Dynamic UCO Controllers and Semiglobal Stabilization of Uncertain Nonminimum Phase Systems by Output Feedback

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1 Introduction

One of the most active research issues in nonlinear feedback theory is the synthesis of feedback laws which robustly stabilize an uncertain system with limited measurement information. In the case of output feedback without uncertainty, one of the major achievements in this area of research has been the "nonlinear separation principle" proved in [6], where it is shown that (semi)global stabilizability via state feedback and a property of uniform observability imply the possibility of semiglobal stabilization via output feedback. To cope with the restricted information structure, the stabilization of [6] includes an approximate state observer (whose role is actually that of producing approximate estimates of a number of "higher order" derivatives of the output) earlier developed in [3] to cope with a similar (though more restricted) stabilization problem. A "robust" version of this stabilization result was given in [5], where it was shown that, in the presence of parameter uncertainties, semiglobal stabilization via output feedback is still possible if a state feedback law is known which robustly globally stabilizes the system and its value, at any time, can be expressed as a (fixed) function of the values, at this time, of a fixed number of derivatives of input and output (a *uniformly completely observable* (UCO) state feedback, in the terminology of [5]).

The design tools introduced in [3] and [5] have been recently used in [2], where a new (iterative) procedure has been proposed for the robust stabilization of certain classes of nonlinear systems. This procedure is not based on the idea of solving separately a problem of state feedback stabilization and a problem of asymptotic state reconstruction. Rather, it is based on the recursive update of a sequence of "dynamic" output feedback stabilizers: specifically, the basic result of [2] is that if a suitable subsystem of lower dimension is robustly stabilizable by dynamic output feedback, so is the entire system. From the point of view of the approach of [5], the condition on which the result of [2] relies (that happens to be necessary in the case of linear systems) can be viewed as a condition for the existence of a *dynamic* feedback driven by functions that are expressible in terms of the output and its derivatives, i.e., driven by UCO functions.

In this chapter we review and extend the result of [2] and we show how this result can also be obtained as a special case of a general stabilization result based on the existence of a *dynamic* feedback driven by UCO functions. More specifically, after some preliminary definitions in Section 2 including our definition of uniform semiglobal practical asymptotic stability, we discuss stabilization of nonminimum phase nonlinear systems by output feedback in Section 3. This discussion is split into two parts: the relative degree one case in Section 3.1, and the higher relative degree case in Section 3.2. The main results of these sections are that if a reduced order, auxiliary system can be stabilized by dynamic output feedback then the original nonminimum phase system can be stabilized by dynamic output feedback. In Section 4 we show how the results of Section 3 can be viewed as special cases of a general result on semiglobal practical asymptotic stabilization by output feedback. In Section 4.1 we present some additional definitions, including the notions of *uniformly completely observable* (UCO) functions and uniform semiglobal practical asymptotic *stabilizability* by dynamic UCO feedback, and a general output feedback stabilization result which expands on the ideas in [5]. This result is specialized to the case of nonminimum phase nonlinear systems in Section 4.2. In this section, we compare and contrast the controllers developed in Section 3 explicitly for the nonminimum phase nonlinear system case to the controllers that result from following the synthesis steps presented in [5].

2 Preliminaries

- For simplicity all nonlinear functions in this chapter will be assumed to be sufficiently smooth so that all needed derivatives exist and are continuous, all differential equations have solutions, etc.

- We will use $\overline{\mathcal{B}}_n(r)$, with $r > 0$, to denote a closed ball of radius r in $\mathbb{R}^n$.

- Unless otherwise noted, $\mu(t)$ is a measurable function taking values in a compact set $\mathcal{P} \subset \mathbb{R}^p$. The set of such functions is denoted $\mathcal{M}_\mathcal{P}$.

- The origin of a nonlinear dynamical system

$$\dot{x} = f(x, \mu(t), k) \; , \tag{8.1}$$

with $x \in \mathbb{R}^n$ and $k \in \mathbb{R}^c$, is said to be *uniformly semiglobally practically asymptotically stable in the parameter* k if

for each pair of strictly positive real numbers $0 < r < R < \infty$ there exist $\bar{k} \in \mathbb{R}^c$, an open set $\mathcal{O} \supset \overline{\mathcal{B}}_n(R)$, a function $V : \mathcal{O} \to \mathbb{R}_{\geq 0}$ that is proper on $\mathcal{O}$ and strictly positive real numbers $0 < q < Q < \infty$ such that

i.) $\overline{\mathcal{B}}_n(R) \subset \{\xi \in \mathcal{O} : V(\xi) \leq Q\}$,

ii.) $\overline{\mathcal{B}}_n(r) \supset \{\xi \in \mathcal{O} : V(\xi) \leq q\}$,

iii.) and

$$\frac{\partial V}{\partial x} f(x, \mu, \bar{k}) < 0 \quad \forall \mu \in \mathcal{P}\,, \forall x \in \{\xi \in \mathcal{O} : q \leq V(\xi) \leq Q\}\,.$$

