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Asymptotic stabilization via output feedback for lower triangular systems with output dependent incremental rate

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Abstract

We study the global asymptotic stabilization by output feedback for systems whose dynamics are in a feedback form where the nonlinear terms admit an incremental rate depending only on the measured output. The output feedback we consider is of the observer-controller type where the design of the controller follows from standard robust backstepping. As far as we know, the novelty is in the observer which is high-gain like with a gain coming from a Riccati equation.

1 Introduction

We consider a nonlinear system with coordinates y_1 to y_n and z_1 to z_m such that its dynamics can be written :

$$\left\{\begin{array}{rcl} \dot{y}_1 &=& f_1(y_1) + y_2 \\ \dot{y}_2 &=& f_2(y_1,y_2) + y_3 \\ &\vdots& \\ \dot{y}_n &=& f_n(y_1,\ldots,y_n) + z_1 + u \\ \dot{z}_1 &=& h_1(y_1,\ldots,y_n,z_1,u) + z_2 \\ \dot{z}_2 &=& h_2(y_1,\ldots,y_n,z_1,z_2,u) + z_3 \\ &\vdots& \\ \dot{z}_m &=& h_m(y_1,\ldots,y_n,z_1,\ldots,z_m,u) \end{array}\right. \qquad (1)$$

where y_1 is the measured output in $\mathbb{R}$, u is the input in $\mathbb{R}$, the functions f_i's are $n+1$ times continuously differentiable and zero at the origin, the functions h_i's are continuously differentiable and zero at the origin and, for all i, u, y, z, ψ and φ, we have :

$$|f_i(y_1, y_2+\psi_2,\ldots,y_i+\psi_i) - f_i(y_1,y_2,\ldots,y_i)| \leq \gamma(y_1)\ (|\psi_2| + \ldots + |\psi_i|)\ , \quad (2)$$

$$|h_i(y_1, y_2+\psi_2,\ldots,y_n+\psi_n, z_1+\varphi_1,\ldots,z_i+\varphi_i,u) - h_i(y_1,y_2,\ldots,y_n,z_1,\ldots,z_i,u)| \leq \gamma(y_1)\ (|\psi_2| + \ldots + |\psi_n| + |\varphi_1| + \ldots + |\varphi_i|)\ , \quad (3)$$

where γ is a $n+1$ times continuously differentiable strictly positive function.

We address the problem of global asymptotic stabilization of the origin with output feedback.

This problem has received a lot of attention. But until recently, the contributions leading to an explicit expression of the feedback were assuming that the f_i's at least were linear in y_2 to y_n (see [5, Section 7], [7, Section 6.3] or [9, 8]) for instance) or that γ was a constant (see [4, 2] for instance).

Actually, for the system (1), it is known how to get a controller from the observer dynamics, with robustification to the observation error. This design is based on the technique of observer backstepping, tackling with the observation errors either via nonlinear damping (see [5, Section 7.1.2]) or via interlacing (see [5, Section 7.4.1]). Such a design allows us to deal with error structures more intricate than those obtained with the linearity or constant γ assumption. In particular it may make possible to take advantage of some sign or gain margin in the observer. The sign margin property for instance has been used in [1] for systems exhibiting a monotonicity property.

The objective of this paper is to use a gain margin property. This leads us to use a high gain like observer. For such observers, it is known (see [4] for instance) that the value of the gain is dictated by the global Lipschitz constant of the non linearities if it exists. Here this Lipschitz "constant" is not constant but depends on the output. This forces us to modify the gain on line. This creates some resemblance with the adapted high-gain observers used typically in universal controllers for (perturbed) linear systems (see [3] for a survey or [12] for a more recent contribution for instance). In fact there is an important difference since our gain up date law depends on the increments of the nonlinearities and not on the non linearities themselves. Actually our update law is a Riccati equation and, for this reason, we view our observer more something like a Kalman filter (compare with [8]) than an adapted high-gain observer.

Unfortunately, as all the previous results for the class

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of systems (1), we do require a "minimum phase" assumption for the inverse dynamics which we phrase as :

Minimum phase assumption :
The system :

$$\left\{\begin{array}{rcl} \dot z_1 &=& h_1(v_1,\ldots,v_n,z_1,v_0-z_1) \;+\; z_2 \\ \dot z_2 &=& h_2(v_1,\ldots,v_n,z_1,z_2,v_0-z_1) \;+\; z_3 \\ &\vdots& \\ \dot z_m &=& h_1(v_1,\ldots,v_n,z_1,\ldots,z_m,v_0-z_1) \end{array}\right. \tag{4}$$

with input $(v_0,\ldots,v_n)$ and state $(z_1,\ldots,z_m)$ is Input-to-State Stable (see [10]).

The dynamic output feedback controller we propose has the structure of an observer-controller. The observer is high-gain like but with an on-line adapted gain. Its design is given in Section 2. The controller, presented in Section 3, is derived with the observer backstepping technique. In section 4, we analyze the behavior of the closed loop system.

