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On Certainty-Equivalence Design of Nonlinear Observer-Based Controllers

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1 Introduction

For nonlinear systems, the implementation of a state feedback with observer estimates may lead to severe forms of instability. For example, the system (see [8]) :

$$\dot{x}_1 = -x_1 + x_2 x_1^2 + u \ , \ \dot{x}_2 = -x_2 + x_1^2 \ , \ y = x_1 \ , \quad (1)$$

with state (x_1, x_2), input u and output y, admits the globally asymptotically stabilizing state feedback :

$$u \ = \ \phi(x_1, x_2) \ = \ -x_2 x_1^2 \quad (2)$$

and the globally exponentially convergent reduced order observer :

$$\dot{\widehat{x}}_2 \ = \ -\widehat{x}_2 \ + \ y^2 \ . \quad (3)$$

However, the certainty-equivalence controller :

$$\dot{\widehat{x}}_2 \ = \ -\widehat{x}_2 \ + \ y^2 \quad , \qquad u \ = \ -\widehat{x}_2 y^2 \quad (4)$$

generates solutions which escape in finite time.

The standard approach to analyzing observer-based controllers is to treat the observer error as a "measurement error" as in [4, 1, 16]. In this paper we eliminate the conservatism of this approach by explicitly including the observer dynamics in stability analysis. This is achieved with a new detectability concept which, when combined with an additional condition on the Lyapunov function for the underlying full-state feedback design, guarantees stability of certainty-equivalence. An application of our result to *strict-feedback* systems [9] shows that, under a mild polynomial growth assumption on nonlinearities, stability can be achieved with a certainty-equivalence implementation of full-state backstepping designs.

For local [3] or semi-global [16] stability, general results are available under weak notions of stabilizability and detectability. In this paper we are interested in a global result, because it fully encompasses the nonlinear features of the system and the phenomena on the boundary of the domain of attraction.

In section 2 we recall some useful technicalities. In section 3, we define the notions of stabilizability and detectability that are crucial for our analysis. The main result is proved in section 4, and applied to several classes of systems.

2 Key technical Lemmas

Our objective in defining stabilizability and detectability here is to allow us to analyze the closed-loop system via an auxiliary system of the form :

$$\dot{x} \ = \ f(x, d) \quad (5)$$

with f continuous. For such a system we have the following sufficient condition for boundedness :

Lemma 1 ([10, Chapter 4]) *Given $d : \mathbb{R} \to \mathbb{R}^d$ as a continuous function, assume the existence of a non-negative, radially unbounded, C^1 function V and of a continuous non-negative function δ satisfying :*

$$\frac{\partial V}{\partial x}(x,t)\, f(x, d(t)) \ + \ \frac{\partial V}{\partial t}(x,t) \ \leq \ \delta(t) \ . \quad (6)$$

Under this condition, if, for some t_0, the function δ is in $L^1([t_0, +\infty))$, then all the solutions of the system (5), starting from time t_0, are right maximally defined and bounded on $[t_0, +\infty)$.

In practice it may be difficult to find a Lyapunov function whose gradient is small enough to satisfy (6). The following lemma shows how a polynomial function V can be redesigned to reduce the magnitude of its gradient :

Lemma 2 ([6, Lemma 2]) *If V is a positive definite, radially unbounded and polynomial function, then $\tilde{V}(x) := \log(1 + V(x))$ satisfies, for some constant $c > 0$,*

$$|x| \frac{\partial \tilde{V}(x)}{\partial x} \ \leq \ c \qquad \forall x \in \mathbb{R}^n \ . \quad (7)$$

3 Assumptions

The system we consider is :

$$\dot{x} \ = \ f(x, u) \quad , \qquad y \ = \ h(x) \quad (8)$$

with state x in $\mathbb{R}^n$, input u in $\mathbb{R}^p$ and output y in $\mathbb{R}^q$ and where the functions f and h are Lipschitz continuous. We assume the availability of an observer and a controller, as made precise in the following definitions.

3.1 Stabilizability

Definition 3 (Stabilizability) *The system (8) is said to be stabilizable if there exist a Lipschitz continuous function $\phi : \mathbb{R}^n \to \mathbb{R}^p$ and a C^1, positive definite and radially unbounded function V such that the function :*

$$W(x) = -\frac{\partial V}{\partial x}(x) f(x, \phi(x)) \quad (9)$$

is positive definite.

