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Strict Lyapunov Functions for Feedforward Systems and Applications

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Abstract

For nonaffine nonlinear feedforward systems classes of control Lyapunov functions are constructed. Explicit formulas are determined in an important particular case. As an application of this design, we prove that the bounded state feedbacks constructed induce the property of nonlinear disturbance-to-state L^p stability.

Key words. Nonlinear system, Lyapunov function, robust stability.

1 Introduction

For classes of feedforward systems, two recursive Lyapunov designs have been proposed in [7] and [1] (see also [5], [8], [10] for some extentions). Both approaches lead to the following basic result: under suitable assumptions, for systems of the form

$$\begin{cases} \dot{x} &= h_0(x) + h_1(x,y)y + h_2(x,y,u)u \\ \dot{y} &= f_0(y) + f_1(x,y)y + f_2(x,y,u)u \end{cases} \tag{1}$$

where $\dot{y} = f_0(y)$ and $\dot{x} = h_0(x)$ are respectively globally asymptotically stable and globally critically stable systems, there exists a proper positive definite function $U(x,y)$ such that

$$\dot{U}|_{(1)}(x,y) \leq -W(y) + G(x,y,u)u \tag{2}$$

where $W(y)$ is a positive definite function. Typically, when the dimension of the variable x is strictly larger than the one of the input u, the right hand side of (2) cannot be made negative definite. This fact is a serious drawback for two technical reasons:

1. The forwarding technique is based on the determination of either decoupling changes of variables (see [5]) or cross-terms (see [10]). In many cases, only approximations of these functions can be determined because exact decoupling changes of variables and exact cross-terms are the solutions of partial differential equations which very often cannot be easily solved and are not well-known when higher-order terms of the system are just approximately known. However, the forwarding approach can be applied even when only approximations of these functions can be found, provided that $W(y)$ is a positive definite function. It follows that the forwarding technique can be applied repeatedly only if strict Lyapunov functions are available at each step.

2. The system (1) in closed-loop with any feedback which renders the right hand side of (2) negative has a priori no disturbance-to-state L^p stability property.

Obviously, this last drawback would not exist if any globally asymptotically stable system would possess the disturbance-to-state L^p stability property with respect to additive perturbations. But it is not so. Indeed in [12], a globally asymptotically stable system which admits unbounded trajectories when is added a specific disturbance is exhibited. This disturbance is a smooth function of the time which does not belong to L^1 but is in L^p for all $p \in]1, +\infty]$.

The present paper is concerned with the problem of overcoming for classes of feedforward systems the limitations mentioned above. In a first part, we determine a class of assignable Lyapunov functions (the assignable Lyapunov functions are a subclass of the control Lyapunov functions, see our basic definitions below) for a particular class of systems of the form (1). In a second part, we exploit the Lyapunov functions constructed to prove for the closed-loop systems the disturbance-to-state L^p stability property ($p = 2$ in an important particular case) by constructing explicitly nonlinear gains.

This second part comes within the framework of the already published results which approach the question of the L^p stability. Some of them are concerned with classes of linear systems subject to actuator saturation, see in particular [2], [4, 3]. In the first of these papers, it is proved that can be obtained local asymptotic stability and finite-gain L^p stability for all the stabilizable linear systems subject to input saturation. In [4, 3], the passivation approach is used to construct, for neutrally stable linear systems, bounded feedbacks which provide L^p stability with respect to persistant perturbations. Moreover, interestingly, it is shown in [3] that even for the simple chain of integrators of dimension two subject to input saturation the problem of L^p stability is far from being obvious.

