Output Feedback Asymptotic Stabilization For Triangular Systems Linear in the Unmeasured State Components

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Abstract

We propose a globally asymptotically stabilizing output feedback for systems whose dynamics are both linear in the unmeasured state components and in a feedback form.

1 Introduction

We consider a nonlinear system whose dynamics admit the following very specific description :

$$\overbrace{\begin{pmatrix} y_1 \\ \vdots \\ y_n \\ \hline \zeta_1 \\ \vdots \\ \zeta_m \end{pmatrix}}^{\cdot} = \mathcal{A}(y_1) \begin{pmatrix} y_1 \\ \vdots \\ y_n \\ \hline \zeta_1 \\ \vdots \\ \zeta_m \end{pmatrix} + \begin{pmatrix} \vdots \\ 0 \\ g \\ \hline d_1 \\ \vdots \\ d_m \end{pmatrix} u \tag{1}$$

where y_1 is the measured output in $\mathbb{R}$, u is the input in $\mathbb{R}$, and $\mathcal{A}(y_1)$ is the matrix

$$\left(\begin{array}{ccc|cccc} a_{11} & a_{12} & 0 & \dots & \dots\dots & & 0 \\ \vdots & & \ddots & \ddots & & & \vdots \\ a_{n1} & \dots & a_{nn} & f & \ddots & & \vdots \\ \hline b_{11} & \dots & b_{1n} & c_{11} & c_{12} & \ddots & \vdots \\ \vdots & & & & & \ddots & 0 \\ b_{(m-1)1} & \dots & b_{(m-1)n} & c_{(m-1)1} & \dots\dots & & c_{(m-1)m} \\ b_{m1} & \dots & b_{mn} & c_{m1} & \dots\dots & & c_{mm} \end{array}\right)$$

whose entries a_{ij}, b_{ij}, c_{ij}, d_i and f and g are $n+1$ continuously differentiable functions of y_1 only. We assume also that, for each ω there exists a strictly positive real number ε, such that we have, for all i,

$$|y_1| \leq \omega \quad \Longrightarrow \tag{2}$$
$$\left\{ \begin{array}{l} a_{i(i+1)}(y_1) \geq \varepsilon \,,\ c_{i(i+1)}(y_1) \geq \varepsilon \,, \\ f(y_1) \geq \varepsilon \,,\ g(y_1) \geq \varepsilon \,. \end{array} \right.$$

We address the problem of global asymptotic stabilization of the origin with output feedback.

This problem has received a lot of attention (see [9, Section 7] or [10, Section 6.3] for instance). But until recently it was imposed that the entries do not depend on y_1 except those in the first column, the a_{i1}'s and b_{i1}'s. Indeed, under this assumption, we get the normal form identified in [12] (see also [10, chapter 6]).

Our interest for the representation (1) with all the entries being y_1 dependent follows from the contribution [2] (see also [7]) giving an intrinsic geometric condition for a general system (without no input)

$$\dot{x} = f(x) \quad , \qquad y_1 = h(x) \tag{3}$$

to admit such a representation. Also in [2], the above problem of global asymptotic stabilization is solved but under the assumptions that all the entries are bounded and the ζ_i components are not present, i.e. the system has no inverse dynamics. In this context, the contribution of this paper is to remove these two assumptions.

Actually, for the system (1), it is well known how to design an observer since y_1 being measured, this system can be seen as a time varying linear system. Hence an observer based on a Kalman filter can be written. In fact this has already been done, even for a more general class of systems in [7, 5] for instance. Also the design of a controller from the observer dynamics, with robustification to the observation error, is known. It is based on the technique of observer backstepping tackling with the observation errors via nonlinear damping (see [9, Section 7.1.2]). However this design is only formal. It gives a solution to our output feedback stabilization problem only if the observer behaves properly. A sufficient condition for this is related to the uniform complete observability of the system (1). The main point of this paper is the proof that this property do hold for the closed loop system.

