Proceedings of the 42nd IEEE
Conference on Decision and Control
Maui, Hawaii USA, December 2003

Global complete observability and output-to-state stability imply the existence of a globally convergent observer

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***Abstract*— In this paper we consider systems which are globally completly observable and output-to-state stable. The former property guarantees the existence of coordinates such that the dynamics can be expressed in observability form. The latter property guarantees the existence of a state norm observer and therefore nonlinearities bounding function and local Lipschitz bound. Both allow us to build an observer from an approximation of an exponentially attractive invariant manifold in the space of the system state and an output driven dynamic extension. The state of this observer has the same dimension as the state to be observed. Its main interest is to provide convergence to zero of the estimation error within the domain of definition of the solutions.**

I. Introduction

We consider a globally completely observable system whose dynamics can be represented globally by:

$$\left\{ \begin{array}{rcl} \dot{x}_0 & = & x_1 \ , \\ & \vdots & \\ \dot{x}_{n-1} & = & x_n \ , \\ \dot{x}_n & = & f_n(x_0, \ldots, x_n) \ , \end{array} \right. \tag{1}$$

where f_n is Lipschitz continuous. For such a system, we wish to establish the existence of a global observer when the only available measurement is:

$$y \ = \ x_0 \ .$$

Such a problem has received a lot of attention from a wide variety of view points. The route we follow here takes its starting point in a contribution[1] of Kazantzis and Kravaris. In [7], they have generalized, to the nonlinear case, Luenberger's early ideas proposed in [9] for linear systems (see also [2, Section 7.4 method II]). However their analysis is a local one and requires too stringent assumptions aiming at getting an analytic observer. Our intent here is to remove these extra assumptions and to deal with the global case. For the latter, we need to add an assumption besides global complete observability.

Assumption 1 (see [8], [11]): The system (1) is output-to-state stable, i.e. there exist C^1 non-negative functions γ_1, γ_2 and V satisfying:

$$|x| \ \leq \ \gamma_2(V(x)) \ , \tag{2}$$

[1]This contribution has been extended in various ways by Kazantzis and Kravaris themselves but also by Xiao and Krener (see [13] and the references therein). But they remain in the same context of looking for a C^∞ observer or at least one admitting a formal power series representation.

and:

$$\dot{\overbrace{V(x)}} \ \leq \ -V(x) \ + \ \gamma_1(x_0) \ . \tag{3}$$

With (2) in Assumption 1 and the continuity of f_n, there exists a C^1 non-decreasing function γ, lower bounded by 1 say, and satisfying:

$$|x_0| + \ldots + |x_n| + |f_n(x_0, \ldots, x_n)| \ \leq \ \gamma(V(x)) \ . \tag{4}$$

It follows that, by defining a new time τ as the solution of[2]:

$$\dot{\tau} \ = \ \gamma(V(x)) \quad , \qquad \tau(0) \ = \ 0 \ ,$$

and by denoting:

$$\mathring{a} \ = \ \frac{da}{d\tau} \ = \ \frac{\dot{a}}{\gamma(V(x))} \ ,$$

the system:

$$\left\{ \begin{array}{rcl} \mathring{x}_0 & = & \dfrac{x_1}{\gamma(V(x))} \ , \\ & \vdots & \\ \mathring{x}_{n-1} & = & \dfrac{x_n}{\gamma(V(x))} \ , \\ \mathring{x}_n & = & \dfrac{f_n(x_0, \ldots, x_n)}{\gamma(V(x))} \end{array} \right. \tag{5}$$

is complete. Actually its solutions do not grow faster than $|\tau|$ both forward and backward in the new time τ. As a consequence, for any strictly Hurwitz $p \times p$ matrix A and any p vector B, the function given as follows is well defined and continuous (see [4, Théorème 3.149]):

$$R(x) \ = \ \int_{-\infty}^{0} \exp(-A\tau)\, B\, x_0(\tau)\, d\tau \ , \tag{6}$$

where, with the notation:

$$x \ = \ (x_0, \ldots, x_n) \ ,$$

$x_0(\tau)$ is the first component of the solution $x(\tau)$ of (5), issued from x. Our interest in R comes from the fact that:

$$z \ = \ R(x)$$

defines a globally attractive invariant manifold of the system (5) coupled with:

$$\mathring{z} \ = \ A\, z \ + \ B\, x_0 \ . \tag{7}$$

[2]We get $\tau(t) \geq t$ for all positive t.