A paper by A. Teel [14] is more closely related to the robustness result of our work. It establishes L^p stability for nonlinear feedforward systems in closed-loop with a feedback with multi-level saturations. However, between this result and ours, there are the following differences: (i) The coupling term we consider (i.e. the term $h_1(x,y)\,y$ in (1)) depends on x (a restrictive growth condition is nonetheless imposed on this dependence). (ii) The class of feedback laws for which we prove our result is the one proposed in the proof of [7, Theorem III.1], which is significantly different from that of multi-level saturation feedbacks proposed in [14]: in particular it contains unbounded elements when $G(\cdot)$ in (2) is independent of u.

Like all the proofs of L^p stability we have encountered in the litterature, ours relies on the knowledge of a strict Lyapunov function. Note that our Lyapunov construction utilizes as a tool the Lyapunov function designed by Liu, Chitour and Sontag in [3] for stable linear systems asymptotically stabilized by a saturated feedback.

At last, observe that the nonlinear L^p-stabilization result we obtain is new even for the particular class of the stabilizable linear systems subject to input saturation which have not simple Jordan blocks.

The paper is organized as follows. A first part (Section 2 and Section 3) is devoted to the design of strictly assignable Lyapunov functions. In Section 2 the general case is treated and Section 3 is concerned with a particular case. In the second part (Section 4) robustness issues are addressed. At last, Section 5 contains some concluding remarks.

Comments and basic definitions.

Smoothness. Unless otherwise stated, we assume throughout the paper that the functions are smooth. Some questions which arise due to a lack of smoothness have been addressed in [7].

Strictly positive constants. Throughout the paper, the symbol c is used to denote generically a strictly positive real number, (i.e. $c + c * c = c$).

Flow. We denote by $\chi(t)$ the solution of $\dot{\chi} = \phi(\chi, t)$ and by χ_0 its value at $t = 0$.

Assignable Lyapunov function. A function $U(\alpha)$ is called an assignable Lyapunov function for the system $\dot{\alpha} = \varphi(\alpha, u)$ if it is positive definite and radially unbounded and there exists $u(\alpha)$ such that

$$\frac{\partial U}{\partial \alpha}(\alpha)\varphi(\alpha, u(\alpha)) \leq 0$$

If this term is negative definite, then $U(\alpha)$ is called strictly assignable Lyapunov function.

Function of class $\mathcal{K}^\infty$. A function $\kappa(\cdot)$ is of class $\mathcal{K}^\infty$ if $\kappa(\cdot)$ is defined over $[0, +\infty[$, zero at zero, strictly increasing and $\lim_{s \to +\infty} \kappa(s) = +\infty$.

Functions L^p, L^∞. A function $f(\cdot)$ is said to belong to L^p with $p \geq 1$, (respectively to L^∞) if $|f|_p = \left(\int_0^\infty |f(t)|^p dt\right)^{\frac{1}{p}}$ (respectively $|f|_\infty = \sup_{t \geq 0} |f(t)|$) is finite.

System L^p stable. We will say that the system $\dot{\alpha} = \phi(\alpha, d)$ is L^p stable if there exist two functions $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$ of class $\mathcal{K}^\infty$ such that, for all initial condition $\alpha(0)$ and all function $d(\cdot)$ of class L^p

$$|\alpha(t)|_p \leq \gamma_1(|d|_p) + \gamma_2(|\alpha(0)|)$$

The proofs. Due to space limits, almost all the proofs of the results of this work have been removed. The reader is referred to [9].

2 Strict negativeness: general case

2.1 Recall of [7, Theorem III.1]

We consider systems of the form

$$\left\{ \begin{array}{rcl} \dot{x} & = & h_0(x) + h_1(x,y)y + h_2(x,y,u)u \\ \dot{y} & = & f_0(y) + f_1(x,y)y + f_2(x,y,u)u \end{array} \right. \tag{3}$$

where u is the input. We introduce the following assumptions.

