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Results on converse Lyapunov functions from class-$\mathcal{KL}$ estimates

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Abstract

We state results on converse Lyapunov functions for differential inclusions where a positive semidefinite function of the solutions satisfies a class-$\mathcal{KL}$ estimate in terms of time and a second positive semidefinite function of the initial condition. The main result is that a smooth converse Lyapunov function, i.e., one whose derivative along solutions can be used to establish the class-$\mathcal{KL}$ estimate, exists if and only if the class-$\mathcal{KL}$ estimate is robust, i.e., it holds for a larger, perturbed inclusion. It remains an open question whether all class-$\mathcal{KL}$ estimates are robust. One sufficient condition for robustness is that the original inclusion is locally Lipschitz. Another is that the two positive semidefinite functions agree and a backward completability condition holds. These special cases unify and generalize many existing results on converse Lyapunov theorems for differential equations and inclusions.

Basic definitions

Given a set $\mathcal{A}$, $\overline{\mathcal{A}}$ stands for the closure of $\mathcal{A}$, $\overline{\text{co}}\mathcal{A}$ the closed convex hull of $\mathcal{A}$ and $\partial\mathcal{A}$ the boundary of $\mathcal{A}$. Also $x \to \partial\mathcal{A}^\infty$ indicates a sequence of points x belonging to $\mathcal{A}$ converging to a point on the boundary of $\mathcal{A}$ or, if $\mathcal{A}$ is unbounded, having the property $|x| \to \infty$. Given a closed set $\mathcal{A} \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$, $|x|_{\mathcal{A}}$ denotes the distance from x to $\mathcal{A}$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class-$\mathcal{K}$ if it is continuous, zero at zero, and strictly increasing. It belongs to class-$\mathcal{K}_\infty$ if, in addition, it is unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class-$\mathcal{KL}$ if, for each $t \geq 0$, $\beta(\cdot, t)$ is nondecreasing and $\lim_{s\to 0^+} \beta(s,t) = 0$, and, for each $s \geq 0$, $\beta(s, \cdot)$ is nonincreasing and $\lim_{t\to\infty} \beta(s,t) = 0$.

1 Background

Where Lyapunov [14] introduced his famous sufficient conditions for asymptotic stability of an equilibrium for

$$\dot{\xi} = f(\xi, t) , \tag{1}$$

we find the first contribution [14, §20, Theorem II] to the converse question: what aspects of asymptotic stability and the function f guarantee the existence of a (smooth) function satisfying Lyapunov's sufficient conditions for asymptotic stability? The answers have proved instrumental, over the years, in establishing robustness of various stability notions and have served as the starting point for many nonlinear control systems design concepts.

One of the first important milestones in the pursuit of smooth converse Lyapunov functions was Massera's 1949 paper [16] that provided a semi-infinite integral construction for time-invariant, continuously differentiable systems with an asymptotically stable equilibrium. Later, in 1954, Malkin observed that Massera's construction worked even for time-varying systems as long as the asymptotic stability and the differentiability of f with respect to the state were uniform in time [15]. Regarding stability Malkin assumed, in effect, the existence of a class-$\mathcal{KL}$ function β such that the solutions $\xi(t, t_\circ, \xi_\circ)$ of the system (1) satisfy

$$|\xi(t, t_\circ, \xi_\circ)| \leq \beta(|\xi_\circ|, t - t_\circ) \qquad \forall t \geq t_\circ \geq 0 \tag{2}$$

at least for initial conditions $\xi_\circ$ sufficiently small. In [4], Barbashin and Krasovskii generalized Malkin's result to the case where (2) holds for all initial conditions. Both Massera [17] and Kurzweil [10], independently in the mid-1950's, weakened the assumptions made by Malkin, and Barbashin and Krasovskii, about the function $f(\xi, t)$. Kurzweil's result allows $f(\xi, t)$ to be only continuous and doesn't assume uniqueness of solutions. In his work he made precise a notion of strong stability of the origin on an open neighborhood $\mathcal{G}$ of the origin which amounts to the existence of a function $\beta \in \mathcal{KL}$ and a locally Lipschitz, positive definite function $\omega : \mathcal{G} \to \mathbb{R}_{\geq 0}$, proper on $\mathcal{G}$, such that, for all $\xi_\circ \in \mathcal{G}$, all solutions of the system (1) satisfy

$$\omega(\xi(t, t_\circ, \xi_\circ)) \leq \beta(\omega(\xi_\circ), t - t_\circ) \qquad t \geq t_\circ \geq 0 \ . \tag{3}$$

Kurzweil showed that this strong stability and continuity of $f(\xi, t)$ imply the existence of a smooth converse Lyapunov function, i.e., a function whose derivative along solutions can be used to deduce (3).

