Proceedings of the 38th
Conference on Decision & Control
Phoenix, Arizona USA • December 1999

Uniting Local and Global Controllers

Christophe Prieur
Département de Mathématiques
École Normale Supérieure de Cachan
61 Avenue Wilson
94230 Cachan, France
prieur@dptmaths.ens-cachan.fr

Laurent Praly
Centre Automatique et Systèmes
École des Mines
35 Rue Saint Honoré
77305 Fontainebleau, France
praly@cas.ensmp.fr

Abstract

We consider control systems for which we know two stabilizing controllers. The former is "optimal" but local, the latter is global. We look for a uniting control law providing a globally stabilizing locally optimal controller. We study several solutions based on continuous, discontinuous, hybrid, time varying controllers. One criterion of selection of a controller is the robustness of the global asymptotic stability to vanishing measurement noise. This leads us in particular to consider a kind of generalization of Krasovskii solutions for hybrid systems.

1 Problem statement and related results

1.1 Introduction

In nonlinear control system theory, we have now numerous tools (backstepping, forwarding, feedback linearization, passivation,...) to design (globally) asymptotically stabilizing feedbacks. But, if such feedbacks give a satisfactory answer to the global asymptotic stabilization problem they are not necessarily intended to address the performance problem. As opposed to this case, for instance via linearization, one may design controllers addressing satisfactorily both the asymptotic stabilization and the performance problems but only locally. This leads us to the idea of uniting a local (optimal) controller with a global (stabilizing) controller, i.e. given 1) a controller u_l able to stabilize locally while providing better performance and 2) a controller u_g providing global asymptotic stability, we are looking for a may be time-varying, possibly hybrid, dynamic controller providing uniform global asymptotic stability for the overall system while matching exactly the local controller u_l when the system state component is in a neighborhood of the origin and matching the global controller u_g when this component is outside a compact set containing the origin.

1.2 Problem statement

Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be a locally Lipschitz function such that $f(0,0) = 0$. We consider the system

$$\dot{x} = f(x,u) . \tag{1}$$

We call uniting problem the following:
Let Ω be a bounded open connected neighborhood of the origin (in $\mathbb{R}^n$) and two continuous controllers $u_l : \mathcal{D} \to \mathbb{R}^m$ and $u_g : \mathbb{R}^n \to \mathbb{R}^m$, where $\mathcal{D} \subset \mathbb{R}^n$ is an open set containing $\mathtt{clos}(\Omega)$, which make the systems $f_l : \mathcal{D} \to \mathbb{R}^n$, $x \mapsto f(x, u_l(x))$ and $f_g : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto f(x, u_g(x))$ to admit the origin as an asymptotically stable equilibrium, globally on their respective domain of definition.

We look for

1. a bounded closed set $A \subset \mathbb{R}^n$ (e.g. an annulus) which separates $\mathbb{R}^n$ in two connected open sets C_l and C_g (e.g. an open ball and the complement of a closed ball) such that we have
$$\Omega \subset C_l \subset \mathcal{D} . \tag{2}$$
2. a control law $\varphi(x,s)$ depending on a parameter s which may be the time, an extra continuous state or a discrete state, satisfying for all s
$$\begin{aligned} \varphi(x,s) &= u_l(x) , \quad \forall x \in C_l , \\ &= u_g(x) , \quad \forall x \in C_g . \end{aligned} \tag{3}$$
and such that there exists a function β of class $\mathcal{KL}$ such that the x-component $X(x,s,t)$ of any trajectory of the closed loop system starting from (x,s) satisfies for all $t \geq 0$:
$$\|X(x,s,t)\| \leq \beta(\|x\|, t-s) . \tag{4}$$

1.3 Related results

Studies on the uniting problem have already been reported in particular in [10] and [7]. In [10] a dynamic time-invariant controller $\varphi(x,s)$ is proposed but it does not satisfy our requirement (3). Specifically, along the trajectories, the proposed control converges with time to the global one u_g. In [7], the solution is given in the form of a continuous static time-invariant controller $\varphi(x)$. It assumes the existence of a continuous path of stabilizing controllers between u_l and u_g. Unfortunately we show by means of an example that this assumption can be violated. Actually, for this particular

0-7803-5250-5/99/$10.00 © 1999 IEEE

example, there is no continuous (and even discontinuous) static time-invariant controller. This shows that dynamic extension may be necessary, this being via time variations, discrete or continuous state.