2 Observer design

To express the observer more easily, we rewrite the system (1) in the following more compact form :

$$\left\{\begin{array}{rcl} \dot x_1 &=& g_1(x_1,u) \;+\; x_2 \\ &\vdots& \\ \dot x_{p-1} &=& g_{p-1}(x_1,\ldots,x_{p-1},u) \;+\; x_p \\ \dot x_p &=& g_p(x_1,\ldots,x_p,u) \\ y_1 &=& x_1 \end{array}\right. \tag{5}$$

where $p=n+m$, x in $\mathbb{R}^{n+m}$ collects the n components y_i's and m components z_i's and the functions g_i's are the f_j's or h_l's respectively. From the inequalities (2) and (3), we have, for all i, u, x and ξ,

$$\begin{array}{l} |g_i(x_1,x_2+\xi_2,\ldots,x_i+\xi_i,u)-g_i(x_1,x_2,\ldots,x_i,u)| \\ \qquad\qquad \leq\; \gamma(y_1)\;(|\xi_1|+\ldots+|\xi_i|)\ . \end{array} \tag{6}$$

The observer we propose is :

$$\left\{\begin{array}{l} \dot{\widehat x}_1 \;= g_1(y_1,u) \;+\; \widehat x_2 \;+\; k_1\, r\, [y_1-\widehat x_1] \\ \quad\vdots \\ \dot{\widehat x}_{p-1} = g_{p-1}(y_1,\widehat x_2,\ldots,\widehat x_{p-1},u) \;+\; \widehat x_p \\ \qquad\qquad\qquad\qquad +\; k_{p-1}\, r^{p-1}\, [y_1-\widehat x_1] \\ \dot{\widehat x}_p \;= g_p(y_1,\widehat x_2,\ldots,\widehat x_p,u) \;+\; k_p\, r^p\, [y_1-\widehat x_1] \\ \dot r \;= \ell(r,y_1) \end{array}\right. \tag{7}$$

where r is an extra state, ℓ is a $n+1$ times continuously differentiable function to be defined below and the k_i's are constant chosen such that (always possible) there exist strictly positive real numbers q and a and a symmetric matrix Q satisfying :

$$Q\,\mathcal{O} \;+\; \mathcal{O}^T Q \;\leq\; -a\,Q \quad , \qquad q\,I \;\leq\; Q \;\leq\; I\ , \tag{8}$$

where :

$$\mathcal{O} \;=\; \begin{pmatrix} -k_1 & 1 & 0 & \ldots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \\ -k_{p-1} & 0 & \ldots & 0 & 1 \\ -k_p & 0 & \ldots & \ldots & 0 \end{pmatrix} \tag{9}$$

The corresponding observation error :

$$\xi \;=\; x-\widehat x \tag{10}$$

satisfies the following equation :

$$\left\{\begin{array}{rcl} \dot\xi_1 &=& \xi_2 \;-\; k_1 r \xi_1 \\ &\vdots& \\ \dot\xi_{p-1} &=& \xi_p \;-\; k_{p-1}\, r^{p-1}\, \xi_1 \\ && \quad -\, [g_{p-1}(y_1,x_2-\xi_2,\ldots,x_{p-1}-\xi_{p-1},u) \\ && \qquad\qquad -g_{p-1}(y_1,x_2,\ldots,x_{p-1},u)] \\ \dot\xi_p &=& -\; k_p\, r^p\, \xi_1 \\ && \quad -\, [g_p(y_1,x_2-\xi_2,\ldots,x_p-\xi_p,u) \\ && \qquad\qquad -g_p(y_1,x_2,\ldots,x_p,u)] \end{array}\right. \tag{11}$$

To go further, we want to make sure that the observer state component r stays bounded away from 0, say larger than 1. For this we impose to the function ℓ to satisfy, for all y_1,

$$\ell(1,y_1) \;>\; 0\ , \tag{12}$$

and we choose the initial condition $r(0)$ strictly larger than 1. Then, as by now routine in the analysis of error dynamics of high gain observers (see [4] for instance), we introduce the following change of coordinates :

$$\varepsilon_i \;=\; \frac{\xi_i}{r^{i+b}}\ . \tag{13}$$

The novelty here is that b is not taken as 0 or -1 as usual. Instead it is a strictly positive real number chosen (sufficiently large) to satisfy :

$$b\,Q \;\geq\; Q\,D \;+\; D\,Q \;\geq\; -b\,Q\ , \tag{14}$$

where D is the diagonal matrix :

$$D \;=\; \mathtt{diag}(1,\ldots,p)\ . \tag{15}$$

With the ε_i's coordinates, we get :

