On The Use Of Dynamic Invariants And Forwarding For Swinging Up A Spherical Inverted Pendulum[1]

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Abstract

We design and implement a controller to swing up a spherical pendulum carried on by a three links robot arm. This controller is the patch of two linear controllers and a nonlinear one. The latter is based on energy and kinetic momentum assignment and relies in part on the forwarding design technique.

1 Problem statement.

We consider a system made of a spherical pendulum carried on by a three links robot arm called $2k\pi$ (see figure 1). We address the problem of swinging up the pendulum (M in figures 1 and 2), i.e. bringing it to its open loop unstable vertical position while having its actuated end point (P in figures 1 and 2) at a prescribed position. More details about the experiment and some implementation data are available in [6, section 6] or can be found at
`http://cas.ensmp.fr/CAS/2kPi/index-e.html`

This exactly same problem has been solved in [6]. There, exploiting the property that the system is flat, our colleagues have obtained a solution by designing an open-loop trajectory steering the pendulum from the downward to the upward equilibrium and designing a tracking controller. But our problem differs from the one studied in [2, section 3.2] where the spherical pendulum is controlled only via a planar 2 D acceleration and only local asymptotic stability of the upward equilibrium is considered, without a requirement on the ultimate position of the actuated end point.

We propose a solution leading to a closed-loop behavior completely different to what is achieved in [6]. Roughly, here, instead of realizing a fast swing, we put the emphasis on reducing the input magnitude during this swing.

As in [6], we postulate that the robot arm is nothing but an actuator delivering a desired 3 D acceleration at the actuated end point (P in figures 1 and 2) from its controlled three torques. Of course this assumption does not hold[1] and to make it more realistic, we have to cope with constraints in the controller design : state constraints, bandwidth constraints and saturation constraints.

With the above postulate, the dynamics of the system is reduced to the dynamics of the free end point M of the pendulum subject to gravity and to the acceleration of the actuated end point P. Let us denote (see figure 2) :

- M the free end point of the pendulum and $\vec{M}$ the vector it defines from the desired rest point of the actuated end point,
- P, the actuated end point and $\vec{P}$ the corresponding vector,
- $\vec{g}$, the normalized gravity force,
- $\vec{b}$ the unit vector $\frac{\overrightarrow{PM}}{PM}$,
- l, the length of the pendulum ($= PM$).
- $\vec{u} = \ddot{\vec{P}}$, the acceleration the robot arm is able to produce at P, i.e. the control in our design,
- $\vec{x} \,.\, \vec{y}$, the scalar product of $\vec{x}$ and $\vec{y}$,
- $\vec{x} \wedge \vec{y}$, the vector product in $\mathbb{R}^3$ of $\vec{x}$ and $\vec{y}$.

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[1] There is no continuous bijection between $\mathbb{R}^3$ and $\mathbb{S}^2 \times \mathbb{S}^1$. This implies that the actuated end point, denoted P below must remain in a prescribed domain for its desired acceleration to be made possible for the robot arm.

Also, the robot arm is not an ideal mechanical system with motors able to deliver arbitrary torques. There are frictions, flexibilities, saturation on the motors, ...