Uniform semiglobal practical asymptotic stability implies:

> for each pair of strictly positive real numbers $0 < r < R < \infty$, there exist $\bar{k} \in \mathbb{R}^c$ and $T > 0$ such that, for all initial conditions in $\overline{\mathcal{B}}_n(R)$, all resulting trajectories $x(t)$ of (8.1) with $k = \bar{k}$ are such that $x(t) \in \overline{\mathcal{B}}_n(r)$ for all $t \geq T$.

It also can be shown to imply:

> for each pair of strictly positive real numbers $0 < r < R < \infty$, there exist $\bar{k} \in \mathbb{R}^c$, a compact set $\mathcal{A} \subseteq \overline{\mathcal{B}}_n(r)$ and an open set $\mathcal{G} \supset \overline{\mathcal{B}}_n(R)$ such that, for the system (8.1) with $k = \bar{k}$, the set $\mathcal{A}$ is uniformly asymptotically stable with basin of attraction $\mathcal{G}$.
>
> By this we mean:
>
> - for each $\epsilon > 0$ there exists $\delta > 0$ such that all trajectories starting in a δ-neighborhood of $\mathcal{A}$ remain in an ϵ-neighborhood of $\mathcal{A}$ for all time, and
> - for each $\epsilon > 0$ and each compact subset of $\mathcal{G}$ there exists $T > 0$ such that all trajectories starting in the compact subset enter within T seconds and remain thereafter in an ϵ-neighborhood of $\mathcal{A}$.

In fact, due to recent converse Lyapunov function results (see [4], [1], [7]), these latter properties are equivalent characterizations of uniform semiglobal practical asymptotic stability. However, we are using the Lyapunov formulation here so that we can more directly appeal to the results on semiglobal practical asymptotic stabilization like [5, Proposition 3.1] where a Lyapunov formulation was used.

3 Stabilization of Nonminimum Phase Systems by Output Feedback

3.1 The Relative Degree One Case

Most methods for robust stabilization of a nonlinear system by relative degree one output feedback rely on the hypothesis that the system has an asymptotically stable zero dynamics. The main reason why this hypothesis is assumed is that most of the methods in question use "high-gain" feedback in order to keep the output small, thereby enforcing a behavior whose asymptotic properties are essentially determined by the asymptotic properties of the zero dynamics. In particular, asymptotic stabilization occurs only if the latter is asymptotically stable, i.e., if the system is minimum phase. Consider robust (with respect to disturbances $\mu(t)$) stabilization of the origin for the system

$$\begin{aligned} \dot{z} &= f_0(z, y, \mu(t)) \\ \dot{y} &= q(z, y, \mu(t)) + b(y)u \end{aligned} \tag{8.2}$$

where $z \in I\!R^{n-1}$, $y \in I\!R$, $u \in I\!R$, $\mu(\cdot) \in \mathcal{M}_{\mathcal{P}}$ and $b(y) \neq 0$ for all y. In the case of uniformly globally asymptotically stable zero dynamics, i.e. (see [4]) when there exists a smooth, positive definite and proper function $V(z)$ such that

$$\frac{\partial V}{\partial z} f_0(z, 0, \mu) < 0 \qquad \forall z \neq 0 \; , \quad \forall \mu \in \mathcal{P} \; ,$$

the control law

$$u = -\frac{1}{b(y)} \, k y \; ,$$

where k is a sufficiently large number, solves the problem of semiglobal practical asymptotic stabilization of the origin. This follows from the fact that, given a compact set in (z, y) not containing the origin, for large enough k the negative definite term $\frac{\partial V}{\partial z} f_0(z, 0, \mu) - ky^2$ in the derivative of the composite Lyapunov function

$$U(z, y) = V(z) + y^2 \; ,$$

i.e., in

$$\frac{\partial V}{\partial z} f_0(z, y, \mu) + 2y[q(z, y, \mu) - ky] \; ,$$

is able to dominate all nonnegative terms on the given compact set.

In the case where the original output does not yield an asymptotically stable zero dynamics, one approach is to look for a new output function, of the form $y - y^*(z)$, for which the resulting system is uniformly minimum phase. Then, by following the reasoning above, the control

$$u = -\frac{1}{b(y)}\, k\,(y - y^*(z))$$

may be used to achieve robust semiglobal practical stabilization of the origin. The potential drawback to this approach is that it requires the measurement, or at least the robust observability via the actual measured output y and the input u, of the term $y^*(z)$.