Much of the research in the 1960's focused on developing converse Lyapunov theorems for systems possessing asymptotically stable closed, not necessarily compact, sets. Taking this approach, the time-varying case can be subsumed into the time-invariant case by augmenting the state-space of (1) as:

$$\dot{x} = \frac{d}{dt}\begin{pmatrix} \xi \\ p \end{pmatrix} = \begin{pmatrix} f(\xi, p) \\ 1 \end{pmatrix} =: F(x) \ , \ x_\circ = \begin{bmatrix} \xi_\circ \\ t_\circ \end{bmatrix} . \tag{4}$$

(One disadvantage in treating time-varying systems as time-invariant ones is that it usually leads to imposing stronger than necessary conditions on the time-dependence of the right-hand side, e.g., continuity where only measurability is needed. An example where a converse theorem is developed for systems with right-hand sides measurable in time, and for Lyapunov and

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Lagrange stability, is [3].) A closed set $\mathcal{A}$ for (4) is uniformly asymptotically stable if there exists of a function $\beta \in \mathcal{KL}$ such that all solutions of (4) with $|x_\circ|_\mathcal{A}$ sufficiently small exist for all forward time and satisfy

$$|\phi(t, x_\circ)|_\mathcal{A} \le \beta(|x_\circ|_\mathcal{A}, t) \qquad \forall t \ge 0 . \tag{5}$$

Some of the early results on converse Lyapunov functions for set stability are summarized in [33]. Particularly noteworthy are the result of Hoppensteadt [9] who generated a C^1 converse Lyapunov function for parameterized differential equations and the result of Wilson [32] who provided a smooth converse Lyapunov function for uniform asymptotic stability of a set.

Noting that every solution $\phi(t, x_\circ)$ of (4) can be written

$$\phi(t, x_\circ) = \begin{bmatrix} \xi(t + t_\circ, t_\circ, \xi_\circ) \\ t + t_\circ \end{bmatrix} \tag{6}$$

where $\xi(\cdot, t_\circ, \xi_\circ)$ is a solution of (1), it follows that (2), (3) and (5) are all particular cases of the estimate

$$\omega(\phi(t, x_\circ)) \le \beta(\omega(x_\circ), t) \qquad \forall t \ge 0 \tag{7}$$

where ω is continuous, positive semi-definite.

In the 1970's, Lakshmikantham and coauthors [12] (see also [11, section 3.4]) provided a Lipschitz converse Lyapunov function for Lipschitz differential equations of the form (4) under an assumption essentially the same as: given two continuous, positive semidefinite functions ω_1 and ω_2, there exists a function $\beta \in \mathcal{KL}$ such that, for all $x_\circ$ with $\omega_2(x_\circ)$ sufficiently small, all solutions exist for all forward time and satisfy

$$\omega_1(\phi(t, x_\circ)) \le \beta(\omega_2(x_\circ), t) \qquad \forall t \ge 0 . \tag{8}$$

The stability concept described by (8), apparently first introduced in [22] and often called stability with respect to two measures, generalizes (7) and thus includes the notions of local uniform asymptotic stability of a point, of a prescribed motion and of a closed set. It also covers the notion of local uniform partial asymptotic stability such as when, for $x_\circ$ sufficiently small,

$$|h(\phi(t, x_\circ))| \le \beta(|x_\circ|, t) \qquad \forall t \ge 0 \tag{9}$$

where $y = h(x)$ is a continuous function of the state. A smooth converse Lyapunov theorem for the global version of (9) was recently derived in [25]. (See [31] for a survey on the partial stability problem.)

Extensions of the above results to differential inclusions started to appear in the late 1970's with some of the most general results appearing only recently. Some motivations for the study of differential inclusions are they describe 1) the solution set for differential equations with arbitrary, measurable bounded disturbances, and 2) important notions of solutions for control systems that use discontinuous feedbacks. (see [7, §8.3].)

The results in [18] pertain to differential inclusions

$$\dot{x} \in F(x) := \overline{\mathrm{co}}\,\{v \in \mathbb{R}^n : v = f(x, d) \ , \ d \in \mathcal{D}\} \tag{10}$$

where $\mathcal{D}$ is compact, $f(x, d)$ is continuous and continuously differentiable with respect to x, and asymptotic stability in the first approximation is assumed, i.e., for

$$\dot{x} \in \left\{ v \in \mathbb{R}^n : v = \frac{\partial f}{\partial x}(0, d)x \ , \ d \in \mathcal{D} \right\} , \tag{11}$$

an estimate of the form $|x(t)| \le k|x(0)|\exp(-\lambda t)$, $k > 0, \lambda > 0$ is assumed for all solutions starting from sufficiently small initial conditions. [18, Theorem 2] states that this implies the existence of a smooth Lyapunov function for local exponential stability and asymptotic stability on the basin of attraction of the origin for the inclusion (10). Related results for inclusions of the type (11) are in [19, 20, 21].