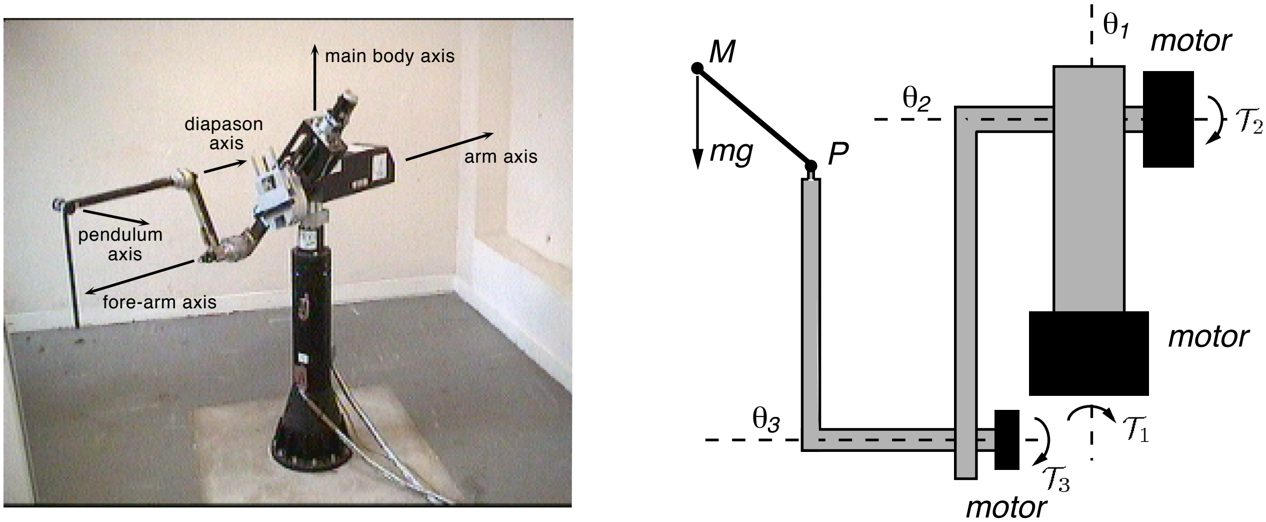

Figure 1: The robot arm $2k\pi$

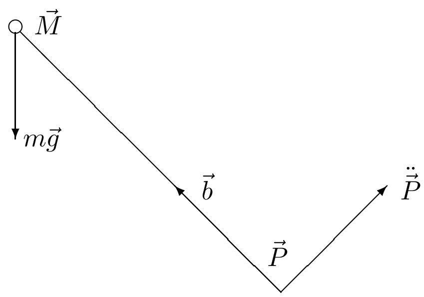

Figure 2: spherical pendulum with controlled acceleration

From Newton's equation, we get the dynamics as :

$$\left\{\begin{array}{rcl} \ddot{\vec{b}} & = & \dfrac{\vec{g}-\vec{u}}{l} - \left(\dfrac{\vec{g}-\vec{u}}{l}\,.\,\vec{b} + \dot{\vec{b}}\,.\,\dot{\vec{b}}\right)\vec{b} \\ \ddot{\vec{P}} & = & \vec{u} \\ \vec{b}\,.\,\vec{b} & = & 1 \\ \dot{\vec{b}}\,.\,\vec{b} & = & 0 \end{array}\right. \tag{1}$$

This system evolves in $\mathbb{S}^2 \times T\mathbb{S}^2 \times \mathbb{R}^3 \times \mathbb{R}^3$ and we want to bring the state from any initial condition to the final rest point :

$$\vec{b} \;=\; -\frac{\vec{g}}{g} \quad , \qquad \dot{\vec{b}} \;=\; \vec{P} \;=\; \dot{\vec{P}} \;=\; 0\ . \tag{2}$$

Extending to the spherical case what has been done in [8] for the circular case, we solve the problem by invoking three controllers :

1. A linear (local) controller based on linearization around the downward equilibrium and in charge of the "take off" from this point.
2. A nonlinear (regional) controller making the pendulum move from a neighborhood of the downward equilibrium to a neighborhood of the upward equilibrium. It actually assigns specific values to the mechanical energy and the kinetic momentum with respect to the vertical axis. Its design relies in part on the forwarding technique.
3. A linear (local) controller based on linearization around the upward equilibrium and in charge of the "landing" on this point.

The controllers 1 and 3 being linear, their design is standard and not developed here. However it is for tuning the controller 3 that we encountered the main difficulties in the implementation. On the other hand the design of the nonlinear controller is more involved. We described its main steps below. A complete description of the design and implementation of the three controllers can be found in [1].

If, at this stage, the reader needs some more motivations for going further in reading this paper, he can have a look at the video which can be downloaded from :
`http://www.stud.enst.fr/~albouy/robot/demo.avi`
or
`http://cas.ensmp.fr/~praly/Publications/XavierX96_1.avi`

2 Ideas behind the design of the non linear controller.

2.1 Assigning open loop invariants for the $(\vec{b}, \dot{\vec{b}})$ coordinates.

As already proposed by many authors [9, 3, 8], ..., at least in the case of a circular pendulum, a way to realize the swing up of the pendulum is to make asymptotically stable the closure of the homoclinic orbit[2] of the pendulum and the desired rest point for the actuated end point P. Indeed, in this case, we are guaranteed that, in finite time, the state of the overall system will be in a neighborhood of the desired equilibrium. In the circular case, this closure of the homoclinic orbit is completely characterized as a level set of the total mechanical energy which is here :

$$E = \frac{1}{2}\dot{\vec{b}}.\dot{\vec{b}} - \frac{\vec{b}.\vec{g}}{l} \quad (3)$$

In the spherical case, the closure of the homoclinic orbits are lying in the 2 dimensional submanifold of the manifold $\mathbb{S}^2 \times T\mathbb{S}^2$ where $\dot{\vec{b}}$ is obtained from $\vec{b} \in \mathbb{S}^2$ by solving the equations :

$$\dot{\vec{b}}.\dot{\vec{b}} = 2\frac{\vec{b}.\vec{g}}{l} + 2\frac{g}{l}, \quad \dot{\vec{b}}.\vec{b} = 0, \quad \dot{\vec{b}}.(\vec{b}\wedge\vec{g}) = 0 . \quad (4)$$