Looking at the structure of the system (8.2), we see that the main information about the z subsystem that is robustly observable through the measurement y and the input u is the term $q(z, y, \mu(t))$ and perhaps its derivatives. The discussion that follows, in this and the next subsection, describes one very efficient way, suggested in [2], to use the information contained in $q(z, y, \mu(t))$ to design a stabilizing feedback law without actually requiring a measurement of $q(z, y, \mu(t))$. We will suppose

Assumption 8.1 *For the* auxiliary system

$$\begin{aligned} \dot{z} &= f_0(z, \bar{u}, \mu(t)) \\ \bar{y} &= q(z, \bar{u}, \mu(t))\ , \end{aligned} \tag{8.3}$$

the controller

$$\begin{aligned} \dot{\varphi} &= L(\varphi) + M\bar{y} \\ \bar{u} &= N(\varphi)\ , \end{aligned} \tag{8.4}$$

with $N(0) = 0$, is such that the origin of the system (8.3),(8.4) is uniformly globally asymptotically stable.

Under this assumption, we can state the following result for the system (8.2) under the action of the controller

$$\begin{aligned} \dot{\varphi} &= L(\varphi) + Mk[y - N(\varphi)] \\ u &= \frac{1}{b(y)}\Big[\frac{\partial N}{\partial \varphi}[L(\varphi) + Mk[y - N(\varphi)]] - k[y - N(\varphi)]\Big]\ . \end{aligned} \tag{8.5}$$

Note that this is simply a dynamic feedback of the original (nonminimum phase) output y.

Theorem 8.2 *Under Assumption 8.1, the origin of the system (8.2), (8.5) is uniformly semiglobally practically asymptotically stable in the control parameter k.*

Proof. The result is established by noting, with the help of the input transformation

$$u = \frac{1}{b(y)}\Big[\frac{\partial N}{\partial \varphi}L(\varphi) + (1 - \frac{\partial N}{\partial \varphi}M)v\Big]\ , \tag{8.6}$$

that the system

$$\begin{aligned} \dot{z} &= f_0(z, y, \mu(t)) \\ \dot{\varphi} &= L(\varphi) - Mv \\ \dot{y} &= q(z, y, \mu(t)) + \frac{\partial N}{\partial \varphi} L(\varphi) + \left(1 - \frac{\partial N}{\partial \varphi} M\right) v \end{aligned} \tag{8.7}$$

with output $\theta = y - N(\varphi)$ has relative degree one with high-frequency gain identically equal to one, is minimum phase and can be written, globally, in a form that matches (8.2). Specifically, in the coordinates (z, ξ, θ) where $\xi := \varphi + M\theta$, we have:

$$\begin{aligned} \dot{z} &= f_0(z, N(\xi - M\theta) + \theta, \mu(t)) \\ \dot{\xi} &= L(\xi - M\theta) + Mq(z, N(\xi - M\theta) + \theta, \mu(t)) \\ \dot{\theta} &= q(z, N(\xi - M\theta) + \theta, \mu(t)) + v\ . \end{aligned} \tag{8.8}$$

By Assumption 8.1, when θ is set to zero, the origin of the (z, ξ) dynamics is uniformly globally asymptotically stable. It follows from the discussion above that the choice $v = -k\theta$ is semiglobally practically stabilizing for the origin of (8.8). And, since $N(0) = 0$, the origin of (8.8) corresponds to the origin of (8.2),(8.5). Moreover, with this choice for v we see from (8.6) and the $\dot{\varphi}$ equation in (8.7) that we recover the control law (8.5). △

Remark 8.1 If a controller of a form more general than (8.4) like

$$\begin{aligned} \dot{\widetilde{\varphi}} &= \widetilde{L}(\widetilde{\varphi}, \bar{y}) \\ \bar{u} &= \widetilde{N}(\widetilde{\varphi}, \bar{y}) \end{aligned}$$

exists (in the case where $\bar{y}$ depends on $\bar{u}$ we would need an assumption that guarantees a solution $\bar{u}$ to the second equation), a controller of the form (8.4) can be obtained by dynamic extension as

$$\begin{aligned} \dot{\widetilde{\varphi}} &= \widetilde{L}(\widetilde{\varphi}, \widetilde{\varphi}_{\nu+1}) \\ \dot{\widetilde{\varphi}}_{\nu+1} &= -m(\widetilde{\varphi}_{\nu+1} - \bar{y}) \\ \bar{u} &= \widetilde{N}(\widetilde{\varphi}, \widetilde{\varphi}_{\nu+1}) \end{aligned}$$

with m a positive number. Instead of achieving uniform *global* asymptotic stability for the auxiliary system, this controller would, in general, achieve uniform semiglobal practical asymptotic stability in the parameter m, at least in the case where the functions $\mu(t)$ are restricted to have uniformly bounded derivatives. While this would complicate the above discussion, the conclusion of the theorem would still be the same. △

Remark 8.2 As discussed in [5], various local conditions can be imposed on the system (8.8) to guarantee uniform semiglobal asymptotic stability, as opposed to only uniform semiglobal *practical* asymptotic stability. △

3.2 The Relative Degree Greater than One Case

The result of the previous section, on stabilization by dynamic output feedback, can be extended to the case of outputs with relative degree greater than one.