1.4 Class of controllers and notions of trajectories

The controllers under consideration in this paper admit the following description (see [9])

$$\begin{aligned} u &= \varphi(x, s_c, s_d, t) \\ \dot{s}_c &= k_c(x, s_c, s_d, t) \\ s_d &= k_d(x, s_c, s_d^-, t) \end{aligned} \tag{5}$$

where s_c evolves in $\mathbb{R}^p$ for some p and s_d in some finite set $\mathcal{F}$, the functions u and k_c are locally bounded, and s_d^- is defined as :

$$s_d^-(t) = \lim_{s \nearrow t} s_d(s) . \tag{6}$$

The above controller is

- dynamic with the presence of s_c and s_d,
- time varying due to the presence of t,
- hybrid due to the presence of the discrete dynamics of s_d.

It gives rise to a non classical ordinary differential equation describing the dynamics of the closed loop system. In particular this system is infinite dimensional since to evaluate $s_d^-(t)$ at time t, we need to know the past values of $s_d(t)$. As a consequence we have to make precise what we mean by trajectory. The most natural definition of trajectory is

Definition 1.1 *Given* (x, s_c, s_d, t_0) *in* $\mathbb{R}^n \times \mathbb{R}^p \times \mathcal{F} \times \mathbb{R}$, *a function* $(X(t), S_c(t), S_d(t))$ *defined on* $[t_0, t_0 + T)$ *is said to be a* classical trajectory *of*

$$\begin{aligned} \dot{x} &= f(x, \varphi(x, s_c, s_d, t)) \\ \dot{s}_c &= k_c(x, s_c, s_d, t) \\ s_d &= k_d(x, s_c, s_d^-, t) \end{aligned} \tag{7}$$

if

1. *X and S_c are absolutely continuous on $[t_0, t_0 + T)$ and, for each t in $[t_0, t_0+T)$, there exists $\varepsilon > 0$ such that S_d is constant on $[t, t + \varepsilon)$.*
2. *For almost all t in $[t_0, t_0 + T)$, we have*

$$\begin{aligned} \dot{X}(t) &= f(X(t), \varphi(X(t), S_c(t), S_d(t), t)) \\ \dot{S}_c(t) &= k_c(X(t), S_c(t), S_d(t), t) \end{aligned} \tag{8}$$

and, for all t in $(t_0, t_0 + T)$, we have[1] *:*

$$S_d(t) = k_d(X(t), S_c(t), S_d^-(t), t) . \tag{9}$$

3. *We have :*

$$(X(t_0), S_c(t_0), S_d(t_0)) = (x, s_c, s_d) . \tag{10}$$

[1] Note that we do not ask for (9) to hold at $t = t_0$.

To make the dependence on the initial condition more explicit, we denote

$$(X(x, s_c, s_d, t_0, t), S_c(x, s_c, s_d, t_0, t), S_d(x, s_c, s_d, t_0, t)).$$

Actually, we are interested in a notion of trajectories which is robust with respect to measurement noise. For this reason, we introduce a notion of generalized trajectory (see also [4, 3, 2]).

Definition 1.2 *Given* (x, s_c, s_d, t_0) *in* $\mathbb{R}^n \times \mathbb{R}^p \times \mathcal{F} \times \mathbb{R}$, *a function* $(X(t), S_c(t), S_d(t))$ *defined on* $[t_0, t_0 + T)$ *is said to be a* weak generalized trajectory *(resp. a* strong generalized trajectory*) of (7) if* $(X(t), S_c(t)) : [t_0, t_0 + T) \to \mathbb{R}^n \times \mathbb{R}^p$ *is continuous, and with* $S_d : [t_0, t_0+T) \to \mathcal{F}$, *we have*

$$(X(t_0), S_c(t_0), S_d(t_0)) = (x, s_c, s_d) , \tag{11}$$

and, for each $J = [\tau_0, \tau_1]$, *compact subinterval of* $[t_0, t_0 + T)$, *and each n in $\mathbb{N}$, we can find a function e_n in* $L^\infty_{loc}([t_0, t_0+T))$, *a point* (x_n, s_{cn}, s_{dn}) *in* $\mathbb{R}^n \times \mathbb{R}^p \times \mathcal{F}$, *and a classical trajectory*