$$\left\{\begin{array}{rcl} \dot\varepsilon_1 &=& r\varepsilon_2 \;-\; rk_1\varepsilon_1 \;-\; (1+b)\frac{\dot r}{r}\varepsilon_1 \\ &\vdots& \\ \dot\varepsilon_{p-1} &=& r\varepsilon_p \;-\; rk_{p-1}\varepsilon_1 \;-\; (p-1+b)\frac{\dot r}{r}\varepsilon_{p-1} \\ && -\left[\dfrac{\begin{array}{r} g_{p-1}(y_1,x_2-r^{2+b}\varepsilon_2,\ldots,x_{p-1}-r^{p-1+b}\varepsilon_{p-1},u) \\ -g_{p-1}(y_1,x_2,\ldots,x_{p-1},u)\end{array}}{r^{p-1+b}}\right] \\ \dot\varepsilon_p &=& -\,rk_p\varepsilon_1 \;-\; (p+b)\frac{\dot r}{r}\varepsilon_p \\ && \qquad -\left[\dfrac{\begin{array}{r} g_p(y_1,x_2-r^{2+b}\varepsilon_2,\ldots,x_p-r^{p+b}\varepsilon_p,u) \\ -g_p(y_1,x_2,\ldots,x_p,u)\end{array}}{r^{p+b}}\right] \end{array}\right. \tag{16}$$

With (8) and (6), we get the inequality (if $r \geq 1$) :

$$
\begin{aligned}
\dot{\overline{\varepsilon^T Q \varepsilon}} \;\leq\; & -a\,r\,\varepsilon^T Q \varepsilon - 2\,\frac{\dot r}{r}\,\varepsilon^T Q\,(D+b\,I)\,\varepsilon && (17)\\
& +2\,\gamma(y_1)\sum_{i=2}^{p}|\varepsilon^T Q_i|\,\frac{r^{2+b}|\varepsilon_2|+\ldots+r^{i+b}|\varepsilon_i|}{r^{i+b}}\;,\\
\leq\; & -a\,r\,\varepsilon^T Q \varepsilon - 2\,\frac{\dot r}{r}\,\varepsilon^T Q\,(D+b\,I)\,\varepsilon\\
& +2\,\gamma(y_1)\sum_{i=2}^{p}|\varepsilon^T Q_i|\sum_{i=2}^{p}|\varepsilon_i|\;, && (18)\\
\leq\; & -a\,r\,\varepsilon^T Q \varepsilon - 2\,\frac{\dot r}{r}\,\varepsilon^T Q\,(D+b\,I)\,\varepsilon\\
& +2\,\gamma(y_1)\,(p-1)\,|\varepsilon^T Q|\,|\varepsilon|\;, && (19)\\
\leq\; & -\left(a\,r+2\,b\,\frac{\dot r}{r}-\frac{2\,(p-1)}{\sqrt{q}}\gamma(y_1)\right)\varepsilon^T Q \varepsilon\\
& -2\,\frac{\dot r}{r}\,\varepsilon^T Q\,D\,\varepsilon\;. && (20)
\end{aligned}
$$

In view of this we choose the function ℓ, i.e. $\dot r$, as :

$$
\dot r \;=\; \ell(r,y_1) \;=\; -\frac{1}{b}\,r\left(\frac{a}{3}\,[r-1]-\frac{2\,(p-1)}{\sqrt{q}}\,\gamma(y_1)\right)\;. \qquad (21)
$$

Since $\gamma(y_1)$ is strictly positive, (12) holds. Also we have the following identities which will be useful in the forthcoming computations :

$$
a\,r+2\,b\,\frac{\dot r}{r}-\frac{2\,(p-1)}{\sqrt{q}}\,\gamma(y_1) \;=\; b\,\frac{\dot r}{r}\;+\;\frac{a}{3}\,[2r+1]\;, \qquad (22)
$$

$$
\frac{a}{3}\,[2r+1] \;=\; -2b\,\frac{\dot r}{r}\;+\;\frac{4\,(p-1)}{\sqrt{q}}\,\gamma(y_1)\;+\;a\;. \qquad (23)
$$

This yields (if $r \geq 1$) :

$$
\dot{\overline{\varepsilon^T Q \varepsilon}} \;\leq\; -\left(b\frac{\dot r}{r}+\frac{a}{3}[2r+1]\right)\varepsilon^T Q \varepsilon \;-\; 2\,\frac{\dot r}{r}\,\varepsilon^T Q\,D\,\varepsilon\;. \qquad (24)
$$

Hence, if $\dot r \geq 0$, with (14), we obtain (if $r \geq 1$) :

$$
\begin{aligned}
\dot{\overline{\varepsilon^T Q \varepsilon}} \;\leq\; & -\left(b\frac{\dot r}{r}+\frac{a}{3}\,[2r+1]\right)\varepsilon^T Q \varepsilon && (25)\\
& +\,b\,\frac{\dot r}{r}\,\varepsilon^T Q \varepsilon\;,\\
\leq\; & -\frac{a}{3}\,[2r+1]\,\varepsilon^T Q \varepsilon\;, && (26)\\
\leq\; & -a\,\varepsilon^T Q \varepsilon\;. && (27)
\end{aligned}
$$