Indeed, by computing the limit for $h \to 0$ of $\frac{R(x(h))-R(x)}{h}$, we can check that R satisfies (7) when evaluated along the solutions of (5). Moreover, integrating (6) by parts, we get:

$$\begin{aligned} R(x) &= -A^{-1}B\,x_0 \\ &-\tfrac{A^{-(n+1)}}{\gamma(V(x))^n}\left(I,\dots\gamma(V(x))^{n-1}A^{n-1}\right)B\begin{pmatrix}x_n\\x_{n-1}\\\vdots\\x_1\end{pmatrix} \\ &-\int_{-\infty}^{0}\exp(-A\tau)\left(\frac{\overset{\circ}{\overbrace{\gamma(V(x))}}}{\gamma(V(x)}\sum_{i=1}^{n}\frac{iBA^{-(i+1)}x_i}{\gamma(V(x))^i} - \right. \\ &\left.\frac{A^{-(n+1)}Bf_n(x_0(\tau),\dots,x_n(\tau))}{\gamma(V(x))^{n+1}}\right)d\tau\ . \end{aligned}$$

It follows that, if the pair (A,B) is controllable and $p \geq n$, there is some hope that, maybe by modifying γ to make the last two lines negligible, the map $(x_0,\overline{x}) \mapsto (x_0,R(x_0,\overline{x}))$ is left invertible, with $\overline{x}$ collecting the unmeasured components of x, i. e.

$$\overline{x} \;=\; (x_1,\dots,x_n)\ .$$

In such a case there would exist a function S defined on the image of this map, subset of $\mathbb{R}\times\mathbb{R}^n$, and satisfying:

$$S(x_0,R(x_0,\overline{x})) \;=\; \overline{x} \qquad \forall\,(x_0,\overline{x})\ .$$

Further, we may expect that this function S can be extended into a continuous function $\mathfrak{S}$ defined on $\mathbb{R}\times\mathbb{R}^n$. For instance, as shown in [9] (see also [2, Theorem 7.10]), $\mathfrak{S}$ does exist when f_n is a linear function, the pair (A,B) is controllable, and the spectrum of A and this of the system (1) are separated. The existence of $\mathfrak{S}$ is also established, in [7], locally around the origin, assumed to be an equilibrium point of (1), under the assumption that f_n is analytic, the pair (A,B) is controllable and a more restrictive condition of spectral separation.

Since the set $\{(z,x)\ :\ z = R(x)\}$ is exponentially attractive, the existence and the continuity of $\mathfrak{S}$ imply that, for each solution $(x(\tau),z(\tau))$, we have:

$$\lim_{\tau\to+\infty}\left(\mathfrak{S}(x_0(\tau),z(\tau))-\overline{x}(\tau)\right) \;=\; 0\ .$$

This says that:

$$\left\{\begin{array}{rcl} \overset{\circ}{z} &=& A\,z\;+\;B\,x_0\ ,\\ \widehat{\overline{x}} &=& \mathfrak{S}(x_0,z)\ , \end{array}\right.$$

with the new time τ, or, if $\gamma(V(x))$ were known,

$$\left\{\begin{array}{rcl} \dot{z} &=& \gamma(V(x))\,[A\,z\;+\;B\,x_0]\ ,\\ \widehat{\overline{x}} &=& \mathfrak{S}(x_0,z)\ , \end{array}\right.$$

with the initial time t, is an observer of $\overline{x}$, with $\widehat{\overline{x}}$ converging to $\overline{x}$, as the new time τ goes to infinity. In terms of the initial time t, this says that the convergence occurs in infinite time if there is no finite escape time, but at the time of the escape, if there is a finite escape time. This observer, but in the case where $\gamma(V(x)) = 1$, is the one presented by Luenberger in [9] for linear systems and by Kazantzis and Kravaris in [7] locally, for nonlinear systems.