$$(X_n(x_n, s_{cn}, s_{dn}, \tau_0, t), S_{cn}(x_n, s_{cn}, s_{dn}, \tau_0, t), S_{dn}(x_n, s_{cn}, s_{dn}, \tau_0, t))$$

of

$$\begin{aligned} \dot{x} &= f(x, \varphi(x + e_n(t), s_c, s_d, t)) \\ \dot{s}_c &= k_c(x + e_n(t), s_c, s_d, t) \\ s_d &= k_d(x + e_n(t), s_c, s_d^-, t) \end{aligned} \tag{12}$$

defined on a right open interval containing J and satisfying

$$\sup_J (X - X_n) + \sup_J (S_c - S_{cn}) + \sup_J (e_n) \le \frac{1}{n} \tag{13}$$

$$(\textit{resp.}\ \sup_J (X - X_n) + \sup_J (S_c - S_{cn}) + \operatorname{esssup}_J (e_n) \le \frac{1}{n})$$

and such that, for all t in J, there exists N, satisfying

$$S_{dn}(t) = S_d(t) \quad \forall n \ge N . \tag{14}$$

We denote by $\sup_J$ the bound of the function on J and by esssup_J the essential bound. In the above definition e_n, plays the role of a measurement noise on x which disturbs the control computation and a generalized trajectory is a limit, when the noise vanishes, of disturbed classical trajectories. (For other motivations for considering generalized trajectories, see [4, p.164-165].)

Of course a classical trajectory is a weak generalized trajectory and a weak generalized trajectory is a strong generalized trajectory.

In this paper, we make the distinction in solving the uniting problem considering only the classical trajectories or taking also into account the strong or weak generalized trajectories. Usually this distinction is not made since we have the following result[2] :

Theorem 1.3 *In the case without discrete dynamics (i.e. without s_d) and when k_c and φ are continuous, each strong generalized trajectory is a classical trajectory.*

The notion of global asymptotic stability for the generalized trajectories is the same as for the classical ones, see (4).

The problem of global asymptotic stabilization for weak generalized trajectories has been condidered per se in [5, 6]. There are strong connections with the problem of uniting local and global controllers we are considering here both in the technicalities and the results. In particular, as we shall see, we have also in our context the need for using an hybrid controller.

1.5 Organization of this paper

We first come back, in section 2.1, on the static time-invariant continuous controller proposed in [7] but we show, in section 2.2, a system to which it cannot be applied. In fact this system motivates us for looking at obstructions for solving the problem via (dis)continuous static time-invariant controllers (see section 2.3). A first way to round this obstruction is via dynamic hybrid control. This is done in section 3. We show that indeed the uniting problem can be solved in terms of weak generalized trajectories but unfortunately not in terms of strong generalized trajectories. So finally, in section 4, we propose a periodic static continuous controller solving the problem in its whole generality.

Due to space limitations, we cannot give the proofs. In particular, since discrete dynamics are present, the proof of Theorem 3.1 requires a whole machinery to handle generalized trajectories. These proofs can be found in the extended version of this paper [8].

2 Static time-invariant controllers

2.1 A solution to the uniting problem

Following the arguments and ideas of [7], we get :

Theorem 2.1 *Let Ω be a bounded open connected neighborhood of the origin in $\mathbb{R}^n$ and u_l and u_g be two continuous functions on $\mathbb{R}^n$. We assume the existence of*

- *$\psi : [0,1] \times \mathbb{R}^n \to \mathbb{R}^m$, a continuous path connecting u_l to u_g, i.e., for all x in $\mathbb{R}^n$, we have :*

$$\psi(0,x) = u_l(x) \quad , \quad \psi(1,x) = u_g(x) \, , \tag{15}$$

- *$V : [0,1] \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, a C^1 function which, for all $s \in [0,1]$, is positive definite and radially unbounded*
- *c a positive real number such that*

$$\Omega \subset \{x : V(0,x) < c\} \tag{16}$$

satisfying

1. *For each s in $[0,1]$ and each x in $\mathbb{R}^n \setminus \{0\}$, we have*

$$\frac{\partial V}{\partial x}(s,x) f(x, \psi(s,x)) \ < \ 0 \ , \tag{17}$$

2. *For each (s,x) satisfying $V(s,x) = c$, we have*

$$\frac{\partial V}{\partial s}(s,x) \ < \ 0 \ . \tag{18}$$

Under these conditions, we can find a locally Lipschitz function γ and a bounded closed set A such that with

$$\varphi(x) \ = \ \psi(\gamma(x), x) \tag{19}$$

we have a solution to the uniting problem in terms of strong generalized trajectories.