A1. *There exist smooth positive definite and proper functions $Q(x)$ and $V(y)$ such that*

$$\tfrac{\partial Q}{\partial x}(x)h_0(x) = -R(x) \leq 0 \quad \forall\, x \tag{4}$$

$$\tfrac{\partial V}{\partial y}(y)[f_0(y) + f_1(x,y)y] \leq -W(y) < 0 \quad \forall\, y \neq 0 \tag{5}$$

A2. *The solution $x = 0$ is the unique solution of*

$$\begin{array}{c} \dot{x} = h_0(x)\,, \quad \frac{\partial Q}{\partial x}(x)h_2(x,0,0) = 0 \\ \frac{\partial Q}{\partial x}(x)h_0(x) = 0 \end{array} \tag{6}$$

A3. *There exist two positive functions $\rho(\cdot)$ and $\kappa(\cdot)$ defined over $[0, +\infty)$ such that*

$$\left|\frac{\partial Q}{\partial x}(x)\, h_1(x,y)\, y\right| \leq \kappa(V(y))W(y)\,[1 + \rho(Q(x))] \tag{7}$$

$$\frac{1}{1+\rho(\cdot)} \notin L^1([0,+\infty)) \quad (8)$$

As an immediate consequence of Theorem III.1 of [7], we have the following result:

Theorem 2.1 *Assume that Assumptions A1 to A3 are satisfied. Then there exists a function $k(\cdot)$ of class $\mathcal{K}^\infty$ such that the derivative of the function*

$$U(x,y) = k(V(y)) + l(Q(x)) \text{ with } l(r) = \int_0^r \frac{1}{1+\rho(s)} ds \quad (9)$$

along the trajectories of (3) satisfies

$$\dot{U}|_{(3)}(x,y) \le -\frac{1}{4}k'(V(y))W(y) + \mathcal{G}(x,y,u)u \quad (10)$$

with $\mathcal{G}(\cdot)$ defined by

$$\begin{array}{rcl}\mathcal{G}(x,y,u) & = & k'(V(y))\frac{\partial V}{\partial y}(y)f_2(x,y,u) \\ & & +l'(Q(x))\frac{\partial Q}{\partial x}(x)h_2(x,y,u)\end{array} \quad (11)$$

Moreover, there exists a family of feedbacks $u_s(x,y)$ such that

$$\mathcal{G}(x,y,u_s(x,y))u_s(x,y) \le 0 \quad (12)$$

$$\mathcal{G}(x,0,u_s(x,0))u_s(x,0) < 0 \Rightarrow \mathcal{G}(x,0,0) \neq 0 \quad (13)$$

Remark 1. It is always possible to determine a feedback $u_s(\cdot)$ such that (12) and (13) hold. Nevertheless, we will see in Section 3.1 that in some particular cases it is usefull to consider feedbacks such that only (12) is satisfied.

2.2 Main result

Thanks to Assumption A2, Theorem 2.1 straightforwardly implies that $u_s(\cdot)$ globally asymptotically stabilizes the origin of (3). So the converse Lyapunov theorem (see [16, Theorem 18.6]) ensures the existence of a Lyapunov function for (3) which is strictly assigned by $u_s(\cdot)$. But this existence result is useless from a practical point of view. The objective of Theorem 2.2 below is to show how the problem of constructing explicitly a strict Lyapunov function for (3) can be reduced to that of finding a strict Lyapunov function for the reduced order system

$$\dot{x} = h_0(x) + h_2(x,0,u_s(x,0))u_s(x,0)$$

Theorem 2.2 *Suppose that the system (3) satisfies Assumptions A1 to A3. Then there exist a proper positive definite function $\mathcal{Q}(x)$, a function $\bar{l}(\cdot)$, positive, zero at zero and such that, with $U(\cdot)$ given in (9), the function*

$$\overline{U}(x,y) = U(x,y) + \bar{l}(\mathcal{Q}(x)) \quad (14)$$

is strictly assignable to (3).