The objective of this paper is to round or solve the problems we have left on our way in the above presentation. These problems are:

1) How to get an upperbound of $\gamma(V(x))$ expressed from the only knowledge of x_0?
2) How to modify γ and the system (7) in order to enforce the existence of a continuous function $\mathfrak{S}:\mathbb{R}\times\mathbb{R}^m\to\mathbb{R}^n$ satisfying:
$$\mathfrak{S}(x_0,R(x_0,\overline{x})) \;=\; \overline{x} \qquad \forall\,(x_0,\overline{x})? \tag{8}$$
3) How to get an expression of $\mathfrak{S}$?

The first problem is addressed in Section II. The other two in Section III. This will allow us to exhibit C^1 functions $\mathfrak{f}$ and $\mathfrak{h}$ such that the n-dimensional system:

$$\left\{\begin{array}{rcl} \dot{\mathcal{X}} &=& \mathfrak{f}(\mathcal{X},y)\ ,\\ \widehat{\overline{x}} &=& \mathfrak{h}(\mathcal{X},y) \end{array}\right.$$

provides an estimate $\widehat{\overline{x}}$ converging in the new time τ to the actual unmeasured state component $\overline{x}$.

II. An upperbound for $\gamma(V(x))$

To get an upperbound for $\gamma(V(x))$, we follow the norm-estimator idea proposed by Sontag and Wang in [11] (see also [8]). From (3) in Assumption 1, we know that V satisfies:

$$\dot{\overbrace{V(x)}} \;\leq\; -V(x)\;+\;\gamma_1(x_0)\ .$$

So let w be obtained as a solution of the system:

$$\dot{w} \;=\; -w\;+\;\gamma_1(x_0) \tag{9}$$

with positive initial condition. For a solution $(x(t),w(t))$ of (1),(9), issued from (x,w), we have:

$$V(x(t)) \;\leq\; w(t)\;+\;\max\{V(x)-w,0\}\exp(-t) \tag{10}$$

for all t for which this solution exists. If $x(t)$ is right maximally defined on $[0,t_0)$, then $w(t)$ is defined at least on the same interval. Also,

1) if t_0 is infinite, then, because of the exponential decay, there exists t_v, depending on (x,w), satisfying:
$$V(x(t)) \;\leq\; w(t)\;+\;1 \qquad \forall t\in[t_v,+\infty)\ .$$
2) If t_0 is finite, we have, from (2) in Assumption 1,
$$\lim_{t\to t_0}V(x(t)) \;=\; +\infty\ .$$
This implies the existence of a time t_v, depending on (x,w), satisfying:
$$\max\{V(x)-w,0\}\exp(-t) \leq \frac{1}{2}V(x(t))\forall t\in[t_v,t_0). \tag{11}$$

With (10), this yields:

$$V(x(t)) \;\leq\; 2\,w(t) \quad \forall t \in [t_v, t_0) \ .$$

From these two cases, we conclude that, for each solution, there exist a real number $v \quad (= V(x))$, and a new time τ_v $(= \tau(t_v))$ satisfying:

$$\begin{aligned} V(x(\tau)) \;&\leq\; w(\tau) \,+\, v \qquad \forall \tau \in [0, \tau_v] \ , \\ &\leq\; 2\,w(\tau) \,+\, 1 \quad \forall \tau \in [\tau_v, \infty) \ . \end{aligned} \tag{12}$$

But it is important to stress that both v and τ_v depend on the initial condition of the solution.