In [13], Lin, Sontag and Wang considered the inclusion (10) with f continuous and locally Lipschitz in x uniformly in d and assumed the estimate (5) globally. They showed, when $\mathcal{A}$ is compact or all solutions exist for all backward time, that (5) for the inclusion (10) implies the existence of a smooth Lyapunov function. In [1], the ideas of [13] were combined with the idea of Kurzweil [10] to establish the existence of a smooth Lyapunov function for the inclusion (10) when there exists a compact set $\mathcal{A}$, a neighborhood $\mathcal{G}$ of $\mathcal{A}$ and function $\omega : \mathcal{G} \to \mathbb{R}_{\ge 0}$ that is locally Lipschitz, positive definite with respect to $\mathcal{A}$ and proper with respect to $\mathcal{G}$ and a function $\beta \in \mathcal{KL}$ such that, for all $x_\circ \in \mathcal{G}$, the solutions of (10) satisfy (7).

The first results on smooth converse theorems for differential inclusions that are only upper semicontinuous (see Definition 1 below) appeared in [5]. In that work, Clarke, Ledyaev and Stern considered inclusions $\dot{x} \in F(x)$ under the assumption that $F(x)$ is nonempty, compact and convex for each $x \in \mathbb{R}^n$ and $F(x)$ is upper semicontinuous. They assumed the estimate (8) with $\omega_1(x) = \omega_2(x) = |x|$, and showed that this implies the existence of a smooth Lyapunov function. Other results on the existence of converse Lyapunov functions can be found in [2, Chapter 6], [6] and [28, 29, 30].

2 Contributions

In this paper, we consider differential inclusions

$$\dot{x} \in F(x) \tag{12}$$

like in [5]: $F(x)$ is a set-valued map from an an open set $\mathcal{G}$ to subsets of $\mathbb{R}^n$ that is upper semicontinuous on $\mathcal{G}$ (see Definition 1) and $F(x)$ is nonempty, compact and convex for each $x \in \mathcal{G}$. The stability property we will assume for (12) we will refer to as "$\mathcal{KL}$-stability with respect to (ω_1, ω_2) on $\mathcal{G}$". Namely, given two continuous functions $\omega_1 : \mathcal{G} \to \mathbb{R}_{\ge 0}$ and $\omega_2 : \mathcal{G} \to \mathbb{R}_{\ge 0}$, we assume the existence of a class-$\mathcal{KL}$ function β such that all solutions of the inclusion (12) starting in $\mathcal{G}$ remain in $\mathcal{G}$ for all forward time and satisfy (8). (See Definition 6). This is like the stability property assumed in [12]. Our main result is (see Theorem 1):

A smooth converse Lyapunov function for $\mathcal{KL}$-stability with respect to (ω_1, ω_2) (see Definition 8) exists if and only if the $\mathcal{KL}$-stability is robust, i.e., it holds for a larger, perturbed inclusion (see Definition 9).

This type of equivalence between robust stability and the existence of a Lyapunov function, reminiscent of the classical "total stability" results for differential equations [8, Theorem 56.4], is already present in the proofs of Kurzweil [10] and Clarke, et al., [5].

It remains an open question whether $\mathcal{KL}$-stability with respect to (ω_1, ω_2) is robust, in general. However, we can state (see Theorem 2):

If the inclusion (12) is locally Lipschitz on $\mathcal{G}$ (see Def. 3) then $\mathcal{KL}$-stability w.r.t. (ω_1, ω_2) on $\mathcal{G}$ is robust.

This is the case for the problems consider by Lakshmikantham, et al. [11], Lin, et. al. [13], and Sontag and Wang [25]. We will also establish (see Theorem 3):

If the inclusion (12) is backward completable by ω-normalization (see Definition 10) then $\mathcal{KL}$-stability with respect to (ω, ω) on $\mathcal{G}$ is robust.

This condition holds for the problems considered by Kurzweil [10] and Clarke, et al. [5] and, more generally, for compact, stable attractors.

Our converse Lyapunov function is built as follows :