3 Strict negativeness: particular case

3.1 The result

In Section 2.2, we have shown that the problem of finding a strictly assignable Lyapunov function for (3) reduces to that of finding a strict Lyapunov function for a reduced order system. In this section, we restrict our attention to the class of systems (3) so that this second problem can be solved. We consider

$$\left\{\begin{array}{rcl}\dot{x} & = & Mx + h_1(x,y)y + h_2(x,y,u)u \\ \dot{y} & = & f_0(y) + f_2(x,y,u)u\end{array}\right. \quad (15)$$

where $h_2(\cdot)$ is such that $h_2(x,0,0)$ is a constant that we denote by

$$D := h_2(x,0,0)$$

In this specific context, we particularize Assumptions A1 to A3 as follows.

A1'. *There exists a symmetric positive definite matrix Q such that*

$$QM + M^\top Q \le 0 \quad (16)$$

There exists a function $V(y)$ positive definite and proper such that

$$\frac{\partial V}{\partial y}(y)f_0(y) = -W(y) < 0 \quad \forall y \neq 0 \quad (17)$$

Moreover, both $V(\cdot)$ and $W(\cdot)$ are lower bounded in a neighborhood of the origin by positive definite quadratic forms.[1]

A2'. *The pair (M,D) is stabilizable.*

A3'. *There exists a positive function $\gamma_1(\cdot)$ such that*

$$|h_1(x,y)y| \le (1+|x|)\gamma_1(|y|)|y|^2 \quad (18)$$

There exist $C \in]0,+\infty]$ and a function $\gamma_2(\cdot)$ of class $\mathcal{K}^\infty$ such that for all $|u_1| \le C$, $|u_2| \le C$,

$$\begin{array}{l}|h_2(x,y,u_1) - h_2(x,y,u_2)| \le \frac{1}{2}\gamma_2(|y| + |u_1-u_2|) \\ |f_2(x,y,u_1) - f_2(x,y,u_2)| \le \frac{1}{2}\gamma_2(|y| + |u_1-u_2|)\end{array} \quad (19)$$

A direct consequence of (16) and Assumption A2' is the following fact (see [15, (93)]).

Fact 3.1 *When Assumptions A1' and A2' hold, then for all $\varepsilon > 0$, there exists a matrix Q_ε symmetric and positive definite such that*

$$Q_\varepsilon(M - \varepsilon DD^\top Q) + (M - \varepsilon DD^\top Q)^\top Q_\varepsilon = -I \quad (20)$$

where I denotes the identity matrix.

[1] If are known functions $V(\cdot)$ and $W(\cdot)$ such that (17) holds but which are not lower bounded in a neighborhood of the origin by positive definite quadratic forms and if the origin of the y-subsystem of (15) with the input set to zero is locally exponentially stable, then, by taking advantage of [6, Appendice G], it is possible to design new functions $V(\cdot)$ and $W(\cdot)$ satisfying the requirements of Assumption A1'.

Proposition 3.2 *Assume that the system (15) satisfies Assumptions A1' to A3'. Then there exist a function $k(\cdot)$ of class $\mathcal{K}^\infty$ and strictly positive constants q and ε such that the function*

$$\overline{U}(x,y) = U(x,y) + \overline{l}(\mathcal{Q}(x)) \tag{21}$$

with

$$U(x,y) = k(V(y)) + l(x^\top Q x)\ ,\ l(r) = \sqrt{1+r} - 1 \tag{22}$$

$$\overline{l}(r) = \ln(1+r)\ ,\ \mathcal{Q}(x) = q(x^\top Q x)^2 + x^\top Q_\varepsilon x \tag{23}$$

is strictly assignable for the system (15). Moreover, $\overline{U}(\cdot)$ is lower bounded in a neighborhood of the origin by a positive definite quadratic form and the derivative of $\overline{U}(\cdot)$ along (15) in closed-loop with $u_s(\cdot)$ is upper bounded by a negative definite quadratic form.

Discussion of the assumptions.