Unfortunately, as all the previous results for this class of systems (with the exception of [13]), we do require a "minimum phase" assumption for the inverse dynamics which we phrase as :

Minimum phase assumption :
The matrix

$$\begin{pmatrix} c_{11}(0) - \dfrac{d_1(0)f(0)}{g(0)} & c_{12}(0) & 0 \dots & 0 \\ \vdots & \vdots & \ddots \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ c_{(m-1)1}(0) - \dfrac{d_{(m-1)}(0)f(0)}{g(0)} & c_{(m-1)2}(0) & \dots\dots & c_{(m-1)m}(0) \\ c_{m1}(0) - \dfrac{d_m(0)f(0)}{g(0)} & c_{m2}(0) & \dots\dots & c_{mm}(0) \end{pmatrix}$$

has all its eigen values with strictly negative real part.

In Section 2, we present our output feedback. In section 3, we analyze the behavior of the closed loop system with some effort spent on establishing a uniform complete observability property.

2 Design of an output feedback controller

2.1 Observer

The observer we propose for the system (1) is similar to a Kalman filter for a linear time-varying system. To express it more easily, we rewrite the system (1) in the following more compact form:

$$\dot{\xi} \;=\; \mathcal{A}(y_1)\,\xi \;+\; \mathcal{B}(y_1)\,u \quad , \qquad y_1 \;=\; C\,\xi \qquad (4)$$

where ξ in $\mathbb{R}^{n+m}$ collects the n components y_i's and m components ζ_i's. The corresponding observer is :

$$\left\{ \begin{array}{rcl} \dot{\hat{\xi}} &=& \mathcal{A}(y_1)\,\hat{\xi} \;+\; \mathcal{B}(y_1)\,u + \mathcal{P}C^T(y_1 - C\hat{\xi}) \\ \dot{\mathcal{P}} &=& \mathcal{P}\mathcal{A}(y_1)^T + \mathcal{A}(y_1)\mathcal{P} - \mathcal{P}CC^T\mathcal{P} + I \end{array} \right. \qquad (5)$$

where $\mathcal{P}$ is initialized as a symmetric positive definite matrix. The corresponding observation error $\tilde{\xi} = \xi - \hat{\xi}$ satisfies the following equation:

$$\dot{\tilde{\xi}} \;=\; \left(\mathcal{A}(y_1) - \mathcal{P}C^TC\right)\tilde{\xi} \qquad (6)$$

A formal computation gives the identity :

$$\overbrace{\tilde{\xi}^T\mathcal{P}^{-1}\tilde{\xi}}^{\cdot} \;=\; -\tilde{\xi}^T\mathcal{P}^{-2}\tilde{\xi} - \tilde{\xi}^TC^TC\tilde{\xi} \qquad (7)$$

It makes sense only when $\mathcal{P}$ is an invertible matrix which we shall have to take care of. But this identity allows us, in the controller design, to use the following inequality where a is a dummy variable, b is a strictly positive real number and C_i is the row vector extracting the ith component of $\tilde{\xi}$:

$$a\,\tilde{\xi}_i \;\leq\; \frac{b}{2}\,a^2\,C_i\,\mathcal{P}\,\mathcal{P}\,C_i^T \;+\; \frac{1}{2b}\,\tilde{\xi}^T\mathcal{P}^{-2}\tilde{\xi} \qquad (8)$$

2.2 Controller

To design the controller, we follow exactly the same steps as in [9, Section 7.1.2]. We work from the part of the observer (5) dealing with the coordinates $(y_1, \hat{y}_2, \dots, \hat{y}_n)$:

$$\overbrace{\begin{pmatrix} y_1 \\ \hat{y}_2 \\ \vdots \\ \hat{y}_n \end{pmatrix}}^{\cdot} = \begin{pmatrix} a_{11} & a_{12} & 0 & \dots\dots & 0 \\ a_{21} & a_{22} & a_{23} & \ddots & \vdots \\ \vdots & & & \ddots \ddots & \vdots \\ a_{n1} & \dots & a_{nn} & & \end{pmatrix} \begin{pmatrix} y_1 \\ \hat{y}_2 \\ \vdots \\ \hat{y}_n \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ g \end{pmatrix} v \qquad (9)$$
$$- \begin{pmatrix} 0 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} \tilde{y}_1 \;+\; \begin{pmatrix} a_{12} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tilde{y}_2$$