For example, for the system (1), a control Lyapunov function associated with ϕ given in (2) is :

$$V(x_1, x_2) = \frac{1}{4} x_1^4 + \frac{1}{2} x_2^2 . \quad (10)$$

Similarly, for the system :

$$\dot{x}_1 = x_1 + x_2 \quad , \qquad \dot{x}_2 = x_2 - x_2^5 + u , \quad (11)$$

a control Lyapunov function and a control law are :

$$V(x_1, x_2) = \frac{1}{2}\left(x_1^2 + (x_2 + 2x_1)^2\right) , \quad (12)$$

$$\phi(x_1, x_2) = -2x_1 - 3x_2 + x_2^5 . \quad (13)$$

For these two examples, Lemma 2 applies because the Lyapunov functions are polynomial. More generally, we have :

Proposition 4 *If, for the following system :*

$$\left\{\begin{array}{rcl} \dot{x}_1 &=& x_2 + f_1(x_1) , \\ \dot{x}_2 &=& x_3 + f_2(x_1, x_2) , \\ &\vdots& \\ \dot{x}_n &=& u + f_n(x_1, \ldots, x_n) , \end{array}\right. \quad (14)$$

the functions f_i's are Lipschitz continuous and satisfy :

$$\begin{aligned} &|f_i(x_1, \ldots, x_i)| \quad (15) \\ &\quad \leq c_i\left(|x_1| + \ldots + |x_i| + |x_1|^{r_1} + \ldots + |x_i|^{r_i}\right) \end{aligned}$$

where the c_i's are real numbers and the r_j's are integers, then there exists a control Lyapunov function satisfying (7).

This result is proved by following the standard backstepping procedure, with virtual controls obtained from a domination design as in [15, Proposition 3.35].

3.2 Detectability

Before giving our definition of detectability, we derive equation (18) below, which encompasses both full- and reduced-order observer structures. The dynamic system :

$$\dot{\widehat{x}} = \varphi(\widehat{x}, u, y) \quad (16)$$

where φ is a C^1 function, is an observer for system (8) if $\widehat{x} \in \mathbb{R}^n$ asymptotically converges to x. In particular, $x = \widehat{x}$ must be an invariant manifold. This leads to the constraint :

$$\varphi(x, u, h(x)) = f(x, u) \qquad \forall (x, u) , \quad (17)$$

which implies the existence of a continuous function k such that (16) can be rewritten as :

$$\dot{\widehat{x}} = f(\widehat{x}, u) + k(\widehat{x}, u, h(x))\,[h(x) - h(\widehat{x})] . \quad (18)$$

For the system (1), an observer of the form (18) is :

$$\left\{\begin{array}{l} \dot{\widehat{x}}_1 = -\widehat{x}_1 + \widehat{x}_2 y^2 + u - k_1(\widehat{x}_1, \widehat{x}_2, u, x_1) e_1 , \\ \dot{\widehat{x}}_2 = -\widehat{x}_2 + y^2 - y^2 e_1 , \end{array}\right. \quad (19)$$

where k_1 is any C^1 function with non negative values and with the notation :

$$e_1 = \widehat{x}_1 - x_1 . \quad (20)$$

Likewise, for the system (11), from [2, Example 1], we get the observer :

$$\left\{\begin{array}{l} \dot{\widehat{x}}_1 = y + \widehat{x}_2 - 2(\widehat{x}_1 - y) , \\ \dot{\widehat{x}}_2 = \widehat{x}_2 - \left[\widehat{x}_2 - \frac{3}{2}(\widehat{x}_1 - y)\right]^5 + u - 3(\widehat{x}_1 - y) , \end{array}\right. \quad (21)$$

which can be rewritten as :

$$\left\{\begin{array}{l} \dot{\widehat{x}}_1 = \widehat{x}_1 + \widehat{x}_2 - 3e_1 , \\ \dot{\widehat{x}}_2 = \widehat{x}_2 - \widehat{x}_2^5 + u \\ -\left[3 - \frac{15}{2}\widehat{x}_2\left(\widehat{x}_2^3 - 3\widehat{x}_2^2 e_1 + \frac{9}{2}\widehat{x}_2 e_1^2 - \frac{27}{8} e_1^3\right) e_1 - e_1^4\right] e_1 . \end{array}\right. \quad (22)$$

Because their conditions for existence are less restrictive (see [13]), we want to consider also reduced-order observers. When the output is used as a global coordinate, the system (8) can be rewritten as :

$$\dot{x}_1 = f_1(x_1, x_2, u) \ , \ \dot{x}_2 = f_2(x_1, x_2, u) \ , \ y = x_1 . \quad (23)$$