1. We imbed the original differential equation or inclusion into a larger, locally Lipschitz inclusion that still exhibits $\mathcal{KL}$-stability w.r.t. (ω_1, ω_2). This idea is due to Kurzweil [10] for the case of differential equations with continuous right-hand side under strong stability of the origin. It is due to Clarke and co-authors [5] for the case of nonempty, compact, convex, upper semicontinuous differential inclusions and global asymptotic stability of the origin. In general, it is possible if and only if the $\mathcal{KL}$-stability with respect to (ω_1, ω_2) is robust.
2. We find class-$\mathcal{K}_\infty$ functions $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$ such that $\widetilde{\alpha}_1(\beta(s,t)) \leq \widetilde{\alpha}_2(s)e^{-2t}$, where β quantifies $\mathcal{KL}$-stability with respect (ω_1, ω_2) for the locally Lipschitz inclusion constructed in step 1. A recent result by Sontag [24, Proposition 7] shows that this can be done.
3. We define a trial Lyapunov function $V_1(x)$ as the supremum, over time and solutions $\phi(\cdot, x)$ of the locally Lipschitz inclusion constructed in step 1, of the quantity $\widetilde{\alpha}_1(\omega_1(\phi(t,x)))e^t$ where $\widetilde{\alpha}_1$ was constructed in step 2. This is a classical construction once the estimate in step 2 is available, at least for locally Lipschitz differential equations (See, e.g., [33, §19].) We show, using many of the tools used in [5] and [13], that this Lyapunov function has all of the desired properties except smoothness. It is locally Lipschitz. It would only be upper semicontinuous, in general, if the supremum were taken over solutions of the original inclusion.
4. We smooth the trial Lyapunov function using ideas from Kurzweil [10] that have been clarified, generalized and used over the years by, for example, Wilson [32], Lin, et al., [13] and Clarke, et al. [5].

To describe our results, in Section 3 we present definitions related to set-valued maps and properties of solutions to differential inclusions. In Section 4 we define $\mathcal{KL}$-stability w.r.t. (ω_1, ω_2) and provide alternative characterizations. Our main results are stated in Section 5 with relations to previous results summarized in Section 6.

3 Preliminaries

Throughout this paper $F(x)$ will be a set-valued map from $\mathcal{G}$ to subsets of $\mathbb{R}^n$ where $\mathcal{G}$ is an open subset of $\mathbb{R}^n$. Also $\mathcal{B}$ denotes the open unit ball in $\mathbb{R}^n$ and

$$F(x) + \varepsilon\mathcal{B} := \{z \in \mathbb{R}^n : \mathrm{dist}(z, F(x)) < \varepsilon\} \ . \quad (13)$$

We use the following definitions (see [7, §5.3, §7.2]):

Definition 1 *The set-valued map F is* upper semicontinuous on $\mathcal{G}$ *if, given $x \in \mathcal{G}$, for each $\varepsilon > 0$ there exists $\delta > 0$ such that, for all $\xi \in \mathcal{G}$ satisfying $|x - \xi| < \delta$ we have $F(\xi) \subseteq F(x) + \varepsilon\mathcal{B}$.*

Definition 2 *The set-valued map F* satisfies the basic conditions on $\mathcal{G}$ *if it is upper semicontinuous on $\mathcal{G}$ and, for each $x \in \mathcal{G}$, $F(x)$ is nonempty, compact and convex.*

Lemma 1 *If the set-valued map F satisfies the basic conditions on $\mathcal{G}$ and $\rho : \mathcal{G} \to \mathbb{R}_{\geq 0}$ is a continuous function such that for all $x \in \mathcal{G}$, we have $\{x\} + \rho(x)\overline{\mathcal{B}} \subset \mathcal{G}$, then the set-valued map*

$$\overline{\mathrm{co}} \left(\bigcup_{\xi \in \{x\} + \rho(x)\overline{\mathcal{B}}} F(\xi) \right) + \rho(x)\overline{\mathcal{B}} \ ,$$

which we denote by $\overline{\mathrm{co}}F(x + \rho(x)\overline{\mathcal{B}}) + \rho(x)\overline{\mathcal{B}}$, satisfies the basic conditions on $\mathcal{G}$.

Definition 3 *Let $\mathcal{O}$ be an open subset of $\mathcal{G}$. The set-valued map F is said to be* locally Lipschitz on $\mathcal{O}$ *if, for each $x \in \mathcal{O}$, there exists a neighborhood $\mathcal{U} \subset \mathcal{O}$ of x and a positive real number L such that*

$$x_1, x_2 \in \mathcal{U} \implies F(x_1) \subseteq F(x_2) + L|x_1 - x_2|\mathcal{B} \ . \quad (14)$$

Given a set-valued map $F(x)$, we can define a solution of the differential inclusion

$$\dot{x} \in F(x) \ . \quad (15)$$

Definition 4 *A function $x : [0,T] \to \mathcal{G}$ $(T > 0)$ is a* solution of the differential inclusion (15) *if it is absolutely continuous and satisfies, for almost all $t \in [0,T]$,*

$$\dot{\widehat{x(t)}} \in F(x(t)) \ . \quad (16)$$

A function $x : [0,T) \to \mathcal{G}$ $(0 < T \leq \infty)$ is a maximal solution of the inclusion (15) *if it does not have an extension which is a solution belonging to $\mathcal{G}$, i.e., either $T = \infty$ or there does not exist a solution $y : [0, T_+] \to \mathcal{G}$ with $T_+ > T$ such that $y(t) = x(t)$ for all $t \in [0,T)$.*

The following basic fact about the existence of maximal solutions is a combination of [7, §7, Theorem 1] and [23, Propositions 1 and 2].