Remark 2.2

- In the above statement, the function γ is obtained as follows:
  - for x in C_l, we let $\gamma(x) = 0$,
  - for x in C_g, we let $\gamma(x) = 1$,
  - for x in A, we choose $\gamma(x)$ as the solution of

$$V(\gamma, x) \ = \ c \, , \tag{20}$$

  where

$$A = \{x : c \leq V(0,x), \ V(1,x) \leq c\} \ . \tag{21}$$
$$C_l = \{x : V(0,x) < c\} \tag{22}$$
$$C_g = \{x : c < V(1,x)\} \tag{23}$$

- With (17) holding for $s = 1$, we impose that $\mathcal{D}$ is actually $\mathbb{R}^n$. In fact this restriction is too strong. We need only that Ω be sufficiently small inside $\mathcal{D}$. This "sufficiently" is linked to the stability properties provided by u_g. Not to make our statement too complicated, we have preferred to impose $\mathcal{D} = \mathbb{R}^n$.

2.2 A topological obstruction

Theorem 2.1 provides a solution to the uniting problem via a static time-invariant continuous controller. We show in this section that we must not restrict our attention to only such kind of feedbacks.

Let the system be :

$$\left\{ \begin{array}{rcl} \dot{x} & = & -y^2 x \\ \dot{y} & = & u \end{array} \right. \tag{24}$$

[2] From [3], we have also:
In the case without discrete dynamics (i.e. without s_d) and when k_c and φ are locally bounded, each weak generalized trajectory is a a Krasovskii trajectory.

The data of the uniting problem we consider are

$$u_l = -y + x , \quad (25)$$
$$u_g = -y - x , \quad (26)$$
$$\Omega = \left\{(x,y) : x^2 + y^2 < \tfrac{1}{2}\right\} . \quad (27)$$

The fact that u_l and u_g are global asymptotic stabilizers can be checked with LaSalle's invariance Theorem and the Lyapunov function $2x^2 + y^4$.

Let A be any closed set which separates $\mathbb{R}^2$ into two connected open sets C_l and C_g with C_l containing Ω. There exists $0 < c_l < c_g$ such that

$$A \subsetneq \left\{(x,y) : c_l^2 \le x^2 + y^2 \le c_g^2\right\} . \quad (28)$$

Assume the existence of a static time-invariant continuous controller $\varphi(x,y)$ solving the uniting problem. Then we have

$$\begin{aligned} \varphi(x,y) &= -y + x \quad \text{if } x^2 + y^2 \le c_l^2 \\ &= -y - x \quad \text{if } c_g^2 \le x^2 + y^2 \end{aligned} \quad (29)$$

and in particular

$$\varphi(c_l, 0) = c_l \quad , \quad \varphi(c_g, 0) = -c_g . \quad (30)$$

Since c_l and c_g are positive, the continuity of u implies the existence of c, strictly positive, such that $\varphi(c,0) = 0$. It follows that $(c,0)$ is an equilibrium of the closed loop system contradicting the fact that $u(x,y)$ is globally asymptotically stabilizing the origin.

We have established that the conclusion of Theorem 2.1 does not hold. Its assumptions are violated. Actually the same argument as above shows that there is no continuous function $\psi(s,(x,y))$ connecting u_l to u_g and providing a globally asymptotically stabilizing controller for each s in $[0,1]$.