In this case a reduced order observer has the form :

$$\left\{\begin{array}{rcl} \dot{\chi}_2 &=& f_2(x_1, \widehat{x}_2, u) + \frac{\partial g}{\partial x_1}(x_1) f_1(x_1, \widehat{x}_2, u) , \\ \widehat{x}_2 &=& \chi_2 - g(x_1) \end{array}\right. \quad (24)$$

where g is a C^1 function. To show that an equation of the form (18) can still be written, we first observe that we have :

$$\begin{aligned} \dot{\widehat{x}}_2 &= f_2(x_1, \widehat{x}_2, u) \quad (25) \\ &\quad + \frac{\partial g}{\partial x_1}(x_1)\left[f_1(x_1, \widehat{x}_2, u) - f_1(x_1, x_2, u)\right] . \end{aligned}$$

Then, by letting formally :

$$\widehat{x}_1 = x_1 \quad , \qquad h(x_1, x_2) = x_2 , \quad (26)$$

we get :

$$\begin{aligned} \dot{\widehat{x}}_1 &= f_1(\widehat{x}_1, \widehat{x}_2, u) \quad (27) \\ &\quad + k_1((\widehat{x}_1, \widehat{x}_2), u, h((\widehat{x}_1, \widehat{x}_2)))\,[h(x_1, x_2) - h(\widehat{x}_1, \widehat{x}_2)] , \end{aligned}$$

with :

$$\begin{aligned} &k_1((\widehat{x}_1, \widehat{x}_2), u, h((x_1, x_2))) \quad (28) \\ &\quad = \int_0^1 \frac{\partial f_1}{\partial x_2}(\widehat{x}_1, \widehat{x}_2 + s(x_2 - \widehat{x}_2), u)\, ds . \end{aligned}$$

So, in this case, the form (18) is obtained with the notation :

$$\widehat{x} = \begin{pmatrix} \widehat{x}_1 \\ \widehat{x}_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ \widehat{x}_2 \end{pmatrix} , \quad (29)$$

and the gain :

$$k(\widehat{x}, u, h(x)) = \begin{pmatrix} I \\ \frac{\partial g}{\partial x_1}(\widehat{x}_1) \end{pmatrix} k_1(\widehat{x}, u, h(x)) . \quad (30)$$

For the system (11) a possible reduced order observer is (see [2, Example 1]) :

$$\dot{x}_2 = -\frac{1}{2}\widehat{x}_2 - \widehat{x}_2^5 - \frac{3}{2}y + u \ , \quad \widehat{x}_2 = x_2 + \frac{3}{2}y \ . \quad (31)$$

It can be rewritten simply as :

$$\dot{x}_1 \ = \ x_1 + \widehat{x}_2 - e_2 \quad , \quad \dot{\widehat{x}}_2 \ = \ \widehat{x}_2 - \widehat{x}_2^5 + u - \frac{3}{2}e_2 \ , \ (32)$$

with :

$$e_2 \ = \ \widehat{x}_2 \ - \ x_2 \ . \quad (33)$$

Finally, for systems in the form (23), it may be useful to design a full-order observer but with x_1 as argument of the functions instead of $\widehat{x}_1$. This leads to an observer in the form :

$$\left\{ \begin{array}{rcl} \dot{\widehat{x}}_1 & = & f_1(x_1,\widehat{x}_2,u) \ + \ k_1(x_1,e_2,\widehat{x}_2)\, e_2 \ , \\ \dot{\widehat{x}}_2 & = & f_2(x_1,\widehat{x}_2,u) \ + \ k_2(x_1,e_1,\widehat{x}_2)\, e_1 \ . \end{array} \right. \quad (34)$$

Again, we recover an equation of the form (18) by letting formally[1] :

$$\widehat{x} \ = \ \left(\begin{array}{c} x_1 \\ \widehat{x}_2 \end{array} \right) \quad , \quad h(\widehat{x}) - h(x) \ = \ \left(\begin{array}{c} e_1 \\ e_2 \end{array} \right) \ , \quad (35)$$

$$k \ = \quad (36)$$

$$\left(\begin{array}{cc} 0 & \int_0^1 \frac{\partial f_1}{\partial x_2}(x_1,\widehat{x}_2 - s e_2, u)\, ds \\ k_2(\widehat{x}_1 + e_1, \widehat{x}_2, x_1) & 0 \end{array} \right) .$$