To simplify our forthcoming notations, we introduce the notation c_* in association with any non-decreasing function $c : \mathbb{R}_+ \to \mathbb{R}_+$. Specifically, $c_* : \mathbb{R}_+ \to \mathbb{R}_+$ is the function associated to c as:

$$c_*(r) \;=\; c(r) \;-\; c(0) \ .$$

We have the property:

$$c(v) \;\leq\; c(4w+2) \;+\; c_*(2[v-2w-1]_+) \ ,$$

where we use the notation:

$$r_+ \;=\; \max\{r, 0\} \ .$$

As a consequence :

$$\gamma(V(x)) \;\leq\; \gamma(4w+2) \;+\; \gamma_*(2[V(x)-2w-1]_+) \ . \tag{13}$$

So, for each solution of (5),(9), we get:

$$\begin{aligned} \frac{\gamma(V(x(\tau)))}{\gamma(4w(\tau)+2)} \quad &\leq \quad 1+\gamma_*(2v-2) \quad \forall \tau \in [0, \tau_v] \ , \\ &\leq \quad 1 \qquad\qquad\qquad \forall \tau \in [\tau_v, \infty) \ , \end{aligned} \tag{14}$$

and therefore, with (4),

$$\begin{aligned} \tfrac{\sum_{i=0}^{n} |x_i(\tau)|+|f_n(x(\tau))|}{\gamma(4w(\tau)+2)} \;&\leq\; 1+\gamma_*(2v-2) \;\; \forall \tau \in [0, \tau_v], \\ &\leq\; 1 \qquad\qquad\quad \forall \tau \in [\tau_v, \infty). \end{aligned}$$

It follows that $\gamma(4w+2)$ is a good candidate to replace $\gamma(V(x))$. This leads us to introduce the notation:

$$\widehat{\gamma}(w) \;=\; \gamma(4w+2) \tag{15}$$

and another new time $\widehat{\tau}$ as the solution of:

$$\dot{\widehat{\tau}} \;=\; \widehat{\gamma}(w) \quad , \qquad \widehat{\tau}(0) \;=\; 0 \ .$$

We have:

$$\mathring{\widehat{\tau}} \;=\; \frac{\widehat{\gamma}(w)}{\gamma(V(x))}$$

and therefore, using (14), along any solution,

$$\limsup_{\tau \to +\infty} \frac{\widehat{\tau}(\tau)}{\tau} \;\geq\; 1 \ .$$

This implies that any boundedness or convergence result established with the time $\widehat{\tau}$ holds also with the time τ. Also, by denoting:

$$\overset{*}{a} \;=\; \frac{da}{d\widehat{\tau}} \;=\; \frac{\dot{a}}{\widehat{\gamma}(w)} \ ,$$

the system:

$$\left\{ \begin{aligned} \overset{*}{x}_0 \quad &= \quad \frac{x_1}{\widehat{\gamma}(w)} \ , \\ &\vdots \\ \overset{*}{x}_{n-1} \quad &= \quad \frac{x_n}{\widehat{\gamma}(w)} \ , \\ \overset{*}{x}_n \quad &= \quad \frac{f_n(x_0, \dots, x_n)}{\widehat{\gamma}(w)} \ , \\ \overset{*}{w} \quad &= \quad -\frac{w-\gamma_1(x_0)}{\widehat{\gamma}(w)} \end{aligned} \right.$$

is complete.

III. Existence and expression of $\mathfrak{S}$

A. Observer design

We have mentioned that, if there exists a continuous function $\mathfrak{S}$ satisfying (8) and if we know this function, then we know an observer asymptotically converging in the new time τ. The route we follow to prove the existence and to express $\mathfrak{S}$ is actually to modify (7). In the following we build this modification step by step with in particular the objective of getting the function R as a linear map with a triangular representation in the $\overline{x}$ coordinates.