The first equation says that the total mechanical energy is equal to the one of the upward rest position $\vec{b} = -\frac{\vec{g}}{g}$. The second equation is nothing but the consequence that $\vec{b}$ being a unit vector, its norm is constant. The third equation is the expression of the fact that $\vec{b}$ remains in a vertical plane, i.e. since the vector $\vec{b}\wedge\vec{g}$ is orthogonal to the vertical plane containing $\vec{b}$, $\dot{\vec{b}}$ must be orthogonal to this vector. In other words, in the spherical case, the manifold containing the closure of the homoclinic orbits is not the whole set :

$$E = \frac{g}{l}, \quad (5)$$

but its intersection with the set

$$J = \dot{\vec{b}}.(\vec{b}\wedge\vec{g}) = 0 \quad (6)$$

where J is the kinetic momentum with respect to the vertical axis. As in the circular case, both E and J are open loop invariants. In particular, we get :

$$\dot{E} = -\frac{\dot{\vec{b}}.\vec{u}}{l} \quad , \qquad \dot{J} = -\frac{\vec{b}\wedge\vec{g}}{l}.\vec{u} \quad . \quad (7)$$

It follows that to make the closure of the homoclinic orbits asymptotically stable it is sufficient to assign these two invariants (E, J) to their prescribed values $(g/l, 0)$. This task can be achieved since the sign of their speed is dictated by the direction of $\vec{u}$.

2.2 Forwarding to cope with $(\vec{P}, \dot{\vec{P}})$.

While swinging up the pendulum, we have to asymptotically stabilize the origin for the $(\vec{P}, \dot{\vec{P}})$ coordinates. Since we have simply :

$$\ddot{\vec{P}} = \vec{u}, \quad (8)$$

like for the two open loop invariants above, the task of regulating $\dot{\vec{P}}$ around the origin can be easily realized.

But on the other hand we do not have directly access to $\dot{\vec{P}}$ from u. This makes the objective of regulation of $\vec{P}$ at the origin more difficult to meet. Nevertheless, we observe that $\vec{P}$ is a state component integrating a function of the other state components, and more specifically here $\dot{\vec{P}}$. So, once the control objective has been met for the $(\dot{\vec{P}}, \vec{b}, \dot{\vec{b}})$ coordinates, we can cope with the position $\vec{P}$ by applying the forwarding technique as described in [7].

2.3 Summary.

To summarize, the non linear controller we design aims at making the set :

$$\mathcal{S} = \left\{(\vec{P}, \dot{\vec{P}}, \vec{b}, \dot{\vec{b}}) : E = \frac{g}{l}, J = 0, \vec{P} = \dot{\vec{P}} = 0\right\} \quad (9)$$

asymptotically stable with a basin of attraction as large as possible. To design such a controller, we proceed in two steps :

1. In a first step, we deal with the $(\dot{\vec{P}}, \vec{b}, \dot{\vec{b}})$ coordinates. We apply a passivity design (see [4, Theorems 2.5.1 and 2.5.2]) to get a regulator of $(E - g/l, J, \dot{\vec{P}})$ at zero.
2. In a second step, we cope with $\vec{P}$ by applying the forwarding technique based on the construction of a Lyapunov function. This construction relies on a change of coordinate exhibiting the stability property, for the overall system, provided by the control law designed in the first step.

3 Controller design.

3.1 The $(\dot{\vec{P}}, \vec{b}, \dot{\vec{b}})$ subsystem.

At this stage we want to make the set :

$$\mathcal{S}_1 = \left\{(\dot{\vec{P}}, \vec{b}, \dot{\vec{b}}) : E = \frac{g}{l}, J = 0, \dot{\vec{P}} = 0\right\} \quad (10)$$

asymptotically stable with a basin of attraction as large as possible.