where k_i is the ith component of $\mathcal{P}C^T$ and we have let :

$$u \;=\; v \;-\; \frac{f(y_1)}{g(y_1)}\,\hat{\zeta}_1 \;. \qquad (10)$$

We can get recursively n functions $\alpha_i(y_1, \hat{y}_1, \dots, \hat{y}_i, \mathcal{P})$ which are $n+1-i$ continuously differentiable respectively and satisfy

$$\alpha_i(0, 0, \dots, 0, \mathcal{P}) \;=\; 0 \;. \qquad (11)$$

In particular α_{i+1} is obtained from the gradient of α_i with respect to all its arguments and therefore with respect to the entries of the matrix $\mathcal{P}$ for which we have the dynamics (5). Also it is in this process of getting these functions α_i's that we need to differentiate the functions of y_1, appearing in (1) and (5), may be up to n times. Finally, we note that, for getting the nonlinear damping terms, we use the inequality (8). This construction leads to the control:

$$v \;=\; \frac{1}{g(y_1)}\,\alpha_n(y_1, \hat{y}_1, \dots, \hat{y}_n, \mathcal{P}) \qquad (12)$$

and provides the variables:

$$\begin{array}{rcl} z_1 &=& y_1 \qquad (13) \\ z_{i+1} &=& \hat{y}_{i+1} \;-\; \alpha_i(y_1, \hat{y}_1, \dots, \hat{y}_i, \mathcal{P}) \;. \qquad (14) \end{array}$$

It gives also formally ((2) and invertibility of $\mathcal{P}$ is invoked) the inequality:

$$\overbrace{\sum_{i=1}^{n} z_i^2 \;+\; \tilde{\xi}^T\mathcal{P}^{-1}\tilde{\xi}}^{\cdot} \;\leq\; -\sum_{i=1}^{n} z_i^2 \;-\; \frac{1}{2}\,\tilde{\xi}^T\mathcal{P}^{-2}\tilde{\xi} \;-\; \tilde{y}_1^2 \;. \qquad (15)$$

Finally our output feedback controller is:

$$\left\{ \begin{array}{rcl} \dot{\hat{\xi}} &=& \mathcal{A}(y_1)\,\hat{\xi} \;+\; \mathcal{B}(y_1)\,u + \mathcal{P}C^T(y_1 - C\hat{\xi}) \\ \dot{\mathcal{P}} &=& \mathcal{P}\mathcal{A}(y_1)^T + \mathcal{A}(y_1)\mathcal{P} - \mathcal{P}CC^T\mathcal{P} + I \\ u &=& \dfrac{1}{g(y_1)}\,\alpha_n(y_1, \hat{y}_1, \dots, \hat{y}_n, \mathcal{P}) \;-\; \dfrac{f(y_1)}{g(y_1)}\,\hat{\zeta}_1 \end{array} \right. \qquad (16)$$

3 Analysis of the closed loop system

3.1 Boundedness and convergence

The dynamics of the closed loop system can be described using the coordinates

$$(y_1, \hat{y}_2, \dots, \hat{y}_n, \zeta_1, \dots, \zeta_m, \tilde{\xi}, \mathcal{P}) \ .$$

They satisfy the following set of 4 equations :

$$\dot{\mathcal{P}} = \mathcal{P}\mathcal{A}(y_1)^T + \mathcal{A}(y_1)\mathcal{P} - \mathcal{P}CC^T\mathcal{P} + I \quad (17)$$

$$\dot{\tilde{\xi}} = \left(\mathcal{A}(y_1) - \mathcal{P}C^TC\right)\tilde{\xi} \quad (18)$$