And, if $\dot r \leq 0$, with (14) and (23), we obtain (if $r \geq 1$) :

$$
\begin{aligned}
\dot{\overline{\varepsilon^T Q \varepsilon}} \;\leq\; & -\left(b\frac{\dot r}{r}+\frac{a}{3}\,[2r+1]\right)\varepsilon^T Q \varepsilon && (28)\\
& -\,b\,\frac{\dot r}{r}\,\varepsilon^T Q \varepsilon\;,\\
\leq\; & -\left(2b\frac{\dot r}{r}+\frac{a}{3}\,[2r+1]\right)\varepsilon^T Q \varepsilon\;, && (29)\\
\leq\; & -\left(\frac{4\,(p-1)}{\sqrt{q}}\gamma(y_1)+a\right)\varepsilon^T Q \varepsilon\;, && (30)\\
\leq\; & -a\,\varepsilon^T Q \varepsilon\;. && (31)
\end{aligned}
$$

To summarize, with our choice for the k_i's and the function ℓ, at each point in the closed loop state space where $r \geq 1$, we have :

$$
\dot{\overline{\varepsilon^T Q \varepsilon}} \;\leq\; -a\,\varepsilon^T Q \varepsilon\;. \qquad (32)
$$

3 Controller design

To design the controller, we work from a part of the observer equation (7) rewritten with the coordinates $(r, y_1, \widehat{y}_2, \ldots, \widehat{y}_n)$:

$$
\left\{
\begin{array}{rcl}
\dot r &=& -\frac{1}{b}\,r\left(\frac{a}{3}\,[r-1]-\frac{2\,(p-1)}{\sqrt{q}}\gamma(y_1)\right)\\
\dot y_1 &=& f_1(y_1)\;+\;\widehat{y}_2\;+\;r^{2+b}\,\varepsilon_2\\
\dot{\widehat{y}}_2 &=& f_2(y_1,\widehat{y}_2)\;+\;\widehat{y}_3\;+\;k_2\,r^{3+b}\,\varepsilon_1\\
&\vdots&\\
\dot{\widehat{y}}_n &=& f_n(y_1,\widehat{y}_2,\ldots,\widehat{y}_n)\;+\;v\;+\;k_n r^{n+1+b}\varepsilon_1
\end{array}
\right. \qquad (33)
$$

where we have let :

$$
u \;=\; v\;-\;\widehat{z}_1\;. \qquad (34)
$$

We follow exactly the same steps as in [5, Section 7.1.2] (see the Appendix for details). This way we get recursively n functions $\alpha_i(r, y_1, \widehat{y}_2, \ldots, \widehat{y}_i)$ which are $n+1-i$ times continuously differentiable respectively and satisfy :

$$
\alpha_i(r,0,0,\ldots,0) \;=\; 0\;. \qquad (35)
$$

In particular α_{i+1} is obtained from the gradient of α_i with respect to all its arguments. So it is in this process of getting these functions α_i's that we need to differentiate may be up to n times the functions appearing in (33), i.e. the f_i and γ. Finally, we note that, for getting the nonlinear damping terms (see [5, p. 289]), we use the inequality (32) (which holds only if $r \geq 1$). This construction leads to the control :

$$
v \;=\; \alpha_n(r, y_1, \widehat{y}_2, \ldots, \widehat{y}_n) \qquad (36)
$$

and provides the variables :

$$
\begin{aligned}
\zeta_1 \;&=\; y_1\;, && (37)\\
\zeta_{i+1} \;&=\; \widehat{y}_{i+1}\;-\;\alpha_i(r, y_1, \widehat{y}_2, \ldots, \widehat{y}_i)\;. && (38)
\end{aligned}
$$

It gives also the inequality (if $r \geq 1$) :

$$
\dot{\overline{y_1^2+\sum_{i=2}^{n}\zeta_i^2+\varepsilon^T Q \varepsilon}} \;\leq\; -y_1^2-\sum_{i=2}^{n}\zeta_i^2-\frac{a}{2}\,\varepsilon^T Q \varepsilon\;. \qquad (39)
$$

Finally our output feedback controller is :

$$\left\{\begin{array}{rcl}
\dot{r} &=& -\frac{1}{b}\,r\left(\frac{a}{3}\,[r-1] - \frac{2\,(p-1)}{\sqrt{q}}\gamma(y_1)\right)\ ,\ r(0)\ >\ 1 \\
\dot{\widehat{y}}_1 &=& f_1(y_1)\ +\ \widehat{y}_2\ +\ k_1\,r\,(y_1-\widehat{y}_1) \\
\dot{\widehat{y}}_2 &=& f_2(y_1,\widehat{y}_2)\ +\ \widehat{y}_3\ +\ k_2\,r^2\,(y_1-\widehat{y}_1) \\
&\vdots& \\
\dot{\widehat{y}}_n &=& f_n(y_1,\widehat{y}_2,\ldots,\widehat{y}_n)\ +\ \widehat{z}_1\ +\ u \\
&& \qquad +\ k_n\,r^n\,(y_1-\widehat{y}_1) \\
\dot{\widehat{z}}_1 &=& h_1(y_1,\widehat{y}_2,\ldots,\widehat{y}_n,\widehat{z}_1,u)\ +\ \widehat{z}_2 \\
&& \qquad +\ k_{n+1}\,r^{n+1}\,(y_1-\widehat{y}_1) \\
&\vdots& \\
\dot{\widehat{z}}_m &=& h_m(y_1,\widehat{y}_2,\ldots,\widehat{y}_n,\widehat{z}_1,\ldots\widehat{z}_m,u) \\
&& \qquad +\ k_{n+m}\,r^{n+m}\,(y_1-\widehat{y}_1) \\
u &=& \alpha_n(r,y_1,\widehat{y}_2,\ldots,\widehat{y}_n)\ -\ \widehat{z}_1\ .
\end{array}\right. \tag{40}$$