Consider a nonlinear system modeled by equations of the form

$$\begin{aligned} \dot{z} &= f(z, \zeta_1, \ldots, \zeta_r, \mu(t)) \\ \dot{\zeta}_1 &= \zeta_2 \\ \dot{\zeta}_2 &= \zeta_3 \\ &\vdots \\ \dot{\zeta}_r &= q(z, \zeta_1, \ldots, \zeta_r, \mu(t)) + b(\zeta)u \\ y &= \zeta_1 \end{aligned} \tag{8.9}$$

in which $z \in I\!R^{n-r}$, $\mu(\cdot) \in \mathcal{M}_{\mathcal{P}}$ and $b(\zeta) \neq 0$ for all ζ. This normal form may result from applying a globally defined, perhaps μ dependent, coordinate transformation to a nonlinear system given in some other form.

The only measurement that we will assume is available is the output y. What we will show is that if a particular reduced system can be stabilized with measurements of ζ and $q(z, \zeta_1, \ldots, \zeta_r, \mu(t))$ then the system (8.9) can be stabilized with measurement of y only.

With the system (8.9), we associate an *auxiliary system*

$$\begin{aligned} \dot{x}_{\mathrm{a}} &= f_{\mathrm{a}}(x_{\mathrm{a}}, u_{\mathrm{a}}, \mu(t)) \\ y_{\mathrm{a}} &= h_{\mathrm{a}}(x_{\mathrm{a}}, u_{\mathrm{a}}, \mu(t)) \end{aligned} \tag{8.10}$$

in which

$$x_{\mathrm{a}} = \begin{pmatrix} x_{\mathrm{a},1} \\ \hline x_{\mathrm{a},2} \end{pmatrix} := \begin{pmatrix} z \\ \hline \zeta_1 \\ \vdots \\ \zeta_{r-2} \\ \zeta_{r-1} \end{pmatrix},$$

and

$$f_{\mathrm{a}}(x_{\mathrm{a}}, u_{\mathrm{a}}, \mu(t)) = \begin{pmatrix} f_{\mathrm{a},1}(x_{\mathrm{a}}, u_{\mathrm{a}}) \\ \hline f_{\mathrm{a},2}(x_{\mathrm{a},2}, u_{\mathrm{a}}) \end{pmatrix} := \begin{pmatrix} f(z, \zeta_1, \ldots, \zeta_{r-1}, u_{\mathrm{a}}, \mu(t)) \\ \hline \zeta_2 \\ \vdots \\ \zeta_{r-1} \\ u_{\mathrm{a}} \end{pmatrix},$$

and

$$h_{\rm a}(x_{\rm a}, u_{\rm a}, \mu) := q(z, \zeta_1, \dots, \zeta_{r-1}, u_{\rm a}, \mu(t)) \,.$$

About this system, we assume the following:

Assumption 8.3 *The controller*

$$\begin{array}{rcl} \dot\varphi &=& L(\varphi, x_{\rm a,2}) + M y_{\rm a} \\ u_{\rm a} &=& N(\varphi, x_{\rm a,2}) \,, \end{array} \tag{8.11}$$

with $N(0,0) = 0$, *is such that the origin of the system (8.10),(8.11) is uniformly globally asymptotically stable.*

Under this assumption, we can state the following result for the system (8.9) under the action of the controller

$$\begin{array}{rcl} \dot\varphi &=& L(\varphi, x_{\rm a,2}) + Mk[\zeta_r - N(\varphi, x_{\rm a,2})] \\ u &=& \dfrac{1}{b(\zeta)}\Big[\dfrac{\partial N}{\partial \varphi}[L(\varphi, x_{\rm a,2}) + Mk[\zeta_r - N(\varphi, x_{\rm a,2})]] + \\ && \qquad \dfrac{\partial N}{\partial x_{\rm a,2}} f_{\rm a,2}(x_{\rm a,2}, \zeta_r) - k[\zeta_r - N(\varphi, x_{\rm a,2})]\Big] \,. \end{array} \tag{8.12}$$

Note that this is a dynamic feedback of the output y and its first $r-1$ derivatives.