For example, for the system :

$$\dot{x}_1 = x_2 + x_1^3 \ , \quad \dot{x}_2 = u \ , \quad y = x_1 \ , \quad (37)$$

a full order observer is :

$$\dot{\widehat{x}}_1 = \widehat{x}_2 + y^3 + 2\,(\widehat{x}_1 - y) \ , \quad \dot{\widehat{x}}_2 = u - (\widehat{x}_1 - y) \ . \quad (38)$$

A direct way to write (18) is :

$$\left\{ \begin{array}{l} \dot{\widehat{x}}_1 = \widehat{x}_2 + \widehat{x}_1^3 - \left[\widehat{x}_1^2 + \widehat{x}_1 y + y + 2\right](\widehat{x}_1 - y) \ , \\ \dot{\widehat{x}}_2 = u - (\widehat{x}_1 - y) \ , \end{array} \right. \quad (39)$$

so with a gain quadratic in $\widehat{x}_1$:

$$k \ = \ -\left(\begin{array}{c} \widehat{x}_1^2 + \widehat{x}_1 y + y + 2 \\ 1 \end{array} \right) . \quad (40)$$

But, following the above, we can also write (18) as :

$$\dot{x}_1 = \widehat{x}_2 + x_1^3 - 2\,(\widehat{x}_2 - x_2) \ , \ \dot{\widehat{x}}_2 = u - (\widehat{x}_1 - x_1) \ , \ (41)$$

with now a gain k constant.

Definition 5 (Uniform Detectability) *The system (8) is said to be uniformly detectable if there exist continuous functions $k_l : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^n \times \mathbb{R}^m$, $k_r : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^m \times \mathbb{R}^n$ and $k_0 : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^n$ and a continuous function $\alpha : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ such that, for each x in $\mathbb{R}^n$, $\alpha(x,\cdot)$ is of class $\mathcal{K}_\infty$, and, for each function u in $L^\infty_{loc}([0,+\infty))$ and each initial condition $(x,\widehat{x})$, the solution $(X(t),\widehat{X}(t))$ of :*

$$\left\{ \begin{array}{l} \dot{x} = f(x,u) \ , \\ \dot{\widehat{x}} = f(\widehat{x},u) \ + \ k(\widehat{x},u,h(x))\,[h(x) - h(\widehat{x})] \ , \end{array} \right. \quad (42)$$

with the notation :

$$k \ = \ k_l\, k_r \ + \ k_0 \ , \quad (43)$$

satisfies :

$$|X(t) - \widehat{X}(t)| \ + \ \int_0^T |h(X(s)) - h(\widehat{X}(s))|\, ds \quad (44)$$

$$+ \int_0^T \left| k_r(\widehat{X}(s),u(s),h(X(s)))[h(X(s)) - h(\widehat{X}(s))] \right| dt$$

$$< \ \alpha(\widehat{x},|x - \widehat{x}|)$$

for all t in $[0,T)$, the right maximal interval of definition. And, when $T = +\infty$, we have :

$$\lim_{t \to +\infty} |X(t) - \widehat{X}(t)| \ = \ 0 \ . \quad (45)$$

□

To illustrate the property (44), we observe that, for the system (1), with the full-order observer (19), we get :

$$\dot{\overbrace{\sqrt{e_1^2 + e_2^2}}} \ = \ -\sqrt{e_1^2 + e_2^2} \ - \ k_1\, e_1^2 \ . \quad (46)$$

This implies that (44) holds with $k_r = 0$.

Another illustration is given by systems for which we can get a reduced-order observer (24) of the form :

$$\dot{x}_2 \ = \ A\,\widehat{x}_2 \ + \ \textstyle\sum_i G_i\, \gamma_i(a_i(y) + H_i \widehat{x}_2) \quad (47)$$

where the scalar functions γ_i's are non-increasing :

$$H_i^T e_2\, [\gamma_i(a_i + H_i \widehat{x}_2) - \gamma_i(a_i + H_i[\widehat{x}_2 - e_2])] \quad (48)$$

$$= -|H_i^T e_2|\, |\gamma_i(a_i + H_i \widehat{x}_2) - \gamma_i(a_i + H_i[\widehat{x}_2 - e_2])|$$

This is the case of (31) or, more generally, of the reduced order observers proposed in [2]. The error system associated with this observer takes the form :

$$\dot{e}_2 \ = \ A\, e_2 \quad (49)$$

$$+ \ \textstyle\sum_i G_i\, (\gamma_i(a_i(y) + H_i \widehat{x}_2) - \gamma_i(a_i(y) + H_i[\widehat{x}_2 - e_2])) \ .$$