4 Analysis of the closed loop system

The dynamics of the closed loop system can be described using the coordinates :

$$(r,\varepsilon,y_1,\widehat{y}_2,\ldots,\widehat{y}_n,z_1,\ldots,z_m)\ .$$

They satisfy the following set of equations :

$$\left\{\begin{array}{rl}
\dot{r} &= -\frac{1}{b}\,r\left(\frac{a}{3}\,[r-1] - \frac{2\,(p-1)}{\sqrt{q}}\gamma(y_1)\right)\ ,\ r(0)\ >\ 1 \\
\dot{\varepsilon}_1 &= r\varepsilon_2\ -\ rk_1\varepsilon_1\ -\ (1+b)\frac{\dot{r}}{r}\varepsilon_1 \\
&\vdots \\
\dot{\varepsilon}_{p-1} &= r\varepsilon_p\ -\ rk_{p-1}\varepsilon_1\ -\ (p-1+b)\frac{\dot{r}}{r}\varepsilon_{p-1} \\
& \quad -\left[\frac{g_{p-1}(y_1,x_2-r^{2+b}\varepsilon_2,\ldots,x_{p-1}-r^{p-1+b}\varepsilon_{p-1},u) - g_{p-1}(y_1,x_2,\ldots,x_{p-1},u)}{r^{p-1+b}}\right] \\
\dot{\varepsilon}_p &= -\,rk_p\varepsilon_1\ -\ (p+b)\frac{\dot{r}}{r}\varepsilon_p \\
& \quad -\left[\frac{g_p(y_1,x_2-r^{2+b}\varepsilon_2,\ldots,x_p-r^{p+b}\varepsilon_p,u) - g_p(y_1,x_2,\ldots,x_p,u)}{r^{p+b}}\right] \\
\dot{y}_1 &= f_1(y_1)\ +\ \widehat{y}_2\ +\ r^{2+b}\,\varepsilon_2 \\
\dot{\widehat{y}}_2 &= f_2(y_1,\widehat{y}_2)\ +\ \widehat{y}_3\ +\ k_2\,r^{3+b}\,\varepsilon_1 \\
&\vdots \\
\dot{\widehat{y}}_n &= f_n(y_1,\widehat{y}_2,\ldots,\widehat{y}_n)\ +\ \alpha_n(r,y_1,\widehat{y}_2,\ldots,\widehat{y}_n) \\
& \qquad +\ k_n\,r^{n+1+b}\,\varepsilon_1 \\
\cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\dot{z}_1 &= z_2 \\
& \quad +\ h_1(y_1,\widehat{y}_2+r^{2+b}\varepsilon_2,\ldots,\widehat{y}_n+r^{n+b}\varepsilon_n, \\
& \qquad ,z_1,-z_1-r^{n+1+b}\varepsilon_{n+1}+\alpha_n) \\
&\vdots \\
\dot{z}_m &= \\
& \quad +\ h_m(y_1,\widehat{y}_2+r^{2+b}\varepsilon_2,\ldots,\widehat{y}_n+r^{n+b}\varepsilon_n, \\
& \qquad ,z_1,\ldots,z_m,-z_1-r^{n+1+b}\varepsilon_{n+1}+\alpha_n)
\end{array}\right. \tag{41}$$

where we have used the notation, for $i \in \{1,\ldots,n\}$,

$$x_i\ =\ \widehat{y}_i\ +\ r^{i+b}\,\varepsilon_i \tag{42}$$

and, for $i \in \{n+1,\ldots,n+m\}$,

$$x_i\ =\ z_{i-n}\ . \tag{43}$$

These dynamics have the following properties :