Let K be a radially unbounded, C^2 function on $\mathbb{R}^2$ whose only stationary point is at the origin. We do not specify what this function is now since we want to keep some flexibility for handling the overall system later on. Consider the function :

$$V_1(\dot{\vec{P}}, \vec{b}, \dot{\vec{b}}) = K(E - \frac{g}{l}, J) + \frac{a}{2}\dot{\vec{P}}.\dot{\vec{P}}, \quad (11)$$

with a a strictly positive real number. This function is radially unbounded on $\mathbb{R}^3 \times \mathbb{S}^2 \times T\mathbb{S}^2$ and is zero on and only on the set $\mathcal{S}_1$. Its derivative along the solutions is :

[2] One which makes just one turn in infinite time from the upward equilibrium back to this position.

$$\dot{V}_1 = \tag{12}$$
$$-\frac{\partial K}{\partial E}(E-\frac{g}{l},J)\,\frac{\dot{\vec{b}}.\vec{u}}{l} \;-\; \frac{\partial K}{\partial J}(E-\frac{g}{l},J)\,\frac{\vec{b}\wedge\vec{g}}{l}.\vec{u} \;+\; a\,\dot{\vec{P}}.\vec{u}\ .$$

This derivative is made non positive by picking $\vec{u}$ as :

$$\vec{u} \;=\; -\Lambda_1(\dot{\vec{P}},\vec{b},\dot{\vec{b}})\,\times \tag{13}$$
$$\times\,\left(-\frac{\partial K}{\partial E}(E-\frac{g}{l},J)\,\frac{\dot{\vec{b}}}{l} - \frac{\partial K}{\partial J}(E-\frac{g}{l},J)\,\frac{\vec{b}\wedge\vec{g}}{l} + a\,\dot{\vec{P}}\right)$$

where Λ_1 is any matrix with positive definite symmetric part. From the property of V_1, the set $\mathcal{S}_1$ is made globally stable. But, with the above analysis, we cannot claim anything about attractiveness nor the domain of attraction. Nevertheless we go on with our design.

3.2 The overall system.

In the previous paragraph, we have taken care of the $(\dot{\vec{P}},\vec{b},\dot{\vec{b}})$ coordinates. To deal with $\vec{P}$, we apply the forwarding technique of [7]. We introduce the change of coordinate (see Appendix A or [1]) :

$$\vec{Q} \;=\; \vec{P} + \frac{\Gamma\dot{\vec{P}} - \frac{\partial K}{\partial E}(E-\frac{g}{l},J)\,\frac{\vec{b}}{l} - \frac{\partial K}{\partial J}(E-\frac{g}{l},J)\,\vec{b}\wedge\dot{\vec{b}}}{a} \tag{14}$$

where Γ is any positive definite symmetric (constant) matrix satisfying, for some $\varepsilon > 0$,

$$(1-\varepsilon)\,\Gamma \;\geq\; I\,|\Phi| \tag{15}$$

where Φ is the matrix :

$$\Phi = \left[\frac{\partial^2 K}{\partial E\partial J}\,\frac{\vec{b}}{l}\right]\otimes\frac{\vec{b}\wedge\vec{g}}{l} + \left[\frac{\partial^2 K}{\partial E\partial J}\,(\vec{b}\wedge\dot{\vec{b}})\right]\otimes\frac{\dot{\vec{b}}}{l} \tag{16}$$
$$+\left[\frac{\partial^2 K}{\partial E^2}\,\frac{\vec{b}}{l}\right]\otimes\frac{\dot{\vec{b}}}{l} + \left[\frac{\partial^2 K}{\partial J^2}\,(\vec{b}\wedge\dot{\vec{b}})\right]\otimes\frac{\vec{b}\wedge\vec{g}}{l}$$
$$+\frac{\partial K}{\partial J}\,\frac{1}{l}\begin{pmatrix} 0 & -b_3 & b_2\\ b_3 & 0 & -b_1\\ -b_2 & b_1 & 0\end{pmatrix}$$

$\otimes$ denoting the usual tensor product. Such an inequality gives actually a constraint on the function K introduced above. It can be shown that, by picking :

$$K(E,J) \;=\; \log\left(1+\frac{l^2}{g^2}E^2 + c\frac{l}{2g^3}J^2\right) \tag{17}$$

where c is an adjusted strictly positive real number, the corresponding matrix Φ is bounded as requested in (15).

The motivation for (14) is actually that, by letting[3] :

$$\Lambda_1 \;=\; (\Phi+\Gamma)^{-1} \tag{18}$$
$$\vec{u} \;=\; -\Lambda_1\left(-\frac{\partial K}{\partial E}\,\frac{\dot{\vec{b}}}{l} - \frac{\partial K}{\partial J}\,\frac{\vec{b}\wedge\vec{g}}{l} + a\,\dot{\vec{P}}\right) + \vec{v} \tag{19}$$

[3]Note that the symmetric part of Λ_1 is made positive definite by (15).