$$\dot{\overbrace{\begin{pmatrix} y_1 \\ \hat{y}_2 \\ \vdots \\ \hat{y}_n \end{pmatrix}}} = \begin{pmatrix} a_{11} & a_{12} & 0 & \dots\dots & 0 \\ a_{21} & a_{22} & a_{23} & \ddots & \vdots \\ \vdots & & & \ddots \ddots & \vdots \\ a_{n1} & \dots & a_{nn} & & \end{pmatrix} \begin{pmatrix} y_1 \\ \hat{y}_2 \\ \vdots \\ \hat{y}_n \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \alpha_n \quad (19)$$

$$- \begin{pmatrix} 0 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} \tilde{y}_1 + \begin{pmatrix} a_{12} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tilde{y}_2$$

$$\dot{\overbrace{\begin{pmatrix} \zeta_1 \\ \vdots \\ \vdots \\ \zeta_m \end{pmatrix}}} = \begin{pmatrix} c_{11} - \frac{d_1 f}{g} & c_{12} & 0 & \dots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ c_{(m-1)1} - \frac{d_{m-1} f}{g} & \dots\dots\dots & & & c_{(m-1)m} \\ c_{m1} - \frac{d_m f}{g} & \dots\dots\dots & & & c_{mm} \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \vdots \\ \vdots \\ \zeta_m \end{pmatrix}$$

$$+ \begin{pmatrix} d_1 \\ \vdots \\ \vdots \\ d_m \end{pmatrix} \left(\frac{\alpha_n}{g} + \frac{f}{g}\tilde{\zeta}_1\right) + \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{pmatrix} \begin{pmatrix} y_1 \\ \hat{y}_2 + \tilde{y}_2 \\ \vdots \\ \hat{y}_n + \tilde{y}_n \end{pmatrix}$$

This closed loop system is given by a right hand side which is continuously differentiable.

Let $\mathfrak{P}$ denote the open subset of $\mathbb{R}^{\frac{(n+m+1)(n+m)}{2}}$ containing those points whose coordinates are entries of a symmetric positive definite $(n+m) \times (n+m)$ matrix. For any

$$(y_1, \hat{y}_2, \dots, \hat{y}_n, \zeta_1, \dots, \zeta_m, \tilde{\xi}, \mathcal{P}) \in \mathbb{R}^{2(n+m)} \times \mathfrak{P}$$

it corresponds a unique solution of the closed loop system starting from this point. Let $[0, t_f)$ be its right maximal interval of definition when it takes its values in $\mathbb{R}^{2(n+m)} \times \mathfrak{P}$. If t_f is finite, we have necessarily that either the solution or $\mathcal{P}(t)^{-1}$ is unbounded on $[0, t_f)$. But also since, for all t in $[0, t_f)$, the component $\mathcal{P}(t)$ of the solution is positive definite, the relations (7) and (8) derived formally can be used. It follows that (15) is also valid. This says in particular that the variables $y_1(t)$ $(= z_1(t))$, $z_i(t)$ and $\mathcal{P}(t)^{-\frac{1}{2}}\tilde{\xi}(t)$ of the solution are bounded on $[0, t_f)$ and that $y_1(t)$ is square integrable on $[0, t_f)$.

So assume for the time being that $\mathcal{P}(t)$ and $\mathcal{P}(t)^{-1}$ are also bounded on $[0, t_f)$. Then $\tilde{\xi}(t)$ and (recursively with the definition (14) of z_i) the functions α_i's and the components $\hat{y}_i$'s are bounded. Finally, by letting:

$$\delta = \begin{pmatrix} d_1 \\ \vdots \\ \vdots \\ d_m \end{pmatrix} \left(\frac{\alpha_n}{g} + \frac{f}{g}\tilde{\zeta}_1\right) \quad (21)$$

$$+ \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{pmatrix} \begin{pmatrix} y_1 \\ \hat{y}_2 + \tilde{y}_2 \\ \vdots \\ \hat{y}_n + \tilde{y}_n \end{pmatrix}$$

we get a function $\delta(t)$ bounded on $[0, t_f)$ and we observe that the $\zeta_i(t)$'s components are solution of:

$$\dot{\zeta} = M(y_1(t))\,\zeta + \delta(t) \ , \quad (22)$$

where M is the matrix

$$\begin{pmatrix} c_{11} - d_1\dfrac{f}{g} & c_{12} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ c_{(m-1)1} - d_{(m-1)}\dfrac{f}{g} & c_{(m-1)2} & \dots & \dots & c_{(m-1)m} \\ c_{m1} - d_m\dfrac{f}{g} & c_{m2} & \dots & \dots & c_{mm}(0) \end{pmatrix} .$$

