(a)}{d_0(a)}\right]\right\}}{a} \times L(|y|) , \tag{38}$$

where σ_2 is a function, lower bounded by b (see (12), to be defined in (54) below). Note that, as L, σ_1 is a Lipschitz

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continuous function of the state of the closed loop system. Our motivation for this choice is that it leads to

$$\dot{V}_{ec} \leq -\left(\frac{d_0 r}{4} + a\sigma_2\right) V_{ec} + d_7 \frac{\sqrt{V_{ec}}}{r^a} \kappa(V_z) . \quad (39)$$

Now, with $\ell \geq 1$ the real number given by the hypothesis (H2.2), let ρ_z be the function, defined on $(0, +\infty)$, as

$$\rho_z(s) = \frac{\kappa(s)^\ell}{s} . \quad (40)$$

As κ, this function ρ_z is C^1 on $(0, +\infty)$. Precisely, we have

$$\rho_z'(s) = \frac{\kappa(s)^{\ell-1}}{s^2} [\ell\kappa'(s)s - \kappa(s)] \geq 0 . \quad (41)$$

So ρ_z is a nondecreasing function. Being nonnegative, it has a limit ρ_{z0} as z goes to 0. So we can extend the definition of ρ_z to $[0, +\infty)$ by letting

$$\rho_z(0) = \rho_{z0} . \quad (42)$$

This way we get a nondecreasing continuous function on $[0, +\infty)$. It follows that $\int_0^v \frac{\kappa(s)^\ell}{s} ds$ defines a C^1 function on $[0, \infty)$ which is radially unbounded. Also, it satisfies

$$\int_0^v \frac{\kappa(s)^\ell}{s} ds \leq \kappa(v)^\ell . \quad (43)$$

With this at hand and with the function V_z given by Hypothesis (H1.1), introduce the function

$$U = c \int_0^{V_z} \frac{\kappa(s)^\ell}{s} ds + \frac{1}{\ell} \left(2\sqrt{V_{ec}}\right)^\ell . \quad (44)$$

where c is another design parameter to be chosen as a strictly positive real number. This function U is positive definite and radially unbounded in $(\overline{x}, \varepsilon, z)$. It is also differentiable at any point except at $(0, 0, z)$. But it is Lipschitz continuous. So it admits an upper right Dini derivative along any solution. We denote this derivative $\dot{U}$. With the hypothesis (H1.1), it satisfies

$$\begin{aligned}\dot{U} \leq & -c \frac{\kappa(V_z)^\ell}{V_z} [V_z - \gamma(|y|)] - \left(2\sqrt{V_{ec}}\right)^{\ell-1} \times \\ & \left[\left(\frac{d_0 r}{8} + \frac{a}{2}\sigma_2\right) \left(2\sqrt{V_{ec}}\right) - \frac{d_7}{r^a} \kappa(V_z)\right] . \quad (45)\end{aligned}$$

Since ρ_z is a nondecreasing function, by considering successively the two cases $V_z \geq 2\gamma(|y|)$ and $V_z < 2\gamma(|y|)$, we get the inequality

$$\frac{\kappa(V_z)^\ell}{V_z} [V_z - \gamma(|y|)] \geq \frac{1}{2} \kappa(V_z)^\ell - \frac{1}{2} \kappa(2\gamma(|y|))^\ell . \quad (46)$$

With Young's inequality and the fact that r remains larger than 1, all along any solution, we get

$$\left(2\sqrt{V_{ec}}\right)^{\ell-1} \frac{d_7}{r^a} \kappa(V_z) \leq d_7 \kappa(V_z)^\ell + \frac{d_7}{r^a} \left(2\sqrt{V_{ec}}\right)^\ell . \quad (47)$$

All this gives

$$\begin{aligned}\dot{U} \leq & -\left(\frac{c}{2} - d_7\right) \kappa(V_z)^\ell + \frac{c}{2} \kappa(2\gamma(|y|))^\ell \\ & - \left(2\sqrt{V_{ec}}\right)^\ell \left(\frac{d_0 r}{8} + \frac{a}{2}\sigma_2 - \frac{d_7}{r^a}\right) . \quad (48)\end{aligned}$$

So we choose c satisfying

$$c \geq 4 d_7(a) . \quad (49)$$

Then let d_8, depending on a, be the square root of the minimum eigenvalue of Q. We have

$$d_8 |y| \leq r^a \sqrt{V_{ec}} \quad (50)$$

This implies

$$\begin{aligned}\dot{U} \leq & -\frac{c}{4} \kappa(V_z)^\ell - \frac{d_0 r}{8} \left(2\sqrt{V_{ec}}\right)^\ell + \frac{c}{2} \kappa(2\gamma(|y|))^\ell \\ & - \left(\frac{a}{2}\sigma_2 - \frac{d_7}{r^a}\right) \frac{1}{r^{a\ell}} (2d_8 |y|)^\ell . \quad (51)\end{aligned}$$