Lemma 2 *If $F(x)$ satisfies the basic conditions on $\mathcal{G}$ then for each $x_\circ \in \mathcal{G}$ there exist solutions of (15) for sufficiently small $T > 0$ satisfying $x(0) = x_\circ$. In addition, every solution can be extended into a maximal solution. Also, if a maximal solution $x(\cdot)$ is defined on a bounded interval $[0,T)$ then $x(t) \to \partial\mathcal{G}^\infty$ as $t \to T$.*

Henceforth, we will use $\phi(\cdot, x)$ to denote a solution of (15) starting at x and we will denote by $\mathcal{S}(x)$ the set of maximal solutions starting at x.

Definition 5 *The differential inclusion (15) is* forward complete on $\mathcal{G}$ *if, for all $x \in \mathcal{G}$, all solutions $\phi \in \mathcal{S}(x)$ are defined (and remain in $\mathcal{G}$) for all $t \geq 0$. The differential inclusion (15) is* backward complete *on $\mathcal{G}$ if $\dot{x} \in -F(x)$ is forward complete on $\mathcal{G}$.*

4 Definition/Characterization of $\mathcal{KL}$ stability

Our main stability definition is the following:

Definition 6 *Let $\omega_i : \mathcal{G} \to \mathbb{R}_{\geq 0}$, $i = 1, 2$, be continuous. The inclusion $\dot{x} \in F(x)$ is* $\mathcal{KL}$-stable with respect to (ω_1, ω_2) on $\mathcal{G}$ *if it is forward complete on $\mathcal{G}$ and there exists $\beta \in \mathcal{KL}$ such that, for each $x \in \mathcal{G}$, all solutions $\phi \in \mathcal{S}(x)$ satisfy $\omega_1(\phi(t,x)) \leq \beta(\omega_2(x), t)$, $\forall t \geq 0$.*

As mentioned earlier, this stability concept was introduced in [22] and considered in [12] and [11]. It is often referred to as stability with respect to two measures. In the case where $\mathcal{A}$ is a closed set, $\omega_1(x) = \omega_2(x) = |x|_{\mathcal{A}}$, and $\mathcal{G} = \mathbb{R}^n$, it has been shown in [13, Proposition 2.5] that $\mathcal{KL}$-stability is equivalent to uniform (local) stability plus uniform (global) boundedness plus uniform (global) attractivity. The same proof technique establishes this result for general $\mathcal{KL}$-stability:

Proposition 1 *Let $\omega_i : \mathcal{G} \to \mathbb{R}_{\geq 0}$, $i = 1, 2$, be continuous. The following are equivalent:*
1. The differential inclusion $\dot{x} \in F(x)$ is $\mathcal{KL}$-stable with respect to (ω_1, ω_2) on $\mathcal{G}$.
2. All of the following hold:
a) The inclusion $\dot{x} \in F(x)$ is forward complete on $\mathcal{G}$.
b) (Uniform stability and global boundedness)*: There exists a class-$\mathcal{K}_\infty$ function γ such that, for each $x \in \mathcal{G}$, all solutions $\phi \in \mathcal{S}(x)$ satisfy $\omega_1(\phi(t,x)) \leq \gamma(\omega_2(x))$ for all $t \geq 0$.*
c) (Uniform global attractivity)*: For each $r > 0$ and $\varepsilon > 0$, there exists $T(r, \varepsilon) > 0$ such that, for each $x \in \mathcal{G}$, all solutions $\phi \in \mathcal{S}(x)$ satisfy*
$$\omega_2(x) \leq r\ , \quad t \geq T \quad \Longrightarrow \quad \omega_1(\phi(t,x)) \leq \varepsilon\ .$$
It can also be shown that when ω is a type of indicator for a compact set $\mathcal{A}$, $\mathcal{KL}$-stability with respect to (ω, ω) follows from local stability of the set plus attractivity to the set from a neighborhood of the set. Uniform boundedness and uniform attractivity are guaranteed by the fact that the attractor is compact. Before stating this result, we make the following definition :

Definition 7 *Given a compact subset $\mathcal{A}$ of an open set $\mathcal{G}$, a function $\omega : \mathcal{G} \to \mathbb{R}_{\geq 0}$ is a* proper indicator for $\mathcal{A}$ on $\mathcal{G}$ *if ω is continuous, $\omega(x) = 0$ if and only if $x \in \mathcal{A}$, and $\lim_{x \to \partial\mathcal{G}^\infty} \omega(x) = \infty$.*

Remark 4.1 For each open set $\mathcal{G}$ and each compact set $\mathcal{A} \subset \mathcal{G}$, there exists a proper indicator function. When $\mathcal{G} = \mathbb{R}^n$ we can take $\omega(x) = |x|_{\mathcal{A}}$. Otherwise, we can take, for example,

$$\omega(x) = \max\left\{ |x|_{\mathcal{A}}\ , \ \frac{1}{|x|_{\mathbb{R}^n \setminus \mathcal{G}}} - \frac{2}{\text{dist}(\mathcal{A}, \mathbb{R}^n \setminus \mathcal{G})} \right\}.$$