With the minimum phase assumption and the facts that the entries of $M(y_1)$ are Lipschitz continuous functions of y_1 and that $y_1(t)$ is bounded on $[0, t_f)$, we get the existence of a positive definite matrix R and of positive real numbers κ (depending on the solution) and ρ such that we have, for all t in $[0, t_f)$,

$$RM(0) + M(0)^TR \leq -2\,\rho\,R \ , \quad (23)$$

$$R^{\frac{1}{2}}|M(y_1(t)) - M(0)|R^{-\frac{1}{2}} \leq \kappa\,|y_1(t)| \ . \quad (24)$$

It follows by completing the squares that we have:

$$\tfrac{1}{2}\dot{\overbrace{\zeta^TR\zeta}}$$

$$\leq -\rho\zeta^TR\zeta + \zeta^TR[M(y_1(t)) - M(0)]\zeta + \zeta^TR\delta(t) \quad (25)$$

$$\leq -\frac{\rho}{3}\zeta^TR\zeta + \frac{3\kappa^2}{4\rho}y_1(t)^2\zeta^TR\zeta + \frac{3}{4\rho}\delta(t)^TR\delta(t)^2 \quad (26)$$

Since $y_1(t)$ is square integrable on $[0, t_f)$ and $\delta(t)$ is bounded on $[0, t_f)$, we get, from [4, Theorem IV.1.9] for instance, that $\zeta(t)$ (i.e. the $\zeta_i(t)$'s components) is bounded on $[0, t_f)$. Hence we have established that, if the functions $\mathcal{P}(t)$ and $\mathcal{P}(t)^{-1}$ are bounded on $[0, t_f)$, then $t_f = +\infty$ and the solution is bounded on $[0, +\infty)$.

Now, invoking LaSalle's invariance principle, we get, from (15), that $z_i(t)$, $\tilde{\xi}(t)$ and, from (11) and (14), $\hat{y}_i(t)$

and therefore $y_i(t)$ converges to 0 as t tends to $+\infty$. This implies also the convergence of $\delta(t)$ and, from (25), the one for $\zeta_i(t)$.

So, to summarize, if we can prove that the functions $\mathcal{P}(t)$ and $\mathcal{P}(t)^{-1}$ are bounded on $[0, t_f)$, then $t_f = +\infty$, the solution is bounded on $[0, +\infty)$ and all its components except $\mathcal{P}$ converge to 0.

3.2 The functions $\mathcal{P}(t)$ and $\mathcal{P}(t)^{-1}$ are bounded

The matrix $\mathcal{P}(t)$ being given by a Riccati equation, it is known from [3] for instance that $\mathcal{P}(t)$ and $\mathcal{P}(t)^{-1}$ are bounded if a uniform complete observability property holds.

Let us come back to the compact notation (4) and recall that the matrix $\mathcal{A}(y_1)$ exhibits the lower triangular structure

$$\mathcal{A}(y_1) \;=\; \begin{pmatrix} \phi_{11}(y_1) & \phi_{12}(y_1) & & \\ \vdots & & \ddots & \\ \phi_{(p_1)1}(y_1) & \dots & \dots & \phi_{(p-1)p} \\ \phi_{p1}(y_1) & \dots & \dots & \phi_{pp} \end{pmatrix} \tag{27}$$

with $p = n+m$ and $\phi_{..} = a_{..}, b_{..}, c_{..}$ or f and where, from (2), we know that, for each ω, there exists a strictly positive real number ε, such that we have, for each i,

$$|y_1| \;\leq\; \omega \qquad \Rightarrow \qquad \phi_{i(i+1)}(y_1) \;\geq\; \varepsilon\ . \tag{28}$$