1. When $h_2(\cdot)$ and $f_2(\cdot)$ do not depend on u, unbounded feedbacks can be deduced from Proposition 3.2.

2. The requirement (18) in Assumption A3' is similar to the growth condition imposed in [10]. In [7, (63)] is given an example which shows that the stabilizability of (15) is not guaranteed any more when the right hand side of (18) is replaced by $(1+|x|^a)\gamma_1(|y|)|y|^2$ with $a > 1$, unless extra assumptions are imposed (see [10]).

4 Robustness property

In this section, we prove that Theorem 2.2 provides us with feedbacks which induce a disturbance-to-state L^p stability property.

4.1 The results

Consider the following system

$$\left\{\begin{array}{rcl} \dot{x} &=& h_0(x) + h_1(x,y)y + \psi(x,y,u,d)d \\ && +h_2(x,y,u)u \\ \dot{y} &=& f_0(y) + f_1(x,y)y + \phi(x,y,u,d)d \\ && +f_2(x,y,u)u \end{array}\right. \tag{24}$$

where $d(t) \in \Re^d$. Besides Assumptions A1 to A3, we introduce four new assumptions.

B1. The disturbance $d(\cdot)$ is a *continuous* function of the time which belongs to L^p.

B2. When $u = 0$, the following inequality holds

$$\dot{V}|_{(24)} \le -W(y) + |d(t)|^p \tag{25}$$

where p is an integer strictly larger than 1.

B3. There exist a positive function $\kappa_2(\cdot)$ and $C \in]0,+\infty]$ such that, for all u smaller in norm than C,

$$|\psi(x,y,u,d)d| \le \kappa_2(V(y))|d|\ ,\quad \forall d \tag{26}$$

$$\left|\tfrac{\partial Q}{\partial x}(x)\right| \le c(1+\rho(Q(x))) \tag{27}$$

$$|\phi(x,y,u,d) - \phi(x,y,0,d)||d| \le \kappa_2(V(y))|d||u| \tag{28}$$

with $\rho(\cdot)$ given by Assumption A3.

B4. The functions $V(\cdot)$ and $W(\cdot)$ are such that

$$\left|\tfrac{\partial V}{\partial y}(y)\right|^{\frac{2p}{p-1}} \le cW(y)\quad ,\quad \forall y : |y| \le 1 \tag{29}$$

There exists a function $\mathcal{Q}(\cdot)$ such that there exists a positive definite function $k_d(\cdot)$ which is not in L^1, not zero at zero and such that

$$k_d\left(\int_0^{Q(x)} \tfrac{1}{1+\rho(s)} ds\right) \tfrac{\left|\frac{\partial Q}{\partial x}\right|^p}{[1+\rho(Q(x))]^p} \le \tfrac{\Re(x)^p}{[|x|\gamma_x(|x|)]^{2p-2}} \tag{30}$$

$$k_d\left(\int_0^{Q(x)} \tfrac{1}{1+\rho(s)} ds\right) \left|\tfrac{\partial \mathcal{Q}}{\partial x}\right|^p \le \Re(x)^{p-2}\left[|x|\gamma_x(|x|)\right]^2 \tag{31}$$

where $\gamma_x(|x|)$ is a positive function such that

$$\left|\frac{\partial \mathcal{Q}}{\partial x}(x)\Delta(x,y)\right| \le |x||y|\gamma_x(|x|)\gamma_y(|y|) \tag{32}$$

with

$$\begin{array}{rcl} \Delta(x,y) &=& h_1(x,y)y \\ && +h_2(x,y,u_s(x,y))u_s(x,y) \\ && -h_2(x,0,u_s(x,0))u_s(x,0) \end{array} \tag{33}$$

Theorem 4.1 *Assume that the system (24) satisfies Assumptions A1 to A3 and B2 to B4. Then, there exist a proper positive definite smooth function $\overline{U}(\cdot)$, a feedback law $u_s(\cdot)$ and a positive definite function $\nu(\cdot)$ such that the Lie derivative of $\overline{U}(\cdot)$ along (24) in closed-loop with $u_s(\cdot)$ satisfies*

$$\dot{\overline{U}}|_{(24)} \le -\nu(x,y) + |d(t)|^p \tag{34}$$

Remark 2. Since (34) is similar to (25), Theorem 4.1 can be applied repeatedly.