We restrict our attention to the case where its solutions $E_2(t)$ satisfy :

$$\int_0^{+\infty} |E_2(t)| dt \ \leq \ \overline{\alpha}(|E_2(0)|) \quad (50)$$

where $\overline{\alpha}$ is a function of class $\mathcal{K}_\infty$ and the G_i's and H_i's satisfy :

$$H_i^T G_j \ = \ 0 \quad \forall i \neq j \quad , \qquad H_i^T G_i \ > \ 0 \ . \quad (51)$$

For such a case, with (48), we get, with D^+ denoting the upper right Dini derivative,

$$D^+|H_i^T e_2| \ \leq \ |H_i^T A e_2| \quad (52)$$

$$- \ H_i^T G_i\, |\gamma_i(a_i(y) + H_i \widehat{x}_2) - \gamma_i(a_i(y) + H_i[\widehat{x}_2 - e_2])| \ .$$

It follows that :

$$\int_0^{+\infty} \left| \begin{array}{l} \gamma_i(a_i(Y(t)) + H_i \widehat{X}_2(t)) \\ \qquad - \gamma_i(a_i(Y(t)) + H_i[\widehat{X}_2(t) - E_2(t)]) \end{array} \right| \quad (53)$$

$$\leq \ \frac{(|H_i| + |A^T H_i|)\ \overline{\alpha}(|E_2(0)|)}{H_i^T G_i} \ .$$

For instance, for (31), we get :

[1] With an abuse of notation since $\widehat{x}_1$, i.e. e_1, is not defined from (23) and (34) only.

$$\dot{\overline{|e_2|}} \leq -\tfrac{1}{2}|e_2| - |\widehat{x}_2^5 - (\widehat{x}_2 - e_2)^5| , \qquad (54)$$

and (44) holds with $k = k_r$, i.e. $k_l = 1$ and $k_0 = 0$.

Finally for (41), since the observer error tends exponentially to 0, uniformly in (x_1, x_2), we get also that (44) holds with $k = k_r$, i.e. $k_l = 1$ and $k_0 = 0$. But for (39), (44) holds only with $k = k_0$, i.e. $k_r = 0$.

4 Stability of Certainty-Equivalence

For a system (8) which is stabilizable and uniformly detectable, an output feedback designed with the certainty-equivalence paradigm is :

$$\dot{\widehat{x}} = f(\widehat{x}, u) + k(\widehat{x}, u, y)\,[y - h(\widehat{x})] , \quad u = \phi(\widehat{x}) . \quad (55)$$

For stability analysis of the closed-loop system, studies in the literature have used (x, e)-coordinates, where e denotes the observer error

$$e = \widehat{x} - x . \qquad (56)$$

A common approach in this framework is to put restrictions on $\frac{\partial f}{\partial u}$ and $\frac{\partial \phi}{\partial x}$. This approach motivated the design of a state feedback ϕ making the system input-to-state stable with respect to e, as in [4, Chapter 6]. Another approach, pursued by [16], is to put restrictions on $\frac{\partial f}{\partial x}$.

A drawback of these approaches is that e is treated as a "measurement error", and no use is made of the structure and convergence properties of the observer. Instead, we use $(\widehat{x}, e)$-coordinates which explicitly include the observer dynamics. This approach was used earlier in adaptive control [11]. It leads to a new set of stability conditions, as derived below.

4.1 Main result

When we use $(\widehat{x}, e)$-coordinates, the closed-loop system is :

$$\begin{cases} \dot{\widehat{x}} = f(\widehat{x}, \phi(\widehat{x})) \\ \qquad + k(\widehat{x}, \phi(\widehat{x}), h(\widehat{x} - e))\,[h(\widehat{x} - e) - h(\widehat{x})] , \\ \dot{e} = [f(\widehat{x}, \phi(\widehat{x})) - f(\widehat{x} - e, \phi(\widehat{x}))] \\ \qquad + k(\widehat{x}, \phi(\widehat{x}), h(\widehat{x} - e))\,[h(\widehat{x} - e) - h(\widehat{x})] . \end{cases} \qquad (57)$$

In this case, a precise result is :

Theorem 6 *Assume the system (8) is stabilizable with a control Lyapunov function V and is uniformly detectable with a gain function k. If V and k are such that, with the decomposition (43), there exists a function $\mathcal{L}$ which is C^1, of class $\mathcal{K}_\infty$ and satisfies, for all (x, e),*

$$\left|\frac{\partial \mathcal{L}(V)}{\partial x}(x)\, k_l(x, \phi(x), h(x - e))\right| \qquad (58)$$
$$+ \left|\frac{\partial \mathcal{L}(V)}{\partial x}(x)\, k_0(x, \phi(x), h(x - e))\right| \leq \gamma(|e|) ,$$

where γ is some continuous function, then the origin is globally asymptotically stable for the closed-loop system.