1. The right hand side is defined on $(0,+\infty)\times\mathbb{R}^{2(n+m)}$ where it is continuously differentiable. It follows that, to each initial condition in $(0,+\infty)\times\mathbb{R}^{2(n+m)}$, it corresponds a unique solution.
2. The expression of $\dot{r}$ has been chosen such that :
$$r\ =\ 1 \qquad \Rightarrow \qquad \dot{r}\ >\ 1\ . \tag{44}$$
It follows that the set $(1,+\infty)\times\mathbb{R}^{2(n+m)}$ is forward invariant and its boundary $\{1\}\times\mathbb{R}^{2(n+m)}$ is repellent. Hence any solution, initialized in this set, remains in it and, if its right maximal interval of definition is bounded, it is unbounded (since its r-component cannot go to 1). So, for any such solution, (39) holds.
3. We have an interconnection structure. The subsystem with coordinates $(r,\varepsilon,y_1,\widehat{y}_2,\ldots,\widehat{y}_n)$ with input z sends the signals :
$$v_0\ =\ -r^{n+1+b}\varepsilon_{n+1}+\alpha_n\ ,\quad v_1\ =\ y_1\ , \tag{45}$$
$$v_2\ =\ \widehat{y}_2+r^{2+b}\varepsilon_2\ ,\ \ldots\ ,\ v_n\ =\ \widehat{y}_n+r^{n+b}\varepsilon_n$$
to the z-subsystem whose dynamics are then given by (4). So, with the arguments introduced in [11] and our minimum phase assumption, it follows that asymptotic stability with domain of attraction $(1,+\infty)\times\mathbb{R}^{2(n+m)}$ holds if the $(r,\varepsilon,y_1,\widehat{y}_2,\ldots,\widehat{y}_n)$-subsystem has, uniformly in z, an asymptotically stable equilibrium with $(1,+\infty)\times\mathbb{R}^{2n+m}$ as domain of attraction (see (50) below for what we mean by this).
4. With the help of (32) and (39), we see that, in the set $(1,+\infty)\times\mathbb{R}^{2n+m}$, the $(r,\varepsilon,y_1,\widehat{y}_2,\ldots,\widehat{y}_n)$-subsystem has only one equilibrium at $(r^*,0,\ldots,0)$ with :
$$r^*\ =\ 1\ +\ \frac{6\,(p-1)}{a\sqrt{q}}\gamma(0)\ . \tag{46}$$
5. The function $r-r^*-r^*\log(r/r^*)$ is C^1, proper and non-negative on $(0,+\infty)$. It is zero if and only if $r=r^*$. It satisfies :
$$\overset{\cdot}{\overbrace{r-r^*-r^*\log(r/r^*)}}$$
$$= -\frac{a}{3b}\left(r-1-\frac{6\,(p-1)}{a\sqrt{q}}\gamma(y_1)\right)(r-r^*)\ , \tag{47}$$
$$= -\frac{a}{3b}\,(r-r^*)^2 \tag{48}$$
$$\qquad -\ \frac{2\,(p-1)}{b\sqrt{q}}\,(r-r^*)\,(\gamma(0)-\gamma(y_1))\ ,$$
$$\leq\ -\frac{a}{6b}\,(r-r^*)^2\ +\ \frac{6\,(p-1)^2}{abq}\,(\gamma(0)-\gamma(y_1))^2 \tag{49}$$

6. With (35) and (38), we know that we can use $(\zeta_2,\ldots,\zeta_n)$ instead of $(\widehat{y}_2,\ldots,\widehat{y}_n)$ as coordinates. Their equilibrium point is also 0.

We conclude that to show the asymptotic stability of the point $(r^*,0,\ldots,0)$ with domain of attraction $(1,+\infty)\times\mathbb{R}^{2(n+m)}$ uniformly in z, it is sufficient to show that for some C^1, unbounded and strictly increasing function $\Phi:[0,+\infty)\to[0,+\infty)$, the derivative of :

$$
\begin{aligned}
&V(r,y_1,\widehat{y}_2,\ldots,\widehat{y}_n,\varepsilon) \;= \qquad\qquad (50)\\
&\quad [r-r^*-r^*\log(r/r^*)] \;+\; \Phi\left(y_1^2+\textstyle\sum_{i=2}^n\zeta_i^2+\varepsilon^TQ\varepsilon\right)
\end{aligned}
$$

is negative definite uniformly in z. In view of (39) and (49), we pick $\varphi:[0,+\infty)\to(0,+\infty)$ as a continuous non decreasing function satisfying, for all y_1,

$$
\varphi(y_1^2)\;\geq\;2\,\frac{6\,(p-1)^2}{abq}\left(\frac{\gamma(0)-\gamma(y_1)}{y_1}\right)^2 . \qquad (51)
$$

Such a choice is possible since $\frac{\gamma(0)-\gamma(y_1)}{y_1}$ is a continuous function. Then in the definition of V above, we use :

$$
\Phi(s)\;=\;\int_0^s\varphi(\sigma)\,d\sigma\,. \qquad (52)
$$

With (39) and (49), we get that, in $(1,+\infty)\times\mathbb{R}^{2n+m}$, we have :

$$
\begin{aligned}
\dot V\;\leq\;&-\frac{a}{6b}(r-r^*)^2+\frac{6\,(p-1)^2}{abq}(\gamma(0)-\gamma(y_1))^2 \qquad (53)\\
&-\varphi\left(y_1^2+\sum_{i=2}^n\zeta_i^2+\varepsilon^TQ\varepsilon\right)\left[y_1^2+\sum_{i=2}^n\zeta_i^2+\frac{a}{2}\varepsilon^TQ\varepsilon\right].
\end{aligned}
$$