Lemma 8.1 *Under Assumption 8.3, the origin of the system (8.9), (8.12) is uniformly semiglobally practically asymptotically stable in the control parameter* k.

Proof. The proof is the same as the proof of Theorem 8.2. With the input transformation

$$u = \frac{1}{b(\zeta)}\Big[\frac{\partial N}{\partial \varphi} L(\varphi, x_{\rm a,2}) + \frac{\partial N}{\partial x_{\rm a,2}} f_{\rm a,2}(x_{\rm a,2}, \zeta_r) + \left(1 - \frac{\partial N}{\partial \varphi} M\right) v\Big] \tag{8.13}$$

we get the system

$$\begin{array}{rcl} \dot x_{\rm a} &=& f_{\rm a}(x_{\rm a}, \zeta_r, \mu(t)) \\ \dot\varphi &=& L(\varphi, x_{\rm a,2}) - Mv \\ \dot\zeta_r &=& h_{\rm a}(x_{\rm a}, \zeta_r, \mu(t)) + \\ && \dfrac{\partial N}{\partial \varphi} L(\varphi, x_{\rm a,2}) + \dfrac{\partial N}{\partial x_{\rm a,2}} f_{\rm a,2}(x_{\rm a,2}, \zeta_r) + \left(1 - \dfrac{\partial N}{\partial \varphi} M\right) v \end{array} \tag{8.14}$$

that, with output $\theta = \zeta_r - N(\varphi, x_{\rm a,2})$, has relative degree one with high-frequency gain identically equal to one, is minimum phase and can be written, globally, in a form that matches (8.2). Specifically, in the coordinates

(x_a, ξ, θ) where $\xi := \varphi + M\theta$, we have:

$$\begin{aligned}
\dot{x}_{\mathrm{a}} &= f_{\mathrm{a}}\Big(x_a, N(\xi - M\theta, x_{\mathrm{a},2}) + \theta, \mu(t)\Big) \\
\dot{\xi} &= L(\xi - M\theta, x_{\mathrm{a},2}) + M h_{\mathrm{a}}\Big(x_{\mathrm{a}}, N(\xi - M\theta, x_{\mathrm{a},2}) + \theta, \mu(t)\Big) \\
\dot{\theta} &= h_{\mathrm{a}}\Big(x_{\mathrm{a}}, N(\xi - M\theta, x_{\mathrm{a},2}) + \theta, \mu(t)\Big) + v \ .
\end{aligned} \tag{8.15}$$

By Assumption 8.3, when θ is set to zero, the origin of the (x_{a}, ξ) dynamics is uniformly globally asymptotically stable. It follows, as before, that the choice $v = -k\theta$ is semiglobally practically stabilizing for the origin of (8.15). And since $N(0,0) = 0$, the origin of (8.15) corresponds to the origin of (8.9),(8.12). Moreover, with this choice for v we see from (8.13) and the $\dot{\varphi}$ equation in (8.14) that we recover the control law (8.12). △

The dynamic controller (8.12) uses the state variables $\zeta_1, \dots, \zeta_r$, i.e., the derivatives up to order $r-1$ of the output y of system (8.9), as input. Thus, in order to find an output feedback controller, these variables must be replaced by appropriate estimates, which can be provided by a dynamical system of the form

$$\dot{\eta} = P\eta + Qy \tag{8.16}$$

in which the matrices Q and P have the form

$$P = \begin{pmatrix} -gc_{r-1} & 1 & 0 & \cdots & 0 \\ -g^2 c_{r-2} & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ -g^{r-1}c_1 & 0 & 0 & \cdots & 1 \\ -g^r c_0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad Q = \begin{pmatrix} -gc_{r-1} \\ -g^2 c_{r-2} \\ \cdot \\ -g^{r-1}c_1 \\ -g^r c_0 \end{pmatrix}. \tag{8.17}$$

As shown in [3], it is convenient to saturate the resulting control law, at least where the estimates of ζ appear, so as to avoid the occurrence of finite escape times for large values of g. For example, we can replace the controller (8.12), which for ease of notation we now write as

$$\begin{aligned} \dot{\varphi} &= C(\varphi, \zeta) \\ u &= K(\varphi, \zeta) \ , \end{aligned} \tag{8.18}$$

with the controller

$$\begin{aligned} \dot{\varphi} &= \sigma_\ell\left(C(\varphi, \eta)\right) \\ u &= \sigma_\ell\left(K(\varphi, \eta)\right) \end{aligned} \tag{8.19}$$

where $\sigma_\ell(\cdot)$ is a (by abuse of notation both a scalar and vector) saturation function

$$\sigma_\ell(v) = v \cdot \min\left\{1, \frac{\ell}{|v|}\right\}.$$

A controller of this type is able to robustly semiglobally practically asymptotically stabilize the plant (8.9). In fact, using the methods of [5] for example, it is possible prove the following result.