1) Estimate of x_1: Let us first introduce the system:

$$\dot{z}_1 \;=\; -a_1\, z_1 \;+\; b_1\, x_0 \;-\; u_1 \tag{16}$$

where a_1, b_1 and u_1 remain to be defined. In particular, u_1 is an extra term added to (7). This equation gives readily, for any C^1 function r_{10},

$$\begin{aligned} \dot{\overbrace{z_1 - r_{10}x_0 + x_1}} \quad &= \quad -a_1 \left(z_1 - \tfrac{b_1 - \dot{r}_{10}}{a_1} x_0 + \tfrac{r_{10}}{a_1} x_1 \right) \\ &\qquad +x_2 - u_1 . \end{aligned}$$

So, by choosing:

$$u_1 \;=\; x_2 \tag{17}$$

and:

$$\frac{b_1 - \dot{r}_{10}}{a_1} \;=\; r_{10} \;=\; a_1 \ ,$$

i.e.:

$$b_1 \;=\; \dot{a}_1 \;+\; a_1^2 \ , \tag{18}$$

we get:

$$\dot{\overbrace{z_1 - a_1 x_0 + x_1}} \;=\; -a_1\,(z_1 - a_1 x_0 + x_1)$$

or:

$$\overset{*}{\overbrace{z_1 - a_1 x_0 + x_1}} \;=\; -\frac{a_1}{\widehat{\gamma}(w)}\,(z_1 - a_1 x_0 + x_1)$$

It follows that the set $\{(z_1,x) : z_1 = a_1 x_0 - x_1\}$ is an invariant manifold of (1),(16) which is exponentially attractive in the original time t and the new time $\widehat{\tau}$ if we choose:

$$a_1 \geq \widehat{\gamma}(w) .$$

This leads us to propose an estimate of x_1 in the form:

$$\widehat{x}_1 = a_1 x_0 - z_1 .$$

The only problem here is that (17) is not a legitimate choice for u_1 since x_2 is not measured. So, for the time being, let us continue the design with u_1 still not specified. However, for the sake of uniformity of notations, we let:

$$u_1 = v_1 .$$

And, for the future, we know that the best choice for v_1 is:

$$v_1 = x_2 .$$

On the other hand, if we restrict ourselves by allowing a_1 to depend only on w, (18) can indeed by realized as:

$$b_1(w,x_0) = a_1'(w)\,[-w+\gamma_1(x_0)] + a_1(w)^2 .$$

So, our proposition for an estimate of x_1 is:

$$\left\{\begin{array}{rcl} \dot{z}_1 &=& -a_1 z_1 + [\dot{a}_1 + a_1^2]\,x_0 - v_1 , \\ \widehat{x}_1 &=& a_1 x_0 - z_1 . \end{array}\right.$$

2) Estimate of x_2: With the estimate $\widehat{x}_1$ of x_1 at hand, we introduce a second system:

$$\dot{z}_2 = -a_2 z_2 + b_2 x_0 - u_2 \tag{19}$$

where again a_2, b_2 and u_2 remain to be defined. For any C^1 functions r_{20} and r_{21}, we get

$$\overbrace{z_2 - r_{20}x_0 - r_{21}x_1 + x_2}^{\cdot} =$$
$$-a_2\left(z_2 - \tfrac{b_2-\dot{r}_{20}}{a_2}x_0 + \tfrac{r_{20}}{a_2}x_1 + \tfrac{r_{21}}{a_2}x_2\right) - \dot{r}_{21}x_1 + x_3 - u_2.$$

This shows that, by choosing:

$$u_2 = x_3 - \dot{r}_{21}\,x_1 \tag{20}$$

and:

$$\begin{array}{rcl} \dfrac{r_{21}}{a_2} &=& 1 , \\ \dfrac{r_{20}}{a_2} &=& -r_{21} , \\ \dfrac{b_2 - \dot{r}_{20}}{a_2} &=& r_{20} , \end{array}$$

i.e.