Proof : Let $(\widehat{X}(t), E(t))$ be any solution of (57) starting from $(\widehat{x}, e)$. Let $[0, T(\widehat{x}, e))$ be its right maximal interval of definition. From the stabilizability assumption and (58), we get, for all $(\widehat{x}, e)$,

$$\dot{\overline{\mathcal{L}(V(\widehat{x}))}} \leq \qquad (59)$$
$$\gamma(|e|)\,|h(\widehat{x}) - h(\widehat{x} - e)|\,[|k_r(\widehat{x}, \phi(\widehat{x}), h(\widehat{x} - e)| + 1] .$$

So, with the detectability property, this implies in particular, for all $t \in [0, T(\widehat{x}, e))$,

$$\mathcal{L}(V(\widehat{X}(t))) \leq \mathcal{L}(V(\widehat{x})) + \gamma(\alpha(\widehat{x}, |e|))\,\alpha(\widehat{x}, |e|) . \quad (60)$$

Since the right hand side can be upperbounded by a class $\mathcal{K}_\infty$ function of $(\widehat{x}, e)$, $T(\widehat{x}, e)$ must be infinite and we have global stability of the origin.

Then, to establish attractiveness, we note that we have :

$$\dot{\overline{V(\widehat{x})}} \leq -W(\widehat{x}) + \Gamma(|\widehat{x}| + |e|) \times \qquad (61)$$
$$\times\ |h(\widehat{x}) - h(\widehat{x} - e)|\,[|k_r(\widehat{x}, \phi(\widehat{x}), h(\widehat{x} - e)| + 1]$$

where, thanks to the global stability, Γ is a function, bounding $\frac{\partial V}{\partial x} k_l$ and $\frac{\partial V}{\partial x} k_0$ along the solution, whose argument is the initial condition. It follows that we have :

$$\int_0^t W(\widehat{X}(s))\, ds \leq V(\widehat{x}) + \Gamma(|\widehat{x}| + |e|)\,\alpha(\widehat{x}, |e|) . \quad (62)$$

With the help of Barbalat's Lemma, we conclude that $W(\widehat{X}(t))$ and therefore $|\widehat{X}(t)|$ tend to 0 as t tends to infinity. With (45), i.e. :

$$\lim_{t \to +\infty} |E(t)| = 0 , \qquad (63)$$

this implies the global asymptotic stability. □

4.2 Applications of Theorem 6

For the system (1), we have seen that the combination of the state feedback (2) and the reduced-order observer (3) does not give a stabilizing output feedback. So let us study now the combination of the state feedback (2) and the full-order observer (19). We have seen that the uniform detectability assumption holds simply with $k_r = 1$. So with (19) and (10), the condition (58) of Proposition 6 is :

$$\mathcal{L}'\left(\frac{x_1^4}{4} + \frac{x_2^2}{2}\right) \left|-x_1^3 k_1 - x_2 x_1^2\right| \leq \gamma \quad \forall (x_1, x_2) . \quad (64)$$

It is met with the gain k_1 satisfying :

$$0 \leq k_1(\widehat{x}_1, \widehat{x}_2, u, x_1) \leq |x_1| \qquad (65)$$

and with the function :

$$\mathcal{L}(v) = \log(1 + v) \qquad (66)$$

which is appropriately C^1 and of class $\mathcal{K}_\infty$. So for instance a globally asymptotically stabilizing output feedback is :

$$\dot{\widehat{x}}_1 = -\widehat{x}_1 , \ \dot{\widehat{x}}_2 = -\widehat{x}_2 + y^2 - y^2(\widehat{x}_1 - y) , \ u = -\widehat{x}_2\, y^2 , \qquad (67)$$

and therefore, more simply :

$$\dot{\widehat{x}}_2 = -\widehat{x}_2 + y^2 + y^3 , \quad u = -\widehat{x}_2\, y^2 . \qquad (68)$$

For exactly the same reasons, from (31) and (13),

$$\left\{ \begin{array}{rcl} \dot{\hat{x}}_2 &=& -\frac{1}{2}\hat{x}_2 - \hat{x}_2^5 - \frac{3}{2}y \,, \\ u &=& -\frac{13}{2}y - 3x_2 + \left[x_2 + \frac{3}{2}y\right]^5 \end{array} \right. \tag{69}$$

is a globally asymptotically stabilizing output feedback for the system (11).