Theorem 8.4 *(See also [2]) Under Assumption 8.3, the origin of the system (8.9), (8.16), (8.19) [with $C(\cdot,\cdot)$ and $K(\cdot,\cdot)$ defined by the identification between (8.12) and (8.18)] is uniformly semiglobally practically stable in the control parameters (k, g, ℓ).*

4 On Dynamic UCO Feedback

The basic observation of [2], summarized in Section 3.2 and on which the result of Lemma 8.1 rests, is that the term $q(z, \zeta_1, \dots, \zeta_{r-1}, \zeta_r, \mu(t))$ in the system (8.9) can be (and, in a nonminimum phase system, has to be) "isolated" from the rest of the system, using measurements only of the output and its first $r-1$ derivatives, and treated as a separate source of information for feedback. Then, having a dynamic controller driven by the output its first $r-1$ derivatives, as in Lemma 8.1, it is straightforward using ideas initially developed in [3] to find a dynamic output feedback controller that induces the desired properties, as in Theorem 8.4.

From this point of view, the contribution in [2] is the identification of a natural (in fact, for linear systems it can be shown to be necessary) condition (Assumption 8.3) that guarantees the existence of a dynamic feedback that is expressible in terms of the output and its derivatives. Then Theorem 8.4 can be viewed as a special case of a more general result that is essentially contained in [5] (see [5, Proposition 3.1 and footnote 5]), namely that semiglobal practical stabilization by dynamic *uniformly completely observable* (UCO) feedback implies semiglobal practical stabilization by dynamic output feedback. We make this result explicit below.

4.1 General Results

Consider multi-input, multi-output nonlinear control systems

$$\begin{aligned} \dot{x} &= f(x, u, \mu(t)) \\ y &= h(x, u, \mu(t)) \end{aligned} \tag{8.20}$$

with $\mu(\cdot) \in \mathcal{M}_{\mathcal{P}}$. The definition of uniformly completely observable (UCO) dynamic feedback, given next, at times implicitly constrains $\mu(t)$ to be sufficiently smooth, where sufficiently smooth has to do with the number of times the output needs to be differentiated to reconstruct the UCO function.

Definition 8.1 *A function* $\varphi(x,u,\mu)$ *is said to be* uniformly completely observable *(UCO) with respect to the system (8.20) if it can be expressed as a function of a finite number of derivatives of the output y and the input u, i.e., if there exist two integers* n_y *and* n_u *and a function* Ψ *such that, for each solution of*

$$\begin{array}{rcl} \dot{x} & = & f(x,u,\mu(t)) \\ u^{(n_u+1)} & = & v \\ y & = & h(x,u,\mu(t)) \end{array} \tag{8.21}$$

we have, for all t where the solution makes sense,

$$\varphi(x(t),u(t),\mu(t)) = \Psi\left(y(t),\dots,y^{(n_y)}(t),u(t),\dots,u^{(n_u)}(t)\right) \tag{8.22}$$

where $y^{(i)}$ *denotes the ith time derivative of y at time t (and similarly for* $u^{(i)}$*).*

Remark 8.3 As in [5, Footnote 6], note the strong requirement that Ψ is independent of $\mu(t)$. On the other hand, note that the functions

$$\zeta_i\ , \qquad q(\zeta_1,\dots,\zeta_r,\mu)$$

for the system (8.9) are UCO since we can write

$$\zeta_i = y^{(i-1)}\ , \qquad q(\zeta_1,\dots,\zeta_r,\mu(t)) = y^{(r)} - b(y)u\ .$$

△

Our next definitions, on uniform semiglobal practical asymptotic *stabilizability* by dynamic UCO or output feedback, are closely related to our definition of uniform semiglobal practical asymptotic stability. However, as was the case in [5], we don't insist that the states of the dynamic compensator eventually become small in the closed-loop. We formulate the definition in Lyapunov function terms but, again, the definition could be formulated in terms of trajectories.