Kurzweil used this type of function, with $\mathcal{A} = \{0\}$, to define his notion of strong stability. ■

The next result, which is similar to [10, Theorem 12], shows that, for differential inclusions satisfying the basic conditions, the basin of attraction $\mathcal{G}$ for a stable, compact attractor $\mathcal{A}$ is open and, for each function ω that is a proper indicator for $\mathcal{A}$ on $\mathcal{G}$, the differential inclusion is $\mathcal{KL}$-stable w.r.t. (ω, ω) on $\mathcal{G}$.

Proposition 2 *Let $F(x)$ satisfy the basic conditions on an open set $\mathcal{O}$ and let $\mathcal{A} \subset \mathcal{O}$ be compact. If the set $\mathcal{A}$ is stable and the set of points $\mathcal{G}$ from which $\mathcal{A}$ is strongly attractive contains a neighborhood of $\mathcal{A}$, i.e.,*

1. Stability: *for each $\varepsilon > 0$ there exists $\delta > 0$ such that, for each $x \in \mathcal{O} \bigcap (\mathcal{A} + \delta\overline{\mathcal{B}})$, each solution $\phi \in \mathcal{S}(x)$ is defined and belongs to $\mathcal{O}$ for all $t \geq 0$ and satisfies $|\phi(t,x)|_{\mathcal{A}} \leq \varepsilon$ for all $t \geq 0$,*
2. Attractivity: *the set $\mathcal{G}$ of points $x \in \mathcal{O}$ such that each solution $\phi \in \mathcal{S}(x)$ is defined and belongs to $\mathcal{O}$ for all $t \geq 0$ and satisfies $\lim_{t\to\infty} |\phi(t,\xi)|_{\mathcal{A}} = 0$ contains a neighborhood of $\mathcal{A}$,*

then the set $\mathcal{G}$ is open and, for each function ω that is a proper indicator for $\mathcal{A}$ on $\mathcal{G}$, the differential inclusion $\dot{x} \in F(x)$ is $\mathcal{KL}$-stable with respect to (ω, ω) on $\mathcal{G}$.

Finally, we characterize $\mathcal{KL}$-stability w.r.t. (ω_1, ω_2) in Lyapunov function terms:

Definition 8 *Let $\omega_i : \mathcal{G} \to \mathbb{R}_{\geq 0}$, $i = 1, 2$, be continuous. A function $V : \mathcal{G} \to \mathbb{R}_{\geq 0}$ is a smooth converse Lyapunov function for $\mathcal{KL}$-stability w.r.t. (ω_1, ω_2) on $\mathcal{G}$ for $F(x)$ if $V(x)$ is smooth on $\mathcal{G}$ and there exist class-$\mathcal{K}_\infty$ functions α_1, α_2 such that, for all $x \in \mathcal{G}$,*

$$\alpha_1(\omega_1(x)) \leq V(x) \leq \alpha_2(\omega_2(x)) \tag{17}$$

and

$$\max_{w \in F(x)} \langle \nabla V(x), w \rangle \leq -V(x)\ . \tag{18}$$

The motivation for this definition is that (18) guarantees that the derivative of $V(x)$ along solutions, denoted $\dot{V}(\phi(t,x))$, satisfies $\dot{V}(\phi(t,x)) \leq -V(\phi(t,x))$ for almost all t in the interval where $\phi(t,x)$ exists and belongs to $\mathcal{G}$. It follows that $V(\phi(t,x)) \leq V(x)e^{-t}$ on this interval and then, using (17) and assuming forward completeness on $\mathcal{G}$, we can deduce $\mathcal{KL}$-stability with respect to (ω_1, ω_2) on $\mathcal{G}$. (By relying on a result like [13, Lemma 4.4], it is possible to deduce $\mathcal{KL}$-stability w.r.t. (ω_1, ω_2) on $\mathcal{G}$ when $V(x)$ on the right-hand side of (18) is replaced by any class-$\mathcal{K}_\infty$ function of $V(x)$.)