The obstruction observed with the system (24) leads to the following necessary condition for the solvability of the uniting problem via static time-invariant continuous feedback

Theorem 2.3 *Let (u_l, u_g, Ω) be the data of a uniting problem. If there exists a static time-invariant continuous control as a solution for this problem then there exists $0 < c_l < c_g$ such that the functions $\tilde{u}_l$ and $\tilde{u}_g$ below are homotopic*

$$\begin{aligned} \tilde{u}_i : \mathbb{S}^{n-1} \to \Sigma &:= \{(x,u),\ f(x,u) \neq 0\} \subset \mathbb{R}^n \times \mathbb{R}^m \\ \xi &\mapsto (c_i\xi, u_i(c_i\xi)) \end{aligned} \quad (31)$$

for $i \in \{l, g\}$.

The necessary condition given in this theorem, written in terms of homotopy, can also be expressed in terms of homology as in [1].

For the system (24), the set Σ is $\mathbb{R}^3$ without the x and y axis. The image of $\mathbb{S}^1$ by $\tilde{u}_l$ in $\mathbb{R}^3$ is an ellipsis in the plane given by $u+y-x=0$ whereas the one by $\tilde{u}_g$ is an ellipsis in the plane given by $u+y+x=0$. We can see that there is no continuous deformation allowing us to go from one ellipsis to the other one without crossing the x or y-axis. So the necessary condition is not met.

We conclude that the class of static time-invariant continuous controllers is not rich enough to address the uniting problem. Before investigating a richer class, we show that, in some cases, the class of static time-invariant discontinuous controllers is also not rich enough.

2.3 Obstruction for a solution with a discontinuous static time-invariant controller

For the case where f is affine in u, i.e.

$$f(x,u) = a(x) + \sum_{i=1}^{m} b_i(x) u_i \quad (32)$$

where a and b are locally Lipschitz, we have

Theorem 2.4 *Let (u_l, u_g, Ω) be the data of a uniting problem for f affine in u. If the uniting problem is solvable, in terms of weak generalized trajectories, with a locally bounded static time-invariant controller, then, for any bounded open connected set $\tilde{\Omega}$, neighborhood of the origin such that*

$$\mathtt{clos}(\tilde{\Omega}) \subset \Omega , \quad (33)$$

the uniting problem, with data $(u_l, u_g, \tilde{\Omega})$, is also solvable in terms of strong generalized trajectories by a static time-invariant continuous controller.

This Theorem implies that, when f is affine in u, if the uniting problem cannot be solved with a continuous static time-invariant controller, it cannot be solved either with a discontinuous static time-invariant controller.

3 Dynamic time-invariant controller with Hysteresis

3.1 A solution to the uniting problem

A very natural way to overcome the difficulties encountered with static time-invariant continuous or discontinuous controllers is to introduce hysteresis taking advantage of the existence of a region where both controllers u_l and u_g are appropriate.

Theorem 3.1 *Let (u_l, u_g, Ω) be the data of a uniting problem. There exists an appropriate bounded closed set A such that the controller below solves the uniting problem in terms of weak generalized trajectories*

$$\begin{aligned} u &= \varphi(x, s_d) \\ s_d &= k_d(x, s_d^-) \end{aligned} \quad (34)$$

where s_d is in $\{0,1\}$ and the functions φ and k_d satisfy

$$\begin{aligned} \varphi(x,0) &= u_l(x) \quad \text{if } x \in \mathtt{clso}(C_l) , \\ \varphi(x,1) &= u_g(x) \quad \text{if } x \in \mathbb{R}^n . \end{aligned} \tag{35}$$

and

$$\begin{aligned} k_d(x, s_d) &= 0 \quad \text{if } x \in \mathtt{clos}(C_l) , \\ &= s_d \quad \text{if } x \in \mathtt{int}(A) , \\ &= 1 \quad \text{if } x \in \mathtt{clos}(C_g) . \end{aligned} \tag{36}$$

3.2 A problem with strong generalized trajectories

With the fact that, with strong generalized trajectories, noise with very large amplitude is allowed, Theorem 3.1 is not true. Precisely, for i in $\{l,g\}$, let X_i denote the solution of :

$$\frac{\partial X_i(x,t)}{\partial t} = f(X_i(x,t), u_i(X_i(x,t))) \ , \ X_i(x,0) = x \ . \tag{37}$$

We have :

Theorem 3.2 *Let A be the compact set and (φ, k_d) be the controller given by Theorem 3.1 as a solution to the uniting problem. If there exist a strictly positive real number ε and two compact sets E_l and E_g, subsets of A, such that, for all x in E_l (resp. E_g), there exists $\tau_x \geq \varepsilon$ such that*

$$X_l(x,\tau_x) \in E_g \qquad (resp. X_g(x,\tau_x) \in E_l) \ , \tag{38}$$
$$X_l(x,t) \ (resp. X_g(x,t)) \ \in \mathtt{int}(A) \quad \forall t \in [0,\tau_x] \tag{39}$$

then $(A, (\varphi, k_d))$ does not solve the uniting problem in terms of strong generalized trajectories.