$$\begin{array}{rcl} r_{21} &=& a_2 , \\ r_{20} &=& -a_2^2 , \\ b_2 &=& -a_2^3 - 2a_2\dot{a}_2 , \end{array}$$

the set $\{(z_2,x) : z_2 = -a_2^2 x_0 + a_2 x_1 - x_2\}$ is an invariant manifold of (1),(19). This leads us to propose an estimate of x_2 in the form:

$$\widehat{x}_2 = -a_2^2\,x_0 + a_2\,x_1 - z_2 .$$

Unfortunately, such as estimate involves x_1 which is unknown. But, from the previous step, we have the estimate $\widehat{x}_1$. So $\widehat{x}_2$ is actually taken as:

$$\widehat{x}_2 = -a_2^2\,x_0 + a_2\,\widehat{x}_1 - z_2 .$$

Here again, the problem we are facing is that u_2 defined in (20) involves x_1 and x_3 which we do not know. So we let:

$$u_2 = v_2 - \dot{r}_{21}\,\widehat{x}_1 ,$$

where, for the future, we know that the best choice for v_2 is:

$$v_2 = x_3 .$$

To recapitulate, we propose an estimate of x_2 as:

$$\left\{\begin{array}{rcl} \dot{z}_2 &=& -a_2\,z_2 - [a_2^3 + 2a_2\dot{a}_2]\,x_0 + \dot{a}_2\,\widehat{x}_1 - v_2 , \\ \widehat{x}_2 &=& -a_2^2\,x_0 + a_2\,\widehat{x}_1 - z_2 . \end{array}\right. \tag{21}$$

where again, if we choose a_2 as a function of w only, we have:

$$\dot{a}_2 = a_2'(w)\,[-w+\gamma_1(x_0)] . \tag{22}$$

3) Estimate of x_i: By proceeding along the same lines for i ranging now from 3 to n, we design an observer for x_i, from the system:

$$\dot{z}_i = -a_i\,z_i + b_i\,x_0 - u_i . \tag{23}$$

For any C^1 functions r_{ij}, we get:

$$\overbrace{z_i - \sum_{j=0}^{i-1} r_{ij}x_j + x_i}^{\cdot} =$$
$$-a_i\left(z_i - \tfrac{b_i-\dot{r}_{i0}}{a_i}x_0 + \textstyle\sum_{j=1}^{i-1}\tfrac{r_{i(j-1)}}{a_i}x_j + \tfrac{r_{i(i-1)}}{a_i}\,x_i\right)$$
$$-\textstyle\sum_{j=1}^{i-1}\dot{r}_{ij}x_j + x_{i+1} - u_i,$$

where, to simplify the notations for $i = n$, we have let formally,

$$x_{n+1} = f_n(x_0,\ldots,x_n) .$$

So, by choosing:

$$u_i = x_{i+1} - \sum_{j=1}^{i-1}\dot{r}_{ij}\,x_j$$

and:

$$\begin{array}{rcll} \dfrac{r_{i(i-1)}}{a_i} &=& 1 , & \\ \dfrac{r_{i(j-1)}}{a_i} &=& -r_{ij} & j\in\{1,\ldots,i-1\} , \\ \dfrac{b_i - \dot{r}_{i0}}{a_i} &=& r_{i0} , & \end{array}$$

i.e.:

$$\begin{array}{rcl} r_{ij} & = & -(-a_i)^{i-j} \ , \\ b_i & = & [i\dot{a}_i + a_i^2]\,(-a_i)^{i-1} \ , \end{array}$$

the set

$$\left\{ (z_i, x) : z_i = -\sum_{j=0}^{i-1} (-a_i)^{i-j} x_j - x_i \right\}$$

is an invariant manifold of (1),(23). This motivates an observer for x_i in the form:

$$\left\{ \begin{array}{rcl} \dot{z}_i & = & -a_i z_i + [i\dot{a}_i + a_i^2]\,(-a_i)^{i-1} x_0 + \\ & & \dot{a}_i \sum_{j=1}^{i-1} (i-j)(-a_i)^{i-j-1} \widehat{x}_j - v_i, \\ \widehat{x}_i & = & -(-a_i)^i x_0 \ - \ \displaystyle\sum_{j=1}^{i-1} (-a_i)^{i-j} \widehat{x}_j \ - \ z_i \ , \end{array} \right. \tag{24}$$

where the best choice for v_i is:

$$v_i \ = \ x_{i+1} \ .$$

In this case, the manifold error ε_i, defined as:

$$\varepsilon_i \ = \ z_i \ + \ (-a_i)^i x_0 \ + \ \sum_{j=1}^{i-1} (-a_i)^{i-j} x_j \ + \ x_i \ ,$$

satisfies:

$$\dot{\varepsilon}_i \ = \ -a_i \varepsilon_i - \sum_{j=1}^{i-1} \overbrace{(-a_i)^{i-j}}^{\cdot} [x_j - \widehat{x}_j] + [x_{i+1} - v_i]. \tag{25}$$