5 Main Results

We are interested in whether $\mathcal{KL}$-stability with respect to (ω_1, ω_2) implies the existence of a smooth converse Lyapunov function for $\mathcal{KL}$-stability with respect to (ω_1, ω_2). This is still an open question, in general. What we will indicate here is that a smooth converse Lyapunov function exists if and only if the $\mathcal{KL}$-stability with respect to (ω_1, ω_2) is robust; that is $\mathcal{KL}$-stability with respect to (ω_1, ω_2) still holds for a set of differential inclusions given by supersets of F. This concept, which is present in the work of Kurzweil [10] and Clarke, et al. [5], is defined more precisely as follows:

Definition 9 *Let $\omega_i : \mathcal{G} \to \mathbb{R}_{\geq 0}$, $i = 1, 2$, be continuous. The inclusion $\dot{x} \in F(x)$ is* robustly $\mathcal{KL}$-stable with respect to (ω_1, ω_2) on $\mathcal{G}$ *if there exists a continuous function $\delta : \mathcal{G} \to \mathbb{R}_{\geq 0}$ such that*

1. $\{x\} + \delta(x)\overline{\mathcal{B}} \subset \mathcal{G}$;

2. the inclusion

$$\dot{x} \in F_{\delta(x)}(x) := \overline{\text{co}} F(x + \delta(x)\overline{\mathcal{B}}) + \delta(x)\overline{\mathcal{B}} \tag{19}$$

is $\mathcal{KL}$-stable with respect to (ω_1, ω_2) on $\mathcal{G}$;

3. $\delta(x) > 0$ for all $x \in \mathcal{G} \backslash \mathcal{A}_\delta$ where

$$\mathcal{A}_\delta := \left\{ \xi \in \mathcal{G} : \sup_{t \geq 0, \phi \in \mathcal{S}_\delta(\xi)} \omega_1(\phi(t, \xi)) = 0 \right\},$$

$\mathcal{S}_\delta(\cdot)$ denoting the set of maximal solutions to (19).

The main feature of the inclusion (19) is that its solution set includes the solution set of the inclusion $\dot{x} \in F(x)$ since $F(x) \subseteq F_{\delta(x)}(x)$. Note that even for differential equations, robust stability will be expressed in terms of stability for a differential inclusion.

The following theorem emphasizes that robust $\mathcal{KL}$-stability with respect to (ω_1, ω_2) is the key property for getting a smooth converse Lyapunov function.

Theorem 1 *Let $\omega_i : \mathcal{G} \to \mathbb{R}_{\geq 0}$, $i = 1, 2$, be continuous and let $F(x)$ satisfy the basic conditions on $\mathcal{G}$. The following statements are equivalent:*

1. *The inclusion $\dot{x} \in F(x)$ is forward complete on $\mathcal{G}$ and there exists a smooth converse Lyapunov function for $\mathcal{KL}$-stability with respect to (ω_1, ω_2) on $\mathcal{G}$ for $F(x)$.*

2. *The inclusion $\dot{x} \in F(x)$ is robustly $\mathcal{KL}$-stable with respect to (ω_1, ω_2) on $\mathcal{G}$.*

We now specify cases where robust $\mathcal{KL}$-stability is guaranteed. The first case is when the inclusion is locally Lipschitz, perhaps on a certain subset of $\mathcal{G}$:

Theorem 2 *Let $\omega_i : \mathcal{G} \to \mathbb{R}_{\geq 0}$, $i = 1, 2$, be continuous and let $F(x)$ satisfy the basic conditions on $\mathcal{G}$. If the inclusion $\dot{x} \in F(x)$ is $\mathcal{KL}$-stable with respect to (ω_1, ω_2) on $\mathcal{G}$ and $F(x)$ is locally Lipschitz on an open set containing $\mathcal{G} \backslash \mathcal{A}$ where*

$$\mathcal{A} := \left\{ \xi \in \mathcal{G} : \sup_{t \geq 0, \phi \in \mathcal{S}(\xi)} \omega_1(\phi(t, \xi)) = 0 \right\}. \tag{20}$$

then the inclusion $\dot{x} \in F(x)$ is robustly $\mathcal{KL}$-stable with respect to (ω_1, ω_2) on $\mathcal{G}$.

Our next result will be that $\mathcal{KL}$-stability w.r.t. (ω_1, ω_2) implies robust $\mathcal{KL}$-stability w.r.t. (ω_1, ω_2) in the case where $\omega_1(x) = \omega_2(x) =: \omega(x)$ and the differential inclusion is backward completable by ω-normalization. The latter is defined as:

Definition 10 *Let $\omega : \mathcal{G} \to \mathbb{R}_{\geq 0}$ be continuous. The differential inclusion $\dot{x} \in F(x)$ is* backward completable by ω-normalization *if there exists a continuous function $\kappa : \mathcal{G} \to [1, \infty)$, a class-$\mathcal{K}$ function γ and a positive real number $c \geq 1$ such that*

$$\kappa(x) \leq \gamma(\omega(x)) + c \tag{21}$$

and the inclusion

$$\dot{x} \in \frac{1}{\kappa(x)} F(x) =: F_N(x) \tag{22}$$

is backward complete on $\mathcal{G}$.