Definition 8.2 *The origin of (8.20) is said to be* uniformly semiglobally practically asymptotically stabilizable by dynamic UCO feedback *if for each pair of strictly positive real numbers* $0 < r < R < \infty$ *there exist:*

- *a UCO function* $\alpha(x,u,\mu)$
- *functions* θ *and* κ,
- *compact sets* $C_{\eta s}$ *and* $C_{\eta l}$, *with* $C_{\eta s}$ *a subset of the interior of* $C_{\eta l}$,
- *an open set* $\mathcal{O} \supset \overline{B}_n(R) \times C_{\eta l}$,
- *a function* $V : \mathcal{O} \to I\!R_{\geq 0}$ *that is proper on* $\mathcal{O}$, *and*

- *strictly positive real numbers* $0 < q < Q < \infty$

such that

i.) $(\overline{\mathcal{B}}_n(R) \times \mathcal{C}_{\eta l}) \subset \{\xi \in \mathcal{O} : V(\xi) \leq Q\}$,

ii.) $(\overline{\mathcal{B}}_n(r) \times \mathcal{C}_{\eta s}) \supset \{\xi \in \mathcal{O} : V(\xi) \leq q\}$,

iii.) and

$$\frac{\partial V}{\partial X} F(X, \mu) < 0 \qquad \forall \mu \in \mathcal{P} \ , \ \forall X \in \{\xi \in \mathcal{O} : q \leq V(\xi) \leq Q\} \tag{8.23}$$

where X *and* $F(X, \mu)$ *are defined by*

$$\dot{X} = \frac{d}{dt} \begin{pmatrix} x \\ \eta \end{pmatrix} = \begin{pmatrix} f(x, u, \mu(t)) \\ \theta\left(\eta, \alpha(x, u, \mu(t))\right) \end{pmatrix} =: F(X, \mu(t)) \tag{8.24}$$

with

$$u = \kappa\left(\eta, \alpha(x, u, \mu(t))\right) \tag{8.25}$$

(and where, for simplicity, we assume the right-hand side of (8.25) is independent of u*).*

Definition 8.3 *The origin of (8.20) is said to be* uniformly semiglobally practically asymptotically stabilizable by dynamic output feedback *if, in the previous definition, we can always take* $\alpha(x, u, \mu) = h(x, u, \mu)$.

Remark 8.4 In these definitions, we could allow the right-hand side of (8.25) to depend on u if we impose an extra condition that guarantees a solution to (8.25). △

It will follow from the proof of [5, Proposition 3.1] (much like what is suggested by [5, Footnote 5]) that we have:

Theorem 8.5 *Let* $\mu(\cdot) \in \mathcal{M}_{\mathcal{P}}$ *be sufficiently smooth with a uniform bound on an appropriate number of derivatives. If the origin of the system (8.20) is uniformly semiglobally practically asymptotically stabilizable by dynamic UCO feedback then it is uniformly semiglobally practically asymptotically stabilizable by dynamic output feedback.*

Sketch of Proof. Fix $0 < r < R < \infty$. From the assumption of uniform semiglobal practical asymptotic stabilizability by dynamic UCO feedback, this fixes a UCO function $\alpha(x, u, \mu)$, a corresponding function Ψ that is used to reconstruct α from derivatives of y and u, functions θ and κ, compact sets $\mathcal{C}_{ns}$ and $\mathcal{C}_{nl}$, an open set $\mathcal{O}$, a function V and strictly positive real numbers $0 < q < Q < \infty$. Now we apply the proof of [5, Proposition 3.1] to the control system

$$\begin{pmatrix} \dot{x} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} f(x, u_1, \mu(t)) \\ \theta(\eta, u_2) \end{pmatrix}$$
$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} h(x, u_1, \mu(t)) \\ \eta \end{pmatrix}$$

where the UCO feedback

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \kappa(\eta, \alpha(x, u_1, \mu(t))) \\ \alpha(x, u_1, \mu(t)) \end{pmatrix}$$

induces the properties for the function V that are assumed in the proof of [5, Proposition 3.1] if we define the objects $\mathcal{K}_{zs}$, $\mathcal{K}_{zl}$, ν_l, c_s and c_l used in the proof of [5, Proposition 3.1] as

$$\mathcal{K}_{zs} := \overline{\mathcal{B}}_n(r) \times \mathcal{C}_{ns} \ , \quad \mathcal{K}_{zl} := \overline{\mathcal{B}}_n(R) \times \mathcal{C}_{nl}$$

and

$$\nu_l := q \ , \quad c_s := Q \ , \quad c_l := Q + 1 \ .$$

From here we follow the proof of [5, Proposition 3.1], but noting that dynamic extension is only needed on the input u_1 and no estimates of the derivatives of $y_2 = \eta$ are needed. △

4.2 Application to Nonminimum Phase Systems

We now apply this general result to the problem considered in Section 3.2. We start with an assumption that is a combination of Assumption 8.3 and Remark 8.1.