So our idea is to select the function σ_2, as a function of y, in order to get the inequality

$$\frac{c}{2} \kappa(2\gamma(|y|))^\ell - \left(\frac{a}{2}\sigma_2 - \frac{d_7}{r^a}\right) \frac{1}{r^{a\ell}} (2d_8 |y|)^\ell \leq 0 . \quad (52)$$

From the hypothesis (H1.3), such a choice is possible since, when $|y| \leq s_0$, it is sufficient to have

$$\frac{\frac{a}{2}\sigma_2 - \frac{d_7}{r^a}}{r^{a\ell}} (2d_8)^\ell \geq \frac{c}{2} k^\ell . \quad (53)$$

A specific expression for σ_2 is for instance

$$\begin{aligned}\sigma_2 = & \max\left\{b, \frac{2d_7(a)}{a}\right\} + \frac{4c}{a\,(2d_8(a))^\ell} r^{a\ell} \times \\ & \max\left\{k^\ell, \left(\frac{\kappa(2\gamma(|y|))}{|y|}\right)^\ell\right\} . \quad (54)\end{aligned}$$

This is a Lipschitz continuous function. This way and with (43), we obtain

$$\begin{aligned}\dot{U} & \leq -\frac{c}{4} \int_0^{V_z} \frac{\kappa(s)^\ell}{s} ds - \frac{d_0 r}{8} \left(2\sqrt{V_{ec}}\right)^\ell , \\ & \leq -\mu U , \quad (55)\end{aligned}$$

with μ some strictly positive real number, guaranteed to exist again since r is larger than 1. So it follows that, along any solution, $U(t)$ is exponentially decreasing and therefore also bounded.

Now consider any closed loop solution. Its state can be taken as $(r, \overline{x}, \varepsilon, z)$. Let it be right maximally defined on $[0, T)$. From the properties of $U(t)$, we know that, as $U(t)$, $V_{ec}(t)$ and $V_z(t)$ are bounded on $[0, T)$. Therefore, for each solution, the components $\overline{x}(t)$, $\varepsilon(t)$ and $z(t)$ are bounded. This, in turn, gives that $\overline{x}_1(t) = y(t)/r^a(t)$ is bounded, i.e. we have

$$|\overline{x}_1(t)| \leq X_1 \quad \forall t \in [0, T) \quad (56)$$

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for some real number X_1, depending on the solution.

Let us show that, by picking a small enough, we can guarantee that the component $r(t)$ is also bounded on $[0,T)$. We know that we have (see (11), (37), (38), (54))

$$\begin{aligned} \dot{r} &= -r\left(b_1 r - b_2 - b_3 L(|y|) - b_4 r^{a\ell} \times \right. \\ &\qquad \left. \max\{k^\ell, \left(\tfrac{\kappa(2\gamma(|y|))}{|y|}\right)^\ell\}\right) \end{aligned} \tag{57}$$

with

$$b_1 = b \, , \tag{58}$$

$$b_2 = \max\left\{b, \frac{2d_7}{a}\right\} \, , \tag{59}$$

$$b_3 = \frac{\max\left\{d_2 \, , \left[d_4 + \frac{2d_5}{d_0}\right]\right\}}{a} \, , \tag{60}$$

$$b_4 = \frac{2d_7}{a} + \frac{4c}{a\,(2d_8)^\ell} \, . \tag{61}$$

With the help of the hypotheses (H2.1), and the inequality

$$|y(t)| \leq X_1 r^a \quad \forall t \in [0,T) \, , \tag{62}$$

we get

$$\begin{aligned} \dot{r} \leq & -r\left(b_1 r - [b_2 + b_3 p] - b_3 X_1^m r^{am} - b_4[2k+p]\, r^{a\ell}\right. \\ & \left. -b_4 X_1^m r^{a(m+\ell)}\right) \, . \end{aligned} \tag{63}$$

But if a satisfies the constraint

$$a\,(m+\ell) < 1 \, , \tag{64}$$

depending only on the system data, with the help of Young's inequality, we can find a positive real number b_5 satisfying

$$\begin{aligned} & b_1 r - [b_2 + b_3 p] - b_3 X_1^m r^{am} - b_4[2k+p]\, r^{a\ell} \\ & -b_4 X_1^m r^{a(m+\ell)} \geq \tfrac{b_1}{2} r - b_5 \, . \end{aligned} \tag{65}$$

This yields simply

$$\dot{r} \leq -r\left(\frac{b_1}{2} r - b_5\right) \, . \tag{66}$$

It follows readily that the component $r(t)$ is also bounded on $[0,T)$.