where $\vec{v}$ is an intermediate control, we get very simply :

$$\dot{\vec{Q}} \;=\; -\frac{\Gamma}{a}\,\vec{v}\quad . \tag{20}$$

This means that the control (13), with the particular matrix Λ_1 given by (18), not only makes the set $\mathcal{S}_1$ globally stable in $\mathbb{R}^3\times\mathbb{S}^2\times T\mathbb{S}^2$ but makes also the set $\mathcal{S}$, given in (9), globally stable in $\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2\times T\mathbb{S}^2$. This is confirmed by considering for instance the Lyapunov function :

$$V_2(\vec{Q},\dot{\vec{P}},\vec{b},\dot{\vec{b}})$$
$$=\; V_1(\dot{\vec{P}},\vec{b},\dot{\vec{b}}) \;+\; \frac{1}{2}\,\vec{Q}.[\Pi\vec{Q}] \tag{21}$$
$$=\; K(E-\frac{g}{l},J) \;+\; \frac{a}{2}\,\dot{\vec{P}}.\dot{\vec{P}} \;+\; \frac{1}{2}\,\vec{Q}.[\Pi\vec{Q}] \tag{22}$$

which, when Π is any symmetric positive definite (constant) matrix, is radially unbounded on $\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2\times T\mathbb{S}^2$ and is zero on and only on the set $\mathcal{S}$. Indeed, it gives :

$$\dot{V}_2 \;= \tag{23}$$
$$-\left(-\frac{\partial K}{\partial E}\,\frac{\dot{\vec{b}}}{l} - \frac{\partial K}{\partial J}\,\frac{\vec{b}\wedge\vec{g}}{l} + a\dot{\vec{P}}\right)\,.$$
$$.\left[\Lambda_1\left(-\frac{\partial K}{\partial E}\,\frac{\dot{\vec{b}}}{l} - \frac{\partial K}{\partial J}\,\frac{\vec{b}\wedge\vec{g}}{l} + a\,\dot{\vec{P}}\right)\right]$$
$$+\left[\left(-\frac{\partial K}{\partial E}\,\frac{\dot{\vec{b}}}{l} - \frac{\partial K}{\partial J}\,\frac{\vec{b}\wedge\vec{g}}{l} + a\,\dot{\vec{P}}\right) - \frac{\Pi\Gamma}{a}\vec{Q}\right].\vec{v}\ .$$

This leads us to choose the intermediate control $\vec{v}$ as :

$$\vec{v} \;=\; -\Lambda_2\left[\left(-\frac{\partial K}{\partial E}\,\frac{\dot{\vec{b}}}{l} - \frac{\partial K}{\partial J}\,\frac{\vec{b}\wedge\vec{g}}{l} + a\,\dot{\vec{P}}\right) - \frac{\Pi\Gamma}{a}\vec{Q}\right] \tag{24}$$

where Λ_2 is is any matrix with positive definite symmetric part. This says that the control we have obtained to make the set $\mathcal{S}$ asymptotically stable is :

$$\vec{u} = -\,(\Phi+\Gamma)^{-1}\left[-\frac{\partial K}{\partial E}\,\frac{\dot{\vec{b}}}{l} - \frac{\partial K}{\partial J}\,\frac{\vec{b}\wedge\vec{g}}{l} + a\,\dot{\vec{P}}\right] \tag{25}$$
$$-\,\Lambda_2\left[\left(-\frac{\partial K}{\partial E}\,\frac{\dot{\vec{b}}}{l} - \frac{\partial K}{\partial J}\,\frac{\vec{b}\wedge\vec{g}}{l} + a\,\dot{\vec{P}}\right) - \frac{\Pi\Gamma}{a}\vec{Q}\right]$$

It contains several parameters : the function K, the real number a, the matrices Λ_2, Π and Γ. They have to be chosen in particular so that (15) holds.

With this choice for $\vec{u}$, we get that $\dot{V}_2$ is non positive globally and that it is zero if and only if :

$$-\frac{\partial K}{\partial E}\,\frac{\dot{\vec{b}}}{l} - \frac{\partial K}{\partial J}\,\frac{\vec{b}\wedge\vec{g}}{l} + a\,\dot{\vec{P}} \;=\; 0 \tag{26}$$
$$\vec{Q} \;=\; 0 \tag{27}$$

By applying LaSalle invariance principle, it is shown in Appendix B that all the solutions converge to one of the following set :

1. The singleton
$$(\vec{P}, \dot{\vec{P}}, \vec{b}, \dot{\vec{b}}) = \left(\frac{\frac{\partial K}{\partial E}(-2g/l, 0)\vec{g}}{la|g|}, 0, \frac{\vec{g}}{|g|}, 0 \right)$$
corresponding to the open loop stable downward equilibrium,
2. The singleton $(\vec{P}, \dot{\vec{P}}, \vec{b}, \dot{\vec{b}}) = (0, 0, -\frac{\vec{g}}{|g|}, 0)$ corresponding to the desired upward equilibrium,
3. The closure of the homoclinic orbits with $\vec{P} = \dot{\vec{P}} = 0$.