Let the solution considered in section 3.1, defined on $[0, t_f)$, be given. Let :

$$\begin{aligned} A(t) \;&=\; \mathcal{A}(y_1(t)) \quad \forall t \in [0, t_f)\ , \qquad (29)\\ &=\; \mathcal{A}(y_1(0)) \quad \forall t \in [t_f, +\infty)\ , \end{aligned}$$

and denote

$$\begin{aligned} \Phi_{ij}(t) \;&=\; \phi_{ij}(y_1(t)) \quad \forall t \in [0, t_f)\ , \qquad (30)\\ &=\; \phi_{ij}(y_1(0)) \quad \forall t \in [t_f, +\infty)\ . \end{aligned}$$

We know that $y_1(t)$ is bounded on $[0, t_f)$ so we get the existence of positive real numbers ε and a such that, for all $t \geq 0$, we have

$$\Phi_{i(i+1)}(t) \;\geq\; \varepsilon \quad , \qquad |A(t)| \;\leq\; a\ . \tag{31}$$

Then, let us study the uniform complete observability of the following auxiliary system

$$\dot{x} \;=\; A(t)\, x \quad , \qquad y \;=\; C\, x\ , \tag{32}$$

under the only assumptions:

1. For all t and (i, j), we have:
$$|\Phi_{ij}(t)| \;\leq\; \Phi\ . \tag{33}$$
2. For all t and i, we have:
$$\Phi_{i(i+1)}(t) \;\geq\; \varepsilon \;>\; 0\ . \tag{34}$$

We shall call upon two technical Lemmas.

The first Lemma says that uniform complete observability is invariant under bounded output feedback. This result can be found for instance in [1, Theorem 4].

Lemma 3.1 *Let $K(t)$ be a bounded function of time. If the system (A, C) is uniformly completely observable, tehnso is the system $(A - KC, C)$.*

The second Lemma is more technical. Its proof is omitted due to space limitations. It can be found in [11]. This result says that, with an extra assumption, if a system is uniformly completely observable from its output and the derivative of its output, then it is also uniformly completely observable from its output only.

Lemma 3.2 *Let $D(t)$ denote*

$$D(t) \;=\; \dot{\overline{C(t)}} \;+\; C(t)\, A(t) \tag{35}$$

Assume :

1. *for all t, we have*
$$|A(t)| \;\leq\; a\ , \quad |C(t)| \;\leq\; c\ , \quad |D(t)| \;\leq\; d\ , \tag{36}$$
2. *the existence of a symmetric matrix (resp. scalar) function $E(t)$ satisfying*
$$0 \;<\; e_{\min}\, I_d \;\leq\; E(t) \;\leq\; e_{\max}\, I_d \tag{37}$$
and
$$\left| \dot{\overline{E(t)^{-1} D(t)}} \right| \;\leq\; f\ . \tag{38}$$

Under these conditions, if the system $\left(A, \begin{pmatrix} C \\ D \end{pmatrix}\right)$ is uniformly completely observable, then so is the system (A, C), i.e. if there exist T and $\eta > 0$ such that, for all t_0 and all solution $X(x, t, t_0)$, we have:

$$\int_{t_0}^{t_0+T} \left(|y(s)|^2 + |\dot{y}(s)|^2\right) ds \;\geq\; \eta\, |x|^2\ , \tag{39}$$

with

$$y(t) \;=\; C(t)\, X(x, t, t_0)\ , \tag{40}$$

then there exist $\bar{T}$ and $\bar{\eta} > 0$ such that, for all t_0 and all solution $X(x, t, t_0)$, we have:

$$\int_{t_0}^{t_0+\bar{T}} |y(s)|^2 ds \;\geq\; \bar{\eta}\, |x|^2\ . \tag{41}$$

With the help of these two Lemmas, we establish the uniform complete observability of the system (32) recursively by getting to this system from reduced order ones :