So all the components being bounded, T must be infinite. This implies that the solution is bounded on $[0,+\infty)$ and $U(t)$ converges to 0 as t goes to ∞. So the state components $(x_i(t), e_i(t), z(t))$ converge to the origin.

IV. Concluding Remarks

In this paper, a high gain linear output feedback is proposed for global asymptotic stabilization of a class of nonlinear systems where the nonlinearities have a linear growth in the (partial) unmeasured state components, with an output dependent growth rate and with ISS inverse dynamics. This feedback involves an on-line tuned gain.

We have seen that the main loss we have by imposing a linear structure, as compared to the best available results using truly nonlinear output feedback, is that the growth rate and the gain from the output to the perturbation via the inverse dynamics should have a growth which cannot be more than polynomial.

The stability analysis has been made possible by :

1) the scaling of some components, as usual when high gain is used. However here we use the modification of the scaling introduced in [10].
2) the application of the Lyapunov version of the nonlinear small-gain techniques [3].

Appendix

The proof of Lemma 1 follows directly from the following technical lemma. Indeed, the existence of the pair (Q,F) in Lemma 1 follows by letting

$$\tau = -\frac{1}{n-1+\frac{a}{2}} \, . \tag{67}$$

The existence of the pair (P,K) follows from the same arguments of the proof of Lemma 2 by letting

$$R = -I - \frac{1}{a}\operatorname{diag}(0,\ldots,n-2,n-1) \, , \tag{68}$$

$$X = P \quad , \quad Y = PK \, , \tag{69}$$

and doing a recursion with

$$A_{j+1} = \begin{pmatrix} 0 & C_j^T \\ 0 & A_j \end{pmatrix} \, , \quad C_{j+1}^T = \begin{pmatrix} 1 & 0 & \ldots & 0 \end{pmatrix} ,$$

$$R_{j+1} = \begin{pmatrix} -1 & 0 \\ 0 & R_j - \frac{1}{a} I_j \end{pmatrix} \, ,$$

$$X_{j+1} = \begin{pmatrix} T_j & -Y_j^T \\ -Y_j & X_j \end{pmatrix} \, , \quad Y_{j+1} = \begin{pmatrix} V_{j+1} \\ U_{j+1} \end{pmatrix} \, .$$

Lemma 2: Let R be the following diagonal matrix :

$$R = -I - \tau\operatorname{diag}(n-1, n-2, \ldots, 0) \tag{70}$$

where $\tau > -1/(n-1)$. Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^n$ be the matrices in the canonical controller form as in (7). Then, the set of pairs (F,Q) satisfying :

$$Q\,(A-BF) + (A-BF)^\top Q < 0 \tag{71}$$

$$RQ + QR < 0 \tag{72}$$

$$Q > 0 \tag{73}$$

is not empty.

Proof of Lemma 2. By letting :

$$X = Q^{-1} \quad , \quad Y^\top = F\,Q^{-1} \tag{74}$$

the statement of Lemma 2 becomes :

The set of pairs (X,Y) satisfying :

$$\begin{aligned} \left[XA^\top + AX\right] - \left[YB^\top + BY^\top\right] &< 0 \\ XR + RX &< 0 \\ X &> 0 \end{aligned} \tag{75}$$

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is not empty.

Now, we prove the above claim by induction on the dimension.

Initialization ($n = 1$): The claim obviously holds with

$$X_1 > 0\ ,\ A_1 = 0\ ,\ B_1 = 1\ ,\ R_1 = -1\ ,\ Y_1 > 0\ . \quad (76)$$

Induction ($n = j + 1$): Assume that the claim holds, in case of $n = j$, with A_j, B_j, R_j, X_j and Y_j. Now, it is shown that the claim holds, in case of $n = j + 1$, with

$$\begin{aligned} A_{j+1} &= \begin{pmatrix} A_j & B_j \\ 0 & 0 \end{pmatrix}\ ,\quad B_{j+1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\ , \\ R_{j+1} &= \begin{pmatrix} R_j - \tau I_j & 0 \\ 0 & -1 \end{pmatrix} \end{aligned} \quad (77)$$

and

$$X_{j+1} = \begin{pmatrix} X_j & S_j \\ S_j^\top & T_j \end{pmatrix}\ ,\quad Y_{j+1} = \begin{pmatrix} U_{j+1} \\ V_{j+1} \end{pmatrix} \quad (78)$$

where T_j and V_{j+1} are real numbers, and S_j and U_{j+1} are matrices of appropriate dimensions to be determined later. Notice that we use 0 to denote either a row vector or a column vector of appropriate length.

Take

$$S_j = -Y_j\ ,\quad U_{j+1} = A_j S_j + B_j T_j \quad (79)$$

As it can be directly verified, as long as $\tau > -1/(j-1)$, by picking the real numbers T_j and V_{j+1} large enough, the above claim is true for $n = j + 1$. Finally, Lemma 2 is established.

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