Theorem 4.2 *Assume that the system (24) satisfies Assumptions A1 to A3 and B1 to B4. Then there exists a state feedback $u_s(x,y)$ such that the following holds.*

1. There exist some functions $\beta_1(\cdot)$, $\beta_2(\cdot)$ of class $\mathcal{K}^\infty$ such that

$$|(x,y)|_\infty \le \beta_1(|(x_0,y_0)|) + \beta_2(|d|_p) \tag{35}$$

2. Moreover, if both $W(\cdot)$ and $\mathcal{Q}(\cdot)$ are lower bounded in a neighborhood of the origin by a positive definite quadratic form, then there exist some functions $\beta_3(\cdot)$, $\beta_4(\cdot)$ of class $\mathcal{K}^\infty$ such that

$$|(x,y)|_p \le \beta_3(|(x_0,y_0)|) + \beta_4(|d|_p) \tag{36}$$

Remark 3. To establish (35), it is just necessary that $\nu(\cdot)$ be positive.

Discussion of the assumptions.

B2. This assumption implies that the uncontrolled y-subsystem is internally stable and possesses the disturbance-to-state L^p stability property.

B3. (i) The reason why this assumption is introduced is clear: roughly speaking, growth assumptions on the disturbances are imposed to select systems for which the finite escape time phenomenon does not occur. (ii) Many versions of this assumption can be proposed; for instance results can be obtained with functions $\psi(\cdot)$ and $\phi(\cdot)$ unbounded with respect to the x part of the state. However, we do not introduce a less restictive version of this assumption because as it is, it gives us the possibility to carry out a reasonably simple proof. (iii) Since C can be $+\infty$, it is not required on $u_s(\cdot)$ to be bounded. In particular, when $\phi(\cdot)$, $\psi(\cdot)$, $h_2(\cdot)$ and $f_2(\cdot)$ do not depend on u, such a choice of feedback can actually be made.

B4. (i) It seems to be difficult to check whether the requirements (30) and (31) are satisfied or not. Fortunately, we will see in Section 4.4 that in the particular context of Section 3, the requirements of this assumption can be easily met when $p = 2$. (ii) Since $\frac{2p}{p-1} \geq 2$ for any $p \geq 1$, requirement (29) is not very restrictive. If $W(\cdot)$ is lower bounded in a neighborhood of the origin by a positive definite quadratic form, it is always satisfied.

4.2 Proof of Theorem 4.1

This proof relies on the following lemmas.

Lemma 4.3 *Let $\Omega(\cdot)$ be any strictly positive function. There exist a Lyapunov function $U(x,y)$, a function $k(\cdot)$ of class $\mathcal{K}^\infty$ with a strictly positive first derivative and a state feedback law $u_s(x,y)$ such that the derivative of $U(\cdot)$ along (24) in closed-loop with $u_s(\cdot)$ satisfies*

$$\begin{aligned} \dot{U}|_{(24)} \leq & -\tfrac{1}{8}k'(V(y))W(y) \\ & +\tfrac{1}{2}\mathcal{G}(x,y,u_s(x,y))u_s(x,y) \\ & +k'(V(y))[2+\Omega(x)^p]|d|^p \\ & +\tfrac{|\frac{\partial Q}{\partial x}(x)|}{1+\rho(Q(x))}\kappa_2(V(y))|d| \end{aligned} \tag{37}$$

Lemma 4.4 *There exist a positive function $\bar{l}(\cdot)$, zero at zero, an invertible function $\widehat{k}(\cdot)$ of class $\mathcal{K}^\infty$ and a function $\nu(\cdot)$ such that (34) holds with*

$$\overline{U}(x,y) = \widehat{k}\left(U(x,y) + \bar{l}(Q(x))\right) \tag{38}$$

4.3 Proof of Theorem 4.2

To establish (35) and (36), we need to prove a preliminary result which draws its inspiration from the last part of the proof of Lemma 2 of [14].