Assumption 8.6 *The controller*

$$\begin{aligned} \dot{\varphi} &= L(\varphi, x_{\mathrm{a},2}, y_{\mathrm{a}}) \\ u_{\mathrm{a}} &= N(\varphi, x_{\mathrm{a},2}, y_{\mathrm{a}}) \ , \end{aligned} \tag{8.26}$$

is such that

1. $N(0,0,0) = 0$ *and, for simplicity,* $N(\varphi, x_{\mathrm{a},2}, h_{\mathrm{a}}(x_{\mathrm{a}}, u_a, \mu))$ *independent of* u_a*,*

2. *the origin of the system (8.10),(8.26) is uniformly globally asymptotically stable;*

3. *the functions* $\mu(\cdot)$ *are restricted so that*

$$\left| \frac{\partial N}{\partial h_{\mathrm{a}}} \frac{\partial h_{\mathrm{a}}}{\partial \mu} \dot{\mu}(t) \right|$$

is bounded in $t \geq 0$, uniformly in $\mu(\cdot)$, on each compact subset of the state-space.

Under this assumption, we can state the following result for the system (8.9) under the action of the controller

$$\begin{aligned} \dot{\varphi} &= L\left(\varphi, x_{\mathrm{a},2}, h_{\mathrm{a}}(x_{\mathrm{a}}, \zeta_r, \mu(t))\right) \\ u &= -k\mathrm{sgn}(b(\zeta))\left(\zeta_r - N(\varphi, x_{\mathrm{a},2}, h_{\mathrm{a}}(x_{\mathrm{a}}, \zeta_r, \mu(t)))\right) . \end{aligned} \tag{8.27}$$

Note that this is a dynamic UCO feedback for the system (8.9) since, as noted in Remark 8.3, $x_{\mathrm{a},2}$, ζ_r and $h_{\mathrm{a}}(x_{\mathrm{a}}, \zeta_r, \mu(t))$ are UCO with respect to the system (8.9).

Lemma 8.2 *Under Assumption 8.6, the origin of the system (8.9), (8.27) is semiglobally practically asymptotically stable in the parameter k.*

Proof. Follows from the discussion in Section 3.1. (See also [5, Lemma 2.2 (Semiglobal backstepping I)].) △

The final result then follows from Theorem 8.5 and Lemma 8.2.

Corollary 8.1 *Under Assumption 8.6, the origin of the system (8.9) is semiglobally practically stabilizable by dynamic output feedback.*

The controller given by Corollary 8.1, which is constructed following the proof of Theorem 8.5, is different from the one given by Theorem 8.4 together with Remark 8.1. In particular, the controller of Corollary 8.1 has the form of an observer

$$\dot{\eta} = P\eta + Qy , \tag{8.28}$$

like in (8.16) but with $\eta \in I\!R^{r+1}$, where η_{r+1} is an estimate of $\dot{\zeta}_r$, plus an estimated and saturated dynamic UCO feedback

$$\begin{aligned} \dot{\varphi} &= \sigma_\ell\left(L\left(\varphi, \hat{x}_{\mathrm{a},2}, \widehat{\dot{\zeta}_r} - b(\hat{\zeta})u\right)\right) \\ \dot{u} &= \sigma_\ell(k_2(v - u)) \\ v &= -k_1\mathrm{sgn}(b(\hat{\zeta}))\left(\hat{\zeta}_r - N(\varphi, \hat{x}_{\mathrm{a},2}, \widehat{\dot{\zeta}_r} - b(\hat{\zeta})u)\right) , \end{aligned} \tag{8.29}$$

like in (8.19). Compared to the controller (8.28),(8.29), the controller (8.16), (8.19) together with remark 8.1 has one less state and can be interpreted as using a reduced-order observer structure to accomplish the goal of robust semiglobal practical asymptotic stabilization.

In [5, Section 6.2], a particular nonminimum phase nonlinear system, whose auxiliary system (using the terminology of the present chapter) is semiglobally asymptotically stabilizable by (static) UCO feedback, was considered as an illustration of the result that semiglobal practical asymptotic stabilization by (static) UCO feedback implies semiglobal practical asymptotic stabilization by dynamic output feedback. The controller used in that section is the type of controller suggested by Corollary 8.1.

5 Conclusions

This chapter presented a simple design method by which it is possible to robustly stabilize, using output feedback, a significant class of uncertain nonlinear systems whose zero dynamics are unstable. The assumption made for such systems was shown to imply the existence of a stabilizing dynamic feedback that is driven by functions that are *uniformly completely observable* (UCO). In this light, the result for nonminimum phase nonlinear systems was shown to be a special case of the more general result that semiglobal practical asymptotic stabilization by dynamic UCO feedback implies semiglobal practical asymptotic stabilization by dynamic output feedback. The controllers developed in this chapter specifically for nonminimum phase nonlinear systems were compared and contrasted to the controllers that prove the general stabilization result.

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