We consider now systems in the strict feedback form (14) or more generally in the form :

$$\left\{ \begin{array}{rl} \dot{x}_1 = & x_2 + f_1(x_1) \,, \\ \dot{x}_2 = & x_3 + f_2(x_1, x_2) \,, \\ & \vdots \\ \dot{x}_m = & x_{m+1} + f_m(x_1, \dots, x_m) + u \,, \\ \dot{x}_{m+1} = & x_{m+2} + f_{m+1}(x_1, \dots, x_{m+1}, u) \,, \\ & \vdots \\ \dot{x}_n = & f_n(x_1, \dots, x_n, u) \,, \\ y = & x_1 \,, \end{array} \right. \tag{70}$$

where the functions f_i's are globally Lipschitz in $(x_2, \dots, x_n)$, uniformly in (x_1, u). We know from [5] there exists an observer with constant gain k. So Theorem 6 applies for example if :

1. the system :

$$\left\{ \begin{array}{l} \dot{x}_{m+1} = x_{m+2} + f_{m+1}(x_1, \dots, x_{m+1}, x_{m+1} + v) \,, \\ \quad \vdots \\ \dot{x}_n = f_n(x_1, \dots, x_n, x_{m+1} + v) \,, \end{array} \right. \tag{71}$$

   is input-to-state stable with $(x_1, \dots, x_m, v)$ as input.

2. for the system :

$$\left\{ \begin{array}{l} \dot{x}_1 = x_2 + f_1(x_1) \,, \\ \dot{x}_2 = x_3 + f_2(x_1, x_2) \,, \\ \quad \vdots \\ \dot{x}_m = x_{m+1} + f_m(x_1, \dots, x_m) + v \,, \end{array} \right. \tag{72}$$

   we know a control Lyapunov function such that the function $\frac{\frac{\partial V}{\partial x}(x_1, \dots, x_m)}{1 + V(x_1, \dots, x_m)}$ is bounded on $\mathbb{R}^m$. With Proposition 4, we know this latter condition is met when the condition (15) holds.

In particular, this establishes for systems which admit an output-feedback form[2] (see [9, (7.29)]) that, when the nonlinearities in x_1 are bounded by polynomials, there is no need for the nonlinear damping technique proposed in [9, chapter 7].

Furthermore since the nonlinear damping is not needed, we do not need to design the control law by exploiting the specific partial triangular structure. A direct consequence is that the above input-to-state stability for the system (71) can be relaxed if we know another design leading to a control Lyapunov function for the overall system with $\frac{\frac{\partial V}{\partial x}(x)}{1+V(x)}$ bounded. To illustrate this point, we consider the system :

$$\dot{x}_1 = x_2 + u \,, \; \dot{x}_2 = f(x_1) + x_3 - u \,, \; \dot{x}_3 = -f(x_1) \tag{73}$$

still with x_1 as output and where f a C^1 function which is zero at the origin. This system is non minimum phase. Except for the presence of $f(x_1)$, this system is an observable linear system. Hence there exist real numbers k_1, k_2 and k_3 such that an appropriate full order observer is :

$$\left\{ \begin{array}{rcl} \dot{\hat{x}}_1 &=& \hat{x}_2 + u - k_1(\hat{x}_1 - y) \,, \\ \dot{\hat{x}}_2 &=& f(y) + \hat{x}_3 - u - k_2(\hat{x}_1 - y) \,, \\ \dot{\hat{x}}_3 &=& -f(y) - k_3(\hat{x}_1 - y) \,. \end{array} \right. \tag{74}$$

It leads to a system in the form (34) as :

$$\left\{ \begin{array}{rcl} \dot{\hat{x}}_1 &=& \hat{x}_2 + u - e_2 \,, \\ \dot{\hat{x}}_2 &=& f(x_1) + \hat{x}_3 - u - k_2 e_1 \,, \\ \dot{\hat{x}}_3 &=& -f(x_1) - k_3 e_2 \,. \end{array} \right. \tag{75}$$

So (44) holds with $k = k_r$, i.e. $k_l = 1$ and $k_0 = 0$.