Lemma 4.5 *If for the system*

$$\dot{\chi} = \phi(\chi, D) \tag{39}$$

with $D(\cdot) \in L^1$, there exists a proper positive definite function $V(\cdot)$ such that

$$\dot{V}|_{(39)} \leq |D| \tag{40}$$

then there exist some continuous functions $\mu_1(\cdot)$, $\mu_2(\cdot)$ of class $\mathcal{K}^\infty$, such that

$$|\chi|_\infty \leq \mu_1(|\chi_0|) + \mu_2(|D|_1) \tag{41}$$

Moreover, if there exists a positive definite function $W(\cdot)$ such that

$$\dot{V}|_{(39)} \leq -W(\chi) + |D| \tag{42}$$

and $W(\cdot)$ is larger on a neighborhood of the origin than $|\chi|^q$ with $q \geq 2$, then there exist some continuous functions $\mu_3(\cdot)$, $\mu_4(\cdot)$ of class $\mathcal{K}^\infty$ such that

$$|\chi|_q \leq \mu_3(|\chi_0|) + \mu_4(|D|_1) \tag{43}$$

Next, one fact should be established: when both $W(\cdot)$ and $\Re(\cdot)$ are lower bounded in a neighborhood of the origin by positive definite quadratic forms, then it is possible to determine a function $\overline{U}(\cdot)$, lower bounded in a neighborhood of the origin by a positive definite quadratic form, such that $\nu(\cdot)$ is lower bounded in a neighborhood of the origin by a positive definite quadratic form. On the one hand, one can prove that $\bar{l}(\cdot)$ can be chosen such that $\bar{l}'(\cdot)$ is strictly positive on a neighborhood of zero and on the other hand both $k'(\cdot)$ and $\widehat{k}'(\cdot)$ are strictly positive on a neighborhood of zero.

At last, Theorem 4.2 can be proved by combining Theorem 4.1 and Lemma 4.5 ($\chi = (x^\top, y^\top)^\top, D = |d_1| + |d_2|^p$).

4.4 Particular case

4.4.1 Result: In the particular context of Section 3, Theorem 4.2 results in a generalization of [14]: L^2 stability can always be proved.

Corollary 4.6 *Consider the subclass of the systems (24) which are of the form (15) and satisfy Assumptions A1' to A3' when $d = 0$. If Assumption B3 is satisfied with $\rho(s) = \sqrt{s+1} - 1$ and $Q(x) = x^\top Qx$, then there exists a feedback law $u_s(\cdot)$ and there exist some functions $\beta_a(\cdot)$, $\beta_b(\cdot)$ of class $\mathcal{K}^\infty$ such that*

$$|(x,y)|_2 \leq \beta_a(|(x_0,y_0)|) + \beta_b(|d|_2) \tag{44}$$

where (x,y) is the solution of (24) in closed-loop with $u_s(\cdot)$ starting at (x_0,y_0).

4.4.2 Counter example: As illustrated by the following example[2], we cannot expect to determine feedbacks which induce the L^3 stability property for systems (15) satisfying Assumptions A1' to A3'. The two dimensional system

$$\left\{ \begin{array}{rcl} \dot{x} & = & y - \frac{y|y||x|}{1+|yx|} + d \\ \dot{y} & = & u \end{array} \right. \qquad (45)$$

satisfies these assumptions but when $d = \frac{10}{\sqrt{1+t}}$, for any input u, the system (45) admits unbounded trajectories.