Let us illustrate Theorem 3.2 by considering the following system in $\mathbb{R}^2$:

$$(\dot{x}, \dot{y}) = u(x,y) \tag{40}$$

The feedback $u_l(x,y) = -(x,y)$ makes the closed loop system globally asymptotically stable. Moreover the following trajectory defined on $[0,+\infty)$ is a trajectory of the closed loop system (40) with $u = u_l$:

$$\forall t \geq 0 \ , \quad (x(t), y(t)) = (2exp(-t), 0) \tag{41}$$

Let F be the closed set

$$F = \{(x,0) \ : \ x \in [1,2]\} \ . \tag{42}$$

Let s be a $\mathcal{C}^\infty$ function on $\mathbb{R}^2$ such that

$$\begin{aligned} s(x,y) &= 1 \quad \text{if} \quad (x,y) \in F \ , &(43) \\ &= 0 \quad \text{if} \quad d((x,y),F) \geq \frac{1}{2} \ . &(44) \end{aligned}$$

Let θ be the function defined, for all $(x,y) \in \mathbb{R}^2$, by :

$$\theta(x,y) = s(x,y)\,\pi \tag{45}$$

Let

$$u_g(x,y) = \mathcal{R}((x,y), \theta(x,y))\, u_l(x,y) \tag{46}$$

where, for all (x,y) in $\mathbb{R}^2$ and for all θ in $[0,2\pi]$, $\mathcal{R}((x,y),\theta)$ denotes the rotation with center (x,y) and angle θ. One can check that u_g makes the origin of (40) globally asymptotically stable.

Let us prove now that the hypothesis of Theorem 3.2 are verified by taking :

$$E_l = \{(2,0)\}, \quad E_g = \{(1,0)\} \ . \tag{47}$$

We note that, for all x in $[1,2]$, we have simply

$$\begin{aligned} u_g(x,0) &= \mathcal{R}((x,0), \theta(x,0))\, u_l(x,0) &(48) \\ &= -u_l(x,0) \ = \ (x,0) \ . &(49) \end{aligned}$$

This implies that, with u_g (resp. u_l), the trajectory through $(1,0)$ (resp. $(2,0)$) is :

$$\forall t \in [0, \log(2)] \ , \ (x(t), y(t)) = (exp(t), 0) \qquad (\text{resp.} \ = (2exp(-t), 0)) \ . \tag{50}$$

So (38) and (39) hold. Hence, the controller given by (34) does not solve the uniting problem in terms of strong generalized trajectories for all closed set $A \subset \mathbb{R}^2$ which separates $\mathbb{R}^2$ in two connected open sets C_l and C_g and such that we have :

$$F \subset A \ . \tag{51}$$

4 Static periodic continuous controller

Instead of enriching the class of controllers with non smooth components, we state here that it is sufficient to introduce time-dependence.

Theorem 4.1 *Let (u_l, u_g, Ω) be the data of the uniting problem. Suppose the existence of two bounded open connected sets Γ_l and Γ_g such that :*

- $\Omega \subset \Gamma_l \subset \Gamma_g \subset \mathcal{D}$
- $\mathtt{clos}(\Gamma_l) \subset \Gamma_g$
- *Γ_l (resp. Γ_g) is stable and attractive for the controller u_l (resp. u_g)*

Under these conditions, we can find an appropriate bounded closed set A and a continuous time-periodic function φ such that the controller $u = \varphi(x,t)$ solves the uniting problem in terms of strong generalized trajectories.