B. Observer properties

To study the properties of the observer we have designed and whose generic expression is given by (24), we introduce the observation error:

$$e_i \ = \ x_i \ - \ \widehat{x}_i \ . \tag{26}$$

With the help of (24), we see it is related to the manifold error by:

$$e_i \ - \ \varepsilon_i \ = \ -\sum_{j=1}^{i-1} (-a_i)^{i-j} e_j \ . \tag{27}$$

To go further, we need to express the e_i's in terms of the ε_i's. For this, let L be the strict lower triangular matrix whose (i, j) entry is $(-a_i)^{i-j}$, i.e.:

$$L \ = \ \begin{pmatrix} 0 & \dots & \dots & \dots & 0 \\ -a_2 & 0 & & & \vdots \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ (-a_n)^{n-1} & (-a_n)^{n-2} & \dots & -a_n & 0 \end{pmatrix} \tag{28}$$

We have:

$$\varepsilon \ = \ (I + L)\, e \ . \tag{29}$$

Also, with these compact notations, (25) reads:

$$\dot{\varepsilon} = -\left(\mathtt{diag}(a_i) - \dot{L}\,(I+L)^{-1} \right) \varepsilon + \mathtt{vect}(x_{i+1} - v_i),$$

and:

$$e \ = \ (I+L)^{-1} \varepsilon.$$

Now, we observe that L is nilpotent, i.e.:

$$L^n \ = \ 0 \ .$$

This implies:

$$(I+L)^{-1} \ = \ I \ + \ \sum_{i=1}^{n-1} (-L)^i \ .$$

So, from the expression of the powers of L, we see that the (i, j) entry of $(I + L)^{-1}$ which we denote ℓ_{ij} in the following,

1) is zero if $j > i$,
2) is 1 if $j = i$,
3) depends only on a_{j+1} to a_i if $j \leq i - 1$.

and similarly the (i, j) entry of $\dot{L}(I+L)^{-1}$, which we denote $\dot{a}_i h_{ij}$, has $\dot{a}_i$ in factor and its other factor h_{ij}

1) is zero if $j \geq i$,
2) depends only on a_{j+1} to a_i if $j \leq i - 1$.

These various remarks show that the overall dynamics admit (x, w, ε) as state and, in the new time $\widehat{\tau}$, can be described by:

$$\left\{ \begin{array}{rcl} \overset{*}{x}_0 & = & \dfrac{x_1}{\widehat{\gamma}(w)} \ , \\ & \vdots & \\ \overset{*}{x}_{n-1} & = & \dfrac{x_n}{\widehat{\gamma}(w)} \ , \\ \overset{*}{x}_n & = & \dfrac{f_n(x_0, \dots, x_n)}{\widehat{\gamma}(w)} \ , \\ \overset{*}{w} & = & -\dfrac{w - \gamma_1(x_0)}{\widehat{\gamma}(w)} \ , \\ \overset{*}{\varepsilon}_i & = & -\dfrac{a_i}{\widehat{\gamma}(w)}\, \varepsilon_i - \displaystyle\sum_{j=1}^{i-1} \overset{*}{a}_i h_{ij}(a_{j+1}, \dots, a_i)\, \varepsilon_j \ + \\ & & \dfrac{x_{i+1} - v_i}{\widehat{\gamma}(w)} \ , \end{array} \right. \tag{30}$$

and the e_i's are given by:

$$e_i \ = \ \varepsilon_i \ + \ \sum_{j=1}^{i-1} \ell_{ij}(a_{j+1}, \dots, a_i)\, \varepsilon_j \ . \tag{31}$$

We our now ready to state one of the main results of the paper.