Step 0 : Consider the system

$$\dot{x}_p = \mathcal{A}_p x_p \quad , \qquad y_p = x_p \tag{42}$$

with $\mathcal{A}_p = \Phi_{pp}$. From inequality (33), we have, for any $T > 0$,

$$\int_{t_0}^{t_0+T} y_p(s)^2 ds$$
$$= \int_{t_0}^{t_0+T} X_p(x_p, s, t_0)^2 \, ds \tag{43}$$
$$\geq \int_{t_0}^{t_0+T} \exp\left(2\int_{t_0}^{s} \Phi_{pp}(\tau)d\tau\right) x_p^2 \, ds \ , \tag{44}$$
$$\geq x_p^2 \int_{t_0}^{t_0+T} \exp\left(-2\Phi(s-t_0)\right) \, ds \ , \tag{45}$$
$$\geq \frac{1-\exp\left(-2\Phi T\right)}{2\Phi} x_p^2 \ . \tag{46}$$

This is exactly the uniform complete observability property.

Step $(p-i)$:
Assume the following system is uniformly completely observable :

$$\begin{aligned} \dot{\overbrace{\begin{pmatrix} x_{p-i+1} \\ \vdots \\ x_p \end{pmatrix}}} &= \mathcal{A}_{p-i+1} \begin{pmatrix} x_{p-i+1} \\ \vdots \\ x_p \end{pmatrix} \\ y_{p-i+1} &= x_{p-i+1} \end{aligned} \tag{47}$$

where $\mathcal{A}_{p-i+1}$ has the same triangular structure as $\mathcal{A}$ in (27).

Sub-step $(p-i)$.1 : We modify the output of (47), multiplying it with $\Phi_{(p-i)(p-i+1)}(t)$,

$$\begin{aligned} \dot{\overbrace{\begin{pmatrix} x_{p-i+1} \\ \vdots \\ x_p \end{pmatrix}}} &= \mathcal{A}_{p-i+1} \begin{pmatrix} x_{p-i+1} \\ \vdots \\ x_p \end{pmatrix} \\ \bar{y}_{p-i+1} &= \Phi_{(p-i)(p-i+1)} \, x_{p-i+1} \end{aligned} \tag{48}$$

With (34), uniform complete observability is preserved.

Sub-step $(p-i)$.2 : We extend the system (48) into

$$\begin{aligned} \dot{x}_{p-i} &= 0 \\ \dot{\overbrace{\begin{pmatrix} x_{p-i+1} \\ \vdots \\ x_p \end{pmatrix}}} &= \mathcal{A}_{p-i+1} \begin{pmatrix} x_{p-i+1} \\ \vdots \\ x_p \end{pmatrix} \\ \bar{\bar{y}}_{p-i+1} &= \begin{pmatrix} x_{p-i} \\ \Phi_{(p-i)(p-i+1)} x_{p-i+1} \end{pmatrix} \end{aligned} \tag{49}$$

Uniform complete observability still holds since the new state component x_{p-i} is directly measured.

Sub-step $(p-i)$.3 : We modify the system (49) by output feedback :

$$\begin{aligned} \dot{x}_{p-i} &= \Phi_{(p-i)(p-i+1)} x_{p-i+1} \\ \dot{\overbrace{\begin{pmatrix} x_{p-i+1} \\ \vdots \\ x_p \end{pmatrix}}} &= \mathcal{A}_{p-i+1} \begin{pmatrix} x_{p-i+1} \\ \vdots \\ x_p \end{pmatrix} \\ &\quad + \begin{pmatrix} \Phi_{(p-i+1)(p-i)} \\ \vdots \\ \Phi_{(p)(p-i)} \end{pmatrix} x_{p-i} \\ \bar{\bar{y}}_{p-i+1} &= \begin{pmatrix} x_{p-i} \\ \Phi_{(p-i)(p-i+1)} x_{p-i+1} \end{pmatrix} \end{aligned} \tag{50}$$

From (33) and Lemma 3.1, we know that this system is also uniformly completely observable.