Remark 4.2 The controller mentioned is the statement above can be obtained as follows :
Let

$$\begin{aligned} C_l &= \Gamma_l \ , &(52) \\ C_g &= \mathtt{int}(\mathbb{R}^n \setminus \Gamma_g) \ , &(53) \\ A &= \mathbb{R}^n \setminus (C_l \cup C_g) \ . &(54) \end{aligned}$$

Let u_- and u_+ be any continuous functions from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that :

$$\begin{aligned} u_-(x) &= u_g(x) \quad \text{if } x \in \mathbb{R}^n \setminus \Gamma_g \,, \\ &= u_l(x) \quad \text{if } x \in \Sigma_g \,. \end{aligned} \tag{55}$$

$$\begin{aligned} u_+(x) &= u_g(x) \quad \text{if } x \in \mathbb{R}^n \setminus \Sigma_l \,, \\ &= u_l(x) \quad \text{if } x \in C_l \,. \end{aligned} \tag{56}$$

Let τ_i be the real numbers defined as

$$\tau_g = \max_{\{x \in \Gamma_g \setminus \Sigma_l\}} \min_{\{s \,:\, X_g(x,t) \in \Sigma_l, \forall t > s\}} \{s\} \tag{57}$$

$$\tau_l = \max_{\{x \in \Sigma_g \setminus C_l\}} \min_{\{s \,:\, X_g(x,t) \in C_l, \forall t > s\}} \{s\} \tag{58}$$

where X_g (resp. X_l) is a trajectory of $\dot{x} = f(x, u_g(x))$ (resp. $\dot{x} = f(x, u_l(x))$). Let $\mathcal{E}$ be the compact set obtained by collecting the pairs (x, u), with $x \in \mathtt{clos}\,(\Sigma_g \setminus \Sigma_l)$ and u in the closed segment with end points $u_l(x)$ and $u_g(x)$. We define τ_c as follows :

$$M = \sup_{(x,u) \in \mathcal{E}} \|f(x,u)\| \,, \tag{59}$$

$$\tau_c = \frac{\mathtt{dist}(\mathtt{clos}(\Sigma_l), \mathbb{R}^n \setminus \Sigma_g)}{2M} \,. \tag{60}$$

With these notations, we choose τ as any real number satisfying

$$\tau > (\tau_g + 2\tau_c + \tau_l) \,. \tag{61}$$

Let γ be a τ-periodic C^∞ function with value 1 on $[0, \tau_g]$ and on $[\tau_g + 2\tau_c + \tau_l, \tau]$, and 0 on $[\tau_g + \tau_c, \tau_g + \tau_c + \tau_l]$. We define the controller as

$$\varphi(x,t) = \gamma(t) u_+(x) + (1 - \gamma(t)) u_-(x) \,. \tag{62}$$

References

[1] J.-M. Coron. A necessary condition for feedback stabilization. *Systems and Control Letters*, 14:98–105, 1990.

[2] J.-M. Coron and L. Rosier. A relation between continuous time-varying and discontinuous feedback stabilization. *J. Math. Syst., Est., and Cont.*, 4:67–84, 1994.

[3] O. Hájek, Discontinuous differential equations, part I. J. Diff. Equations 32 (1979) 149-170.

[4] H. Hermes. *Discontinuous vector fields and feedback control.* Differential Equations and Dynamic Systems. J. K. Hale and J.P. La Salle, Academic Press, New York and London, 1967.

[5] Y.S. Ledyaev, E.D. Sontag, A Lyapunov characterization of robust stabilization. Nonlinear Analysis 37(1999): 813-840.

[6] Y.S. Ledyaev, E.D. Sontag, A remark on robust stabilization of general asymptotically controllable systems. Proc. Conf. on Information Sciences and Systems (CISS 97), Johns Hopkins, Baltimore, MD, March 1997, pp. 246-251.

[7] P. Morin, R.M. Murray, L. Praly, Nonlinear Rescaling of Control Laws with Application to Stabilization in the Presence of Magnitude Saturation *NOLCOS'98*, July 1998.

[8] C. Prieur, L. Praly, Uniting Local and Global Controllers with Robustness to Vanishing Measurement Noise. In preparation, July 1999.

[9] L. Tavernini. Differential automata and their discrete simulators. *Nonlin. An., Th., Meth., App.*, 11(6):665–683, 1997.

[10] A. R. Teel and N. Kapoor. Uniting local and global controllers. *Proceedings of the European Control Conference,* Brussels 1997.