Now, to design the state feedback, it is appropriate to introduce new coordinates :

$$\chi_3 = x_3 \,, \quad \chi_2 = x_1 + x_2 + x_3 \,, \quad \chi_1 = x_2 + x_3 \,. \tag{76}$$

Then, via a backstepping design for the (χ_1, χ_2)-subsystem, completed by the forwarding modulo $L_g V$ procedure of [14], we can get the following pair for the stabilizability :

$$V(\chi_3, \chi_2, \chi_1) = V_2(\chi_2) + \tfrac{1}{2}(\chi_1 + \chi_2)^2 + \left(\chi_3 - \int_0^{\chi_2} \tfrac{f(2s)}{s} ds\right)^2 , \tag{77}$$

$$\phi(x_1, x_2, x_3) = x_3 + (\chi_2 + \chi_1) + \chi_1 + V_2'(\chi_2) + \left(\chi_3 - \int_0^{\chi_2} \tfrac{f(2s)}{s} ds\right) \left(-\tfrac{f(2\chi_2)}{\chi_2} + \tfrac{f(2\chi_2) - f(\chi_2 - \chi_1)}{\chi_2 + \chi_1}\right) . \tag{78}$$

Actually this V is not a strict Lyapunov function, but La Salle invariance principle applies. Its interest is that condition (58) of Theorem 6 holds with (66), if f meets a growth condition such that we can find a function V_2 with the appropriate properties and satisfying :

$$\max\left\{ \left| \frac{f(2\chi_2)}{\chi_2} \right|^2 , |V_2'(\chi_2)| \right\} \le c(1 + V_2(\chi_2)) \tag{79}$$

for some real number c. It is satisfied for instance by :

$$f(x) = x \exp(x) \,. \tag{80}$$

Then by applying Theorem 6 completed with La Salle invariance principle, we can show that a globally asymptotically stabilizing output feedback is :

$$\left\{ \begin{array}{rcl} \dot{\hat{x}}_1 &=& \hat{x}_2 + u - k_1(\hat{x}_1 - y) \,, \\ \dot{\hat{x}}_2 &=& f(y) + \hat{x}_3 - u - k_2(\hat{x}_1 - y) \,, \\ \dot{\hat{x}}_3 &=& -f(y) - k_3(\hat{x}_1 - y) \,, \\ u &=& \phi(y, \hat{x}_2, \hat{x}_3) \,. \end{array} \right. \tag{81}$$

When applying Theorem 6 to the system (70), it is important to consider the use of (34) for writing the detectability assumption, i.e. the use of $x_1 = y$ and not of $\hat{x}_1$ both for writing the observer and the control. This point is another departure from the nominal separation principle. In fact when u is taken as $u = \phi(\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n)$ with the estimate $\hat{x}_1$ instead of the true output y, we may get instability. For instance, for the system (37), an appropriate pair for the stabilizabil-

[2] The function f_i's are linear in $(x_2, \dots, x_n)$.

ity is :

$$V(x_1,x_2) = \frac{1}{2}\left(x_1^2+(x_2+x_1+x_1^3)^2\right) , \quad (82)$$

$$\phi(x_1,x_2) = -(x_2+x_1+x_1^3)-(1+3x_1^2)(x_2+x_1^3) \quad (83)$$

So Lemma 2 applies. This implies that the condition (58) of Theorem 6 holds if the observer gain is linearly bounded in $|\widehat{x}|$. This is the case of (41) corresponding to a computation of the control as :

$$u = \phi(x_1,\widehat{x}_2) . \quad (84)$$

But it is not the case with (39) corresponding to :

$$u = \phi(\widehat{x}_1,\widehat{x}_2) . \quad (85)$$

Actually, it turns out that the certainty-equivalence controller (39),(85) leads to closed-loop solutions which escape in finite time.

5 Further remarks on Theorem 6

In the condition (58), we have not taken advantage of the stability margin quantified by the function W and given by the state feedback. It is by using this margin that this condition can be relaxed. In particular, we can again approach the problem as the one of enforcing via the state feedback an input-to-state stability property. But this time it is not with respect to a measurement noise but with respect to the disturbance e present in :

$$\dot{x} = f(x,u) + k(x,u,h(x-e))\,[h(x)-h(x-e)] . \quad (86)$$

This implies that the state feedback can be designed only once the observer is known. Fortunately, such an input-to-state stability property can be obtained at least for strict feedback systems by applying the nonlinear damping technique of [7].

Instead of trying to get this input-to-state stability property, it has also been proposed to modify the observer. This is done for instance for output feedback systems in the interlaced controller-observer design of [9, Section 7.4.1], and for systems which are linear in the unmeasured state components as in [12].

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