Since φ is non decreasing and satisfies (51), this yields :

$$
\begin{aligned}
\dot V\;\;\leq\;\;&-\frac{a}{6b}(r-r^*)^2\;+\;\frac{6\,(p-1)^2}{abq}(\gamma(0)-\gamma(y_1))^2\\
&-\varphi\left(y_1^2\right)y_1^2\\
&-\varphi\left(y_1^2+\sum_{i=2}^n\zeta_i^2+\varepsilon^TQ\varepsilon\right)\left[\sum_{i=2}^n\zeta_i^2+\frac{a}{2}\varepsilon^TQ\varepsilon\right] (54)\\
\leq\;\;&-\frac{a}{6b}(r-r^*)^2\;-\;\frac{1}{2}\varphi\left(y_1^2\right)y_1^2\\
&-\varphi\left(\sum_{i=2}^n\zeta_i^2+\varepsilon^TQ\varepsilon\right)\left[\sum_{i=2}^n\zeta_i^2+\frac{a}{2}\varepsilon^TQ\varepsilon\right] (55)
\end{aligned}
$$

The right hand side of (55) is non positive and zero if and only if we are at $(r^*,0,\ldots,0)$. Hence we have established the asymptotic stability of this point uniformly in z. Also this inequality holding everywhere in the set $(1,+\infty)\times\mathbb{R}^{2n+m}$ which is forward invariant, this whole set is the domain of attraction.

To conclude, for the system (1) satisfying the inequalities (2) and (3) and the minimum phase assumption, the dynamic output feedback we have proposed provides asymptotic stability of the point $(r^*,0,\ldots,0)$ with domain of attraction $(1,+\infty)\times\mathbb{R}^{2(n+m)}$.

5 Conclusion

We have shown that, by combining an adapted high gain observer and observer backstepping, we can design globally asymptotically stabilizing output feedbacks for systems admitting the form (1) where the nonlinearities have an incremental rate depending only on the measured output as specified by the inequalities (2) and (3).

The main contribution here is in the observer gain up date law. The key to get such an update law is in the coordinate scaling commonly used in the analysis of high gain observer. In our case, this scaling, $\varepsilon_i=\frac{\xi_i}{r^{i+b}}$ depends not only on the rank i in the integrator chain but also on b, a parameter directly related the "observer poles" (see (8) and (14)).

Appendix : Construction of the functions α_i's

For the sake of completeness, we reproduce here with some adaptation what can be found in [5, Section 7.1.2].

Consider the system :

$$
\left\{\begin{array}{rcl}
\dot r &=& -\frac{1}{b}\,r\left(\frac{a}{3}[r-1]-\frac{2\,(p-1)}{\sqrt{q}}\gamma(y_1)\right)\\
\dot y_1 &=& f_1(y_1)\;+\;\widehat{y}_2\;+\;r^{2+b}\varepsilon_2\\
\dot{\widehat{y}}_2 &=& f_2(y_1,\widehat{y}_2)\;+\;\widehat{y}_3\;+\;k_2\,r^{3+b}\varepsilon_1\\
&\vdots&\\
\dot{\widehat{y}}_i &=& f_i(y_1,\widehat{y}_2,\ldots,\widehat{y}_i)\;+\;v_i\;+\;k_ir^{i+1+b}\varepsilon_1
\end{array}\right. \qquad (56)
$$

where the ε_j's are components of a vector ε. Aiming at establishing a result by recurrence, we assume the existence of functions α_j which are $n+1-j$ times continuously differentiable respectively, satisfy :

$$
\alpha_j(r,0,0,\ldots,0)\;=\;0\,. \qquad (57)
$$

and are such that, by letting :

$$
\zeta_{j+1}\;=\;\widehat{y}_{j+1}\;-\;\alpha_j(r,y_1,\widehat{y}_2,\ldots,\widehat{y}_j)\,, \qquad (58)
$$

we have :

$$
\begin{aligned}
&\overbrace{y_1^2+\sum_{j=2}^i\zeta_j^2+\varepsilon^TQ\varepsilon}^{\cdot} \qquad (59)\\
&\leq\;-y_1^2\;-\;\sum_{j=2}^i\zeta_j^2\;-\;\frac{a(2n-i)}{2n}\varepsilon^TQ\varepsilon\;+\;2\,\zeta_i\,(v_i-\alpha_i)
\end{aligned}
$$

Now, we consider the system (56) with the $\dot{\widehat{y}}_i$-equation replaced by :

$$
\dot{\widehat{y}}_i\;=\;f_i(y_1,\widehat{y}_2,\ldots,\widehat{y}_i)\;+\;\widehat{y}_{i+1}\;+\;k_ir^{i+1+b}\varepsilon_1 \qquad (60)
$$

with $\widehat{y}_{i+1}$ satisfying :

$$
\dot{\widehat{y}}_{i+1}\;=\;f_{i+1}(y_1,\widehat{y}_2,\ldots,\widehat{y}_{i+1})\;+\;v_{i+1} \qquad (61)
$$

$$+ k_{i+1} r^{i+2+b} \varepsilon_1 .$$

With (58), (59) gives :