Sub-step $(p-i)$.4 : We note that the output $\bar{\bar{y}}_{p-i+1}$ of system (50) can be expressed as

$$\bar{\bar{y}}_{p-i+1} = \begin{pmatrix} x_{p-i} \\ \dot{x}_{p-i} \end{pmatrix} \tag{51}$$

Using Lemma 3.2 with (33) and the notations :

$$D(t) = \begin{pmatrix} 0 & \Phi_{(p-i)(p-i+1)}(t) & 0 & \dots & 0 \end{pmatrix} \tag{52}$$
$$E(t) = \Phi_{(p-i)(p-i+1)}(t) \ , \tag{53}$$
$$E(t)^{-1}D(t) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \end{pmatrix} \ , \tag{54}$$

we conclude that the system

$$\begin{aligned} \dot{x}_{p-i} &= \Phi_{(p-i)(p-i+1)} x_{p-i+1} \\ \dot{\overbrace{\begin{pmatrix} x_{p-i+1} \\ \vdots \\ x_p \end{pmatrix}}} &= \mathcal{A}_{p-i+1} \begin{pmatrix} x_{p-i+1} \\ \vdots \\ x_p \end{pmatrix} \\ &\quad + \begin{pmatrix} \Phi_{(p-i+1)(p-i)} \\ \vdots \\ \Phi_{(p)(p-i)} \end{pmatrix} x_{p-i} \\ y_{p-i} &= x_{p-i} \end{aligned} \tag{55}$$

is also uniformly completely observable.

Sub-step $(p-i)$.5 : We modify the system (55) once again by output feedback, preserving uniform complete observability:

$$
\begin{array}{rcl}
\dot{x}_{p-i} & = & \Phi_{(p-i)(p-i)} x_{p-i} + \Phi_{(p-i)(p-i+1)} x_{p-i+1} \\
\overbrace{\begin{pmatrix} x_{p-i+1} \\ \vdots \\ x_p \end{pmatrix}}^{\cdot} & = & \mathcal{A}_{p-i+1} \begin{pmatrix} x_{p-i+1} \\ \vdots \\ x_p \end{pmatrix} \\
 & & + \begin{pmatrix} \Phi_{(p-i+1)(p-i)} \\ \vdots \\ \Phi_{(p)(p-i)} \end{pmatrix} x_{p-i} \\
y_{p-i} & = & x_{p-i}
\end{array}
\qquad (56)
$$

This latter system is in the form :

$$
\begin{array}{rcl}
\overbrace{\begin{pmatrix} x_{p-i} \\ \vdots \\ x_p \end{pmatrix}}^{\cdot} & = & \mathcal{A}_{p-i} \begin{pmatrix} x_{p-i} \\ \vdots \\ x_p \end{pmatrix} \\
y_{p-i} & = & x_{p-i}
\end{array}
\qquad (57)
$$

Proceeding from $i = 1$ to $i = p-1$, we arrive at the end of step $p-1$ at the system (32), which is thus uniformly completely observable.

Then, let $P(t)$ be the unique solution of the following matrix differential equation:

$$
\dot{P} \;=\; PA^T(t) + A(t)P - PCC^TP + I\,, \quad P(0) = \mathcal{P}(0)\;, \qquad (58)
$$

where $\mathcal{P}(0)$ is the initial value of the component $\mathcal{P}$ of the solution considered in section 3.1.

The system (32) being uniformly completely observable, the following lemma is a restatement of [6, Lemma 3]

Lemma 3.3 *If (33) and (34) hold, then there exist two strictly positive real numbers $p_{\min}$ and $p_{\max}$ such that, for all $t \geq 0$, we have:*

$$
p_{\min} I \leq P(t) \leq p_{\max} I\;. \qquad (59)
$$

Now coming back to the study of the solution considered in section 3.1. We get that, with (29) and by comparing (17) and (58), the uniqueness of solutions implies

$$
\mathcal{P}(t) \;=\; P(t) \qquad \forall t \in [0, t_f)\;. \qquad (60)
$$

This implies readily that we have

$$
p_{\min} I \leq \mathcal{P}(t) \leq p_{\max} I \qquad \forall t \in [0, t_f)\;. \qquad (61)
$$

So we have established the property which we were needing to prove boundedness and convergence.

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