Only the last two sets give rise to points $(E, J, \dot{\vec{P}}, \vec{Q})$ which are stationary point of V_2. It follows that the first set is unstable. Actually, with the help of [5], a local linear analysis shows that the corresponding point can have a stable manifold of measure at most zero. Since the other two sets are contained in the closure of the homoclinic orbits, we can conclude that the desired $\mathcal{S}$ is made attractive for all points except may be those in a set of zero measure.

4 Implementation and results.

For the implementation of the control law, the environment provided with the robot arm $2k\pi$:

- gives us the overall state, reconstructed, from angle measurements, via an had hoc numeric differentiation and smoothing.
- computes the torques from the desired acceleration of the actuated end point.
- takes care of all the real time tasks.

This allows us to concentrate our attention only on the state feedback itself. This feedback is obtained by patching three controllers (see [1]) :

1. A linear controller stabilizing asymptotically an oscillatory trajectory which is an open loop solution. This solution evolves in the neighborhood of the downward equilibrium and is contained in a prescribed vertical plane,
2. The nonlinear controller described above,
3. A linear controller stabilizing asymptotically the upward equilibrium.

The switches between these controllers are dictated by criteria evaluating how well the task they are assigned is fulfilled.

In the implementation, the main difficulties we have encountered are :

- The friction which we have had to compensate partially in computing the torques from the desired acceleration of the actuated end point.
- The measurement noise coming mainly from the fact that the various speeds are reconstructed and not measured. This has led us to introduce filters, not on the primitive signals, but more specifically on some functions of them which are involved in our controllers.
- The possible instability of the vertical plane. This occurs typically when the energy and kinetic momentum are closed to their desired values (i.e. $\vec{u} \approx 0$) and the energy is positive. This has led us to decompose the swing in two steps :
    1. A first step where the energy is assigned to a value close to 0 (not high enough) and such that the vertical plane is stable.
    2. A second step where the energy is assigned the nominal value.
- The fact that, if the non linear controller does not achieve his task very properly, the subsequent linear controller may ask for too strong controls or create too large amplitudes of the actuated end point.

Nevertheless all these difficulties have been overcome and a result can be seen on the video which can be downloaded from :
`http://www.stud.enst.fr/~albouy/robot/demo.avi`
or
`http://cas.ensmp.fr/~praly/Publications/XavierX96_1.avi`

5 Acknowledgements.

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A Expression of $\vec{Q}$

Following [7], we look for a new coordinate $\vec{Q}$ to be used in place of $\vec{P}$. This coordinate should be such that, when the control is given by (13), the derivative $\dot{\vec{Q}}$ is zero. More specifically, let us look for $\vec{Q}$ in the form :

$$\vec{Q} \;=\; \vec{P} \;-\; \mathcal{M}\left(\dot{\vec{P}}, \vec{b}, \dot{\vec{b}}\right) \tag{28}$$

where $\mathcal{M}$ is a C^1 function. The constraint

$$\dot{\vec{Q}} \;=\; 0 \tag{29}$$

translates into :

$$\dot{\overbrace{\mathcal{M}\left(\dot{\vec{P}}, \vec{b}, \dot{\vec{b}}\right)}} \;=\; \dot{\vec{P}} \; . \tag{30}$$

This says that we need to express $\dot{\vec{P}}$ as a derivative of a function of $(\dot{\vec{P}}, \vec{b}, \dot{\vec{b}})$. In order to find this expression, let us recall the data :

– Dynamics :

$$\left\{ \begin{array}{rcl} \ddot{\vec{P}} & = & \vec{u} \\ \frac{\vec{b}\wedge\vec{g}}{l} & = & \vec{b}\wedge\ddot{\vec{b}} + \frac{\vec{b}\wedge\vec{u}}{l} \end{array} \right. \tag{31}$$

– Control law :

$$\Lambda_1^{-1}\vec{u} \;=\; \frac{\partial K}{\partial E}\,\frac{\dot{\vec{b}}}{l} \;+\; \frac{\partial K}{\partial J}\,\frac{\vec{b}\wedge\vec{g}}{l} \;-\; a\,\dot{\vec{P}} \tag{32}$$