$$\overbrace{y_1^2 + \sum_{j=2}^{i+1} \zeta_j^2 + \varepsilon^T Q \varepsilon}^{\cdot} \qquad (62)$$
$$\leq -y_1^2 - \sum_{j=2}^{i} \zeta_j^2 - \frac{a(2n-i)}{2n} \varepsilon^T Q \varepsilon + 2\,\zeta_{i+1} \left(\zeta_i + \dot{\widehat{y}}_{i+1} - \dot{\alpha}_i \right)$$

where in particular we have :

$$\begin{aligned}\dot{\alpha}_i \;=\; & \tfrac{\partial \alpha_i}{\partial r} \left[-\tfrac{1}{b}\, r \left(\tfrac{a}{3} \left[r-1 \right] - \tfrac{2(p-1)}{\sqrt{q}} \gamma(y_1) \right) \right] \\ & + \tfrac{\partial \alpha_i}{\partial y_1} \left[f_1(y_1) + \widehat{y}_2 + r^{2+b} \varepsilon_2 \right] \\ & \vdots \\ & + \tfrac{\partial \alpha_i}{\partial \widehat{y}_i} \left[f_i(y_1, \widehat{y}_2, \ldots, \widehat{y}_i) + \widehat{y}_{i+1} + k_i r^{i+1+b} \varepsilon_1 \right] \end{aligned} \qquad (63)$$

We observe that the term $\zeta_i + \dot{\widehat{y}}_{i+1} - \dot{\alpha}_i$ admits the following decomposition :

$$\zeta_i + \dot{\widehat{y}}_{i+1} - \dot{\alpha}_i \;=\; v_{i+1} \;+\; \mu_i(r, y_1, \widehat{y}_2, \ldots, \widehat{y}_{i+1}) \;+\; \nu_i(r, y_1, \widehat{y}_2, \ldots, \widehat{y}_i)\, \varepsilon_1 \;+\; \tfrac{\partial \alpha_i}{\partial y_1} r^{2+b} \varepsilon_2 \qquad (64)$$

with a straightforward identification of the function μ_i and ν_i. Also note that (57) implies :

$$\frac{\partial \alpha_i}{\partial r}(r, 0, 0, \ldots, 0) \;=\; 0 \,. \qquad (65)$$

And, since the f_j's are zero at the origin, with (57) and (65), it follows in particular :

$$\mu_i(r, 0, \ldots, 0) \;=\; 0 \,. \qquad (66)$$

Finally, by completing the squares, we get :

$$2\zeta_{i+1} \left(\nu_i \varepsilon_1 \;+\; \tfrac{\partial \alpha_i}{\partial y_1} r^{2+b} \varepsilon_2 \right) \qquad (67)$$
$$\leq \tfrac{8nq}{a} \zeta_{i+1}^2 \left(\nu_i^2 \;+\; \tfrac{\partial \alpha_i}{\partial y_1}^2 r^{4+2b} \right) \;+\; \tfrac{a}{2n} \varepsilon^T Q \varepsilon \,.$$

Using this inequality in (62), we obtain :

$$\overbrace{y_1^2 + \sum_{j=2}^{i+1} \zeta_j^2 + \varepsilon^T Q \varepsilon}^{\cdot} \qquad (68)$$
$$\leq \; -y_1^2 - \sum_{j=2}^{i} \zeta_j^2 - \frac{a(2n-(i+1))}{2n} \varepsilon^T Q \varepsilon + 2\,\zeta_{i+1} \left[v_{i+1} + \mu_i(r, y_1, \widehat{y}_2, \ldots, \widehat{y}_{i+1}) + \frac{nq}{2a} \zeta_{i+1} \left(\nu_i^2 \;+\; \frac{\partial \alpha_i}{\partial y_1}^2 r^{4+2b} \right) \right] \,.$$

So by defining α_{i+1} as :

$$\alpha_{i+1}(r, y_1, \widehat{y}_2, \ldots, \widehat{y}_{i+1}) \;=\; -\tfrac{1}{2} \zeta_{i+1} - \left[\mu_i(r, y_1, \widehat{y}_2, \ldots, \widehat{y}_{i+1}) + \tfrac{nq}{2a} \zeta_{i+1} \left(\nu_i^2 \;+\; \tfrac{\partial \alpha_i}{\partial y_1}^2 r^{4+2b} \right) \right] \qquad (69)$$

we get (compare with (59)) :

$$\overbrace{y_1^2 + \sum_{j=2}^{i+1} \zeta_j^2 + \varepsilon^T Q \varepsilon}^{\cdot} \qquad (70)$$
$$\leq \; -y_1^2 \;-\; \sum_{j=2}^{i+1} \zeta_j^2 \;-\; \frac{a(2n-(i+1))}{2n} \varepsilon^T Q \varepsilon + 2\,\zeta_{i+1} \left(v_{i+1} - \alpha_{i+1} \right) \,.$$

Note also that α_{i+1} is $n-i$ times continuously differentiable and satisfies :

$$\alpha_{i+1}(r, 0, \ldots, 0) \;=\; 0 \,. \qquad (71)$$

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