Theorem 1: Given the functions $V(x)$, γ_1, γ_2 and f_n, we can find expressions for the functions a_i's and v_i's such that for system (30) the set:

$$\mathcal{A} \ = \ \{(x, w, \varepsilon) : \, V(x) \leq 2w + 1 \, , \ \varepsilon = 0\} \tag{32}$$

is globally asymptotically stable.

IV. CONCLUSION

The observer we propose is:

$$\left\{\begin{array}{rcl} \dot{z}_i & = & -a_i\, z_i \;+\; [i\dot{a}_i + a_i^2]\,(-a_i)^{i-1}\, x_0 \;+ \\ & & \dot{a}_i \displaystyle\sum_{j=1}^{i-1} (i-j)(-a_i)^{i-j-1}\widehat{x}_j \;-\; v_i \;, \\ \widehat{x}_i & = & -(-a_i)^i x_0 \;-\; \displaystyle\sum_{j=1}^{i-1}(-a_i)^{i-j}\widehat{x}_j \;-\; z_i \;, \\ \dot{w} & = & -w + \gamma_1(x_0) \end{array}\right. \tag{33}$$

where the v_i's have to be selected as :

$$\begin{array}{rcl} v_i & = & \widehat{x}_{i+1} \quad \forall i \in \{1, \ldots, n-1\}\;, \\ v_n & = & 2\widehat{\gamma}(w)\, \mathtt{sat}\left(\dfrac{f_n(x_0, \widehat{x}_1, \ldots, \widehat{x}_n)}{2\widehat{\gamma}(w)}\right)\;. \end{array}$$

Recall that the a_i's involved here being functions of w, the notation $\dot{a}_i$ means simply:

$$\dot{a}_i \;=\; \frac{da_i}{dw}(w)\,[-w + \gamma_1(x_0)]\;.$$

By rewriting this observer (33) in the compact form:

$$\left\{\begin{array}{rcl} \dot{\mathcal{x}} & = & F(\mathcal{x}, x_0)\;, \\ \widehat{x} & = & H(\mathcal{x}, x_0)\;, \end{array}\right. \tag{34}$$

and by denoting by $X(x,t)$ the solutions of the system (1) and by $\mathcal{X}(\mathcal{x}, t; x_0)$ the solutions of this system (34), coupled to (1), we have established the following statement.

Theorem 2: For the globally completely observable system (1), under Assumption 1, we can construct[3] functions a_i's and $\widehat{\gamma}$, the a_i's being C^1, such that for each solution $X(x,t)$ of (1), right maximally defined on $[0,T)$, with $T \leq +\infty$, and for each initial condition $\mathcal{x}$, the associated solution $\mathcal{X}(\mathcal{x}, t; x_0)$ of (34) is defined also on $[0,T)$ and satisfies:

$$\lim_{t\to T} |X(x,t) - H(\mathcal{X}(\mathcal{x}, t; x_0), X_0(x,t))| \;=\; 0\;. \tag{35}$$

This result says that we have obtained an observer which gives an estimate of the system state which converges to the actual value in infinite time, if there is no finite escape time of this actual value and at the time of the escape if there is a finite escape time.

[3]To be precise, the functions a_i's are C^1 functions that can be expressed from the problem data γ_1 and γ_2 and from the function γ, chosen to satisfy:

$$|x_0| + \ldots + |x_n| + |f_n(x_0, \ldots, x_n)| \;\leq\; \gamma(V(x))$$

and

$$\sup_{|\eta_i|\leq 1}\left\{\frac{|f_n(x_0, x_1+\eta_1, \ldots, x_n+\eta_n) - f_n(x_0, x_1, \ldots, x_n)|}{\sqrt{\displaystyle\sum_{i=1}^{n}\eta_i^2}}\right\} \leq \gamma(V(x))\,.$$

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