Then, since K is a function of E and J only and we have (7), we obtain :

$$\left\{ \begin{array}{rcl} \frac{\partial K}{\partial E}\,\frac{\dot{\vec{b}}}{l} & = & \dot{\overbrace{\frac{\partial K}{\partial E}\,\frac{\vec{b}}{l}}} \;+\; \Phi_1 u \\ \frac{\partial K}{\partial J}\,\frac{\vec{b}\wedge\vec{g}}{l} & = & \dot{\overbrace{\frac{\partial K}{\partial J}\,\vec{b}\wedge\dot{\vec{b}}}} \;+\; \Phi_2 u \end{array} \right. \tag{33}$$

where Φ_1 and Φ_2 are some functions of $(\dot{\vec{P}}, \vec{b}, \dot{\vec{b}})$. Using these relations in the expression of the control law yields :

$$\left(\Lambda_1^{-1} - \Phi_1 - \Phi_2\right)\ddot{\vec{P}} \;=\; \dot{\overbrace{\left(\frac{\partial K}{\partial E}\,\frac{\vec{b}}{l} + \frac{\partial K}{\partial J}\,\vec{b}\wedge\dot{\vec{b}} - a\,\vec{P}\right)}} \tag{34}$$

But, by choosing Λ_1 as (see (18)) :

$$\Lambda_1 \;=\; \left(\Phi_1 + \Phi_2 + \Gamma\right)^{-1} \;, \tag{35}$$

where Γ is a constant matrix, we get our result in the form (see(14)) :

$$-\Gamma\,\dot{\vec{P}} \;+\; \dot{\overbrace{\left(\frac{\partial K}{\partial E}\,\frac{\vec{b}}{l} + \frac{\partial K}{\partial J}\,\vec{b}\wedge\dot{\vec{b}}\right)}} \;=\; a\,\dot{\vec{P}} \; . \tag{36}$$

B The closed loop limit sets

We study here the bounded solutions of (1), (26) and (27) with $\vec{u} = 0$.

With $\vec{u} = 0$, we know that E, J and therefore K, $\frac{\partial K}{\partial E}$ and $\frac{\partial K}{\partial J}$ are constant. Then the time derivative of (26) gives :

$$\frac{\partial K}{\partial E}\,\ddot{\vec{b}} \;+\; \frac{\partial K}{\partial J}\,\dot{\vec{b}}\wedge\vec{g} \;=\; 0 \tag{37}$$

or, with (1),

$$\frac{\partial K}{\partial E}\left[\frac{\vec{g}}{l} - \left(\frac{\vec{g}}{l}\,.\,\vec{b} + \dot{\vec{b}}\,.\,\dot{\vec{b}}\right)\vec{b}\right] \;+\; \frac{\partial K}{\partial J}\,\dot{\vec{b}}\wedge\vec{g} \;=\; 0 \tag{38}$$

By multiplying by $\dot{\vec{b}}$, this implies $\frac{\partial K}{\partial E}\; g\,.\,\dot{\vec{b}} = 0$. So :

1. either the constant $\frac{\partial K}{\partial E}$ is zero. In this case (37) gives :

   (a) either the constant $\frac{\partial K}{\partial J}$ is zero. But then $E = \frac{g}{l}$, $J = 0$ since this point is the only stationary point of K. This says that, for the solutions we are studying, $\vec{b}$ evolves in the closure of the homoclinic orbits. Also, from (14), (26) and (27), we get $\vec{P} = \dot{\vec{P}} = 0$.

   (b) or $\dot{\vec{b}}\wedge\vec{g}$ is constant and zero. In this case $\vec{b}\wedge\vec{g}$ is constant and so is $\dot{\vec{P}}$ from (26). But since the solutions under investigation are bounded, $\dot{\vec{P}}$ cannot be constant without being zero. This in its turn implies that $\vec{b}\wedge\vec{g}$ is zero, i.e. $\vec{b}$ is colinear with $\vec{g}$. But $\vec{b}$ being of unit norm, we must have $\vec{b} = \pm\frac{\vec{g}}{g}$ and $\dot{\vec{b}} = 0$. This implies $E = \mp\, g/l$ and $J = 0$. And from (14) and (27), we get $\vec{P} = 0$.