In this definition, the existence of $\kappa(x)$ making (22) backward complete on $\mathcal{G}$ is always guaranteed. Indeed, from [7, §5, Lemma 15], $\sup_{v \in F(x)} |v|$ can be upper bounded by a function $\kappa_\circ(x)$ that is continuous on $\mathcal{G}$. So, for instance, in the case where $\mathcal{G} = \mathbb{R}^n$, by picking $\kappa(x) = \kappa_\circ(x)$ we get $\sup_{v \in F_N(x)} |v| \leq 1$ which implies that the inclusion (22) is backward complete on $\mathbb{R}^n$. The difficulty comes from requiring that $\kappa(x)$ simultaneously satisfies (21). However, when $\omega(x)$ is proper on $\mathcal{G}$ this difficulty disappears. More generally we have, when $\mathcal{G}$ is the Cartesian product $\mathcal{G} := \mathcal{G}_1 \times \mathbb{R}^{n_2}$ with $\mathcal{G}_1$ an open subset of $\mathbb{R}^{n_1}$ and by writing

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \in F(x) \quad : \tag{23}$$

Proposition 3 *If 1) F satisfies the basic conditions on $\mathcal{G}$; 2) $\lim_{x_1 \to \partial \mathcal{G}_1^\infty} \inf_{x_2 \in \mathbb{R}^{n_2}} \omega(x_1, x_2) = \infty$; 3) there exist positive real numbers ℓ and b such that*

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in F(x) \quad \Longrightarrow \quad |v_2| \leq \ell |x_2| + b$$

then the inclusion (23) is backward completable by ω-normalization.

With backward completability, $\mathcal{KL}$-stability w.r.t. (ω, ω) implies robust $\mathcal{KL}$-stability w.r.t. (ω, ω):

Theorem 3 *Let $\omega : \mathcal{G} \to \mathbb{R}_{\geq 0}$ be continuous and let $F(x)$ satisfy the basic conditions on $\mathcal{G}$. If the inclusion $\dot{x} \in F(x)$ is backward completable by ω-normalization and $\mathcal{KL}$-stable with respect to (ω, ω) on $\mathcal{G}$ then it is robustly $\mathcal{KL}$-stable with respect to (ω, ω) on $\mathcal{G}$.*

6 Corollaries

We briefly make connections to previous results on converse Lyapunov theorems. With the combination of Theorems 1 and 2 we recover the converse Lyapunov function results of [13] and [25, Theorem 2]. We also obtain a smooth, global version of [11, Theorem 3.4.1]. With the combination of Theorems 1 and 3 together with Propositions 1 and 3 we recover Kurzweil's main result [10, Theorem 7]. By replacing Proposition 1 in this list with Proposition 2 we recover the main result of Clarke, et. al. [5, Theorem 1.2] as well the result of Kurzweil for time-invariant, continuous differential equations (cf. [10, Theorem 12]).

Finally, we mention a corollary that is relevant for problems of semiglobal practical asymptotic stabilization of nonlinear control systems (see, e.g., [26]). We suppose:

Assumption 1 *There exist two compact sets $\mathcal{C}_1$, $\mathcal{C}_2$, two strictly positive real numbers ρ, T and an open set $\mathcal{O}$ such that 1) $\mathcal{C}_1 + \rho\overline{\mathcal{B}} \subset \mathcal{C}_2 \subset \mathcal{O}$, 2) $F(x)$ satisfies the basic conditions on $\mathcal{O}$ and is Lipschitz on $\mathcal{C}_1 + \rho\overline{\mathcal{B}}$, 3) for all $x \in \mathcal{C}_2$, all solutions $\phi \in \mathcal{S}(x)$ are defined and belong to $\mathcal{O}$ for all $t \geq 0$ and belong to $\mathcal{C}_1$ for $t \geq T$.*

Proposition 4 *Under assumption 1 the set*

$$\mathcal{A} := \{\xi \in \mathcal{C}_1 : \phi(t,\xi) \in \mathcal{C}_1 \ , \ \forall \phi \in \mathcal{S}(\xi) \ , \ \forall t \geq 0\}$$

is a nonempty, compact stable attractor with basin of attraction containing $\mathcal{C}_2$.

As a consequence, Proposition 2 applies for this set $\mathcal{A}$. Also, for each function ω that is a proper indicator for $\mathcal{A}$ on its strong domain of attraction, Proposition 3 allows us to apply Theorem 3 and then Theorem 1. So we can state the following converse Lyapunov function theorem for finite-time convergence to a compact set from a larger compact set :

Corollary 1 *Under Assumption 1, there exist a compact set $\mathcal{A} \subseteq \mathcal{C}_1$ and an open set $\mathcal{G} \supset \mathcal{C}_2$ such that, for each function $\omega : \mathcal{G} \to \mathbb{R}_{\geq 0}$ that is a proper indicator for $\mathcal{A}$ on $\mathcal{G}$, there exists a smooth converse Lyapunov function for $\mathcal{KL}$-stability w.r.t. (ω,ω) on $\mathcal{G}$ for $F(x)$.*

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