5 Conclusion

We have designed strictly assignable Lyapunov functions for systems obtained by the addition of one integration. This design can be repeatedly applied and gives explicit formulas in an important particular case. As an application of this Lyapunov design, we obtain a result of robustness. To summarize, let us list all the robustness properties induced for feedforward systems by our strategy of design: (i) Property of nonlinear disturbance-to-state L^2 stability. (ii) Robustness with respect to coupling disturbances bounded by higher order functions. (iii) Robustness due to the local exponential stability property of the closed-loop systems (obtained for feedforward systems which admit an asymptotically stabilizable linearization at the origin).

References

[1] M.J. Jankovic, R. Sepulchre, P.V. Kokotovic : *Global stabilization of an enlarged class of cascade nonlinear systems.* IEEE Trans. on Aut. Contr., vol.41, no.12, pp.1723-1735, 1996.

[2] Z. Lin, A. Saberi, A. Teel : *Simultaneous L_p-Stabilization and Internal Stabilization of Linear Systems Subject to Input Saturation - State Feedback Case.* Systems and Control Letters Vol.25 (1995) 219-226.

[3] W. Liu, Y. Chitour, E. Sontag : *Remarks on Finite Gain Stabilizability of Linear Systems Subject to Input Saturation.* Proceedings of the 32rd IEEE conference on decision and control. December 1993.

[4] W. Liu, Y. Chitour, E. Sontag : *On Finite Gain Stability of Linear Systems Subject to Input Saturation.* Siam J. Control and Optimization, July 1996.

[5] F. Mazenc : *Stabilization of Feedforward Systems Approximated by a Nonlinear Chain of Integrators.* Systems and Control Letters Vol.32, no.4, pp.223-229, 1997.

[6] F. Mazenc : *Stabilisation de trajectoires, Ajout d'intégration, Commande saturées.* Thèse en Mathématiques et Automatique. École des Mines de Paris. 1996.

[7] F. Mazenc, L. Praly : *Adding an integration and Global asymptotic stabilization of feedforward systems.* IEEE Trans. on Aut. Contr., vol.41, no.11, pp.1559-1578, 1996.

[8] F. Mazenc, L. Praly : *Asymptotic Tracking of a State Reference for Systems with a Feedforward Structure.* Automatica 36 (2) (2000) pp. 179-187.

[9] F. Mazenc, L. Praly : *Strict Lyapunov Functions for Feedforward Systems and Applications.* Submitted to ESAIM: Control, Optimisation and Calculus of Variations

[10] F. Mazenc, R. Sepulchre, M. Jankovic : *Lyapunov functions for stable cascades and applications to global stabilization.* IEEE Trans. Aut. Contr., vol. 44, No.8, August 1999.

[11] A. Megretski : *L_2 BIBO output feedback stabilization with saturated control.* 13th World Congress of IFAC, Volume D,1996, p435-440.

[12] J.B. Pomet, L. Praly : *A result on robust boundedness.* System and Control Letters 10 (1988) 83-92.

[13] A. Teel : *Feedback stabilization : nonlinear solutions to inherently nonlinear problems.* Memorandum No. UCB/ERL M92/65. 12 June 1992

[14] A. Teel : *On L_2 performance induced by feedbacks with multiple saturations.* ESAIM: Control, Optimisation and Calculus of Variations, vol. 1, pp. 225–240, September, 1996.

[15] A. Teel : *A nonlinear small gain theorem for the analysis of control systems with saturation.* IEEE Trans. on Aut. Contr., vol. 41, no. 9, pp. 1256-1270, 1996.

[16] T. Yoshizawa : *Stability theory by Lyapunov's second method.* The mathematical Society of Japan, 1966.

[2] The higher order term $\frac{y|y||x|}{1+|yx|}$ is continuous but not smooth. It is possible to replace it by a smooth term, but this would involve a more complicate analysis.