2. or $g\,.\,\dot{\vec{b}} = 0$. In this case, since $\dot{\vec{b}}$ is also orthogonal to $\vec{b}$, $\dot{\vec{b}}$ is colinear with $\vec{b}\wedge\vec{g}$, i.e.

   $$\dot{\vec{b}} \;=\; \lambda\,\vec{b}\wedge\vec{g} \tag{39}$$

   Then (37) yields :

   $$\frac{\partial K}{\partial E}\left(\dot{\lambda}\,\vec{b}\wedge\vec{g} + \lambda\,\dot{\vec{b}}\wedge\vec{g}\right) \;=\; -\frac{\partial K}{\partial J}\,\dot{\vec{b}}\wedge\vec{g} \tag{40}$$

   With the identity :

   $$\vec{a}\wedge\left(\vec{b}\wedge\vec{c}\right) \;=\; (\vec{a}\,.\,\vec{c})\,\vec{b} \;-\; \left(\vec{a}\,.\,\vec{b}\right)\vec{c} \tag{41}$$

   we obtain :

   $$\frac{\partial K}{\partial E}\,\dot{\lambda}\,\vec{b}\wedge\vec{g} = \left(\frac{\partial K}{\partial E}\lambda + \frac{\partial K}{\partial J}\right)\lambda\left[g^2\,\vec{b} - (\vec{g}\,.\,\vec{b})\vec{g}\right] \; . \tag{42}$$

But $\vec{b}\wedge\vec{g}$ being orthogonal to $\vec{b}$ and $\vec{g}$, the terms on each side must be zero. So we get in particular :

$$\left(\frac{\partial K}{\partial E}\lambda + \frac{\partial K}{\partial J}\right)\lambda\left[g^2\,\vec{b} - (\vec{g}\,.\,\vec{b})\vec{g}\right] \;=\; 0\;. \qquad (43)$$

Since the cases $\frac{\partial K}{\partial E}=0$, $\dot{\vec{b}}\wedge\vec{g}=0$ (implied by $\lambda = 0$) and $\vec{b}\wedge\vec{g}=0$ (implied by $\vec{b}$ and $\vec{g}$ colinear) have been studied already, we consider the following implication of this equation :

$$\lambda \;=\; -\frac{\frac{\partial K}{\partial J}}{\frac{\partial K}{\partial E}} \qquad (44)$$

With this expression, we get $\dot{\vec{P}} = 0$ from (26). Hence $\vec{P}$ is constant and, with (14) and (27), we have :

$$\begin{aligned} \texttt{const.} \;&=\; \frac{\partial K}{\partial E}\,\frac{g\wedge\vec{b}}{l} \;+\; \frac{\partial K}{\partial J}\,g\wedge(\vec{b}\wedge\dot{\vec{b}}) && (45)\\ &=\; -\left[\frac{\frac{\partial K}{\partial E}}{l\lambda} + \frac{\partial K}{\partial J}\,g\,.\,\vec{b}\right]\dot{\vec{b}} && (46) \end{aligned}$$

where we have used $g\,.\,\dot{\vec{b}} = 0$. (46) says that $\dot{\vec{b}}$ is constant, i.e. $\ddot{\vec{b}} = 0$. From (1) we get that $\vec{b}$ and $\vec{g}$ are colinear and conclude as in the case 1b above.

References

[1] X. Albouy, “Stabilisation d’un pendule sphérique inversé : Cas du robot équilibriste $2k\pi$,” *Rapport de Stage d’option scientifique*, École Polytechnique, Palaiseau, France, Juillet 1999.

[2] A. Bloch, N. Leonard, J. Marsden, Matching and stabilization by the method of controlled Lagrangians. *Proceedings of the 37th IEEE Conference on Decision and Control*, December 1998

[3] C.C. Chung, J. Hauser, “Nonlinear Control of a Swinging Pendulum,” *Automatica*, Vol. 31, No. 6, pp. 851–862, 1995.

[4] D. Jacobson, *Extensions of linear quadratic control, optimization and matrix theory*, Academic Press, New York 1977.

[5] M. Krstić, R. Sepulchre, L. Praly, On the measure of global invariant manifold for fixed points. *Technical report CCEC 95-0228*, University of California, Santa Barbara, February 1995

[6] Y. Lenoir, P. Martin, P. Rouchon, $2k\pi$, the juggling robot. *Proceedings of the 37th IEEE Conference on Decision and Control*, December 1998

[7] F. Mazenc, L. Praly, “Adding integrations, saturated controls and global asymptotic stabilization for feedforward systems,” *IEEE Transactions on Automatic Control*, Vol.41, N. 11, November 1996.

[8] M.W. Spong, L. Praly, “Control of underactuated mechanical systems using switching and saturation,” *Proceedings of the Block Island Workshop on Control Using Logic Based Switching*, Springer-Verlag, 1996

[9] M. Wiklund, A. Kristenson,, K.-J. Astrom, A New Strategy for Swinging up an Inverted Pendulum, *Proc. IFAC Symposium*, Sydney, Australia, 1993.