GROWTH RATE CONDITIONS FOR GLOBAL ASYMPTOTIC STABILITY OF CASCADED SYSTEMS

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Abstract: In the last decade several results on the stabilization of cascaded autonomous systems have appeared in the literature. The sufficient conditions for the stability or stabilizability of these systems are often related to certain growth rate conditions on the functions which define the dynamics of the system. In this note we analyze three complementary classes of nonlinear autonomous systems according to the growth rates of the functions which determine their motion. For each case, we give further results to guarantee global asymptotic stability (GAS) of the cascade and relate our contributions to other important concepts such as Input-to-State Stability (ISS) and growth rate conditions previously reported. *Copyright © 1998 IFAC*

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Notation. In this paper the solution of a differential equation $\dot{x} = f(x)$ where $f : \mathbb{R}^n \to \mathbb{R}^n$, with initial conditions $x_0 \in \mathbb{R}^n$ with $x_0 = x(0)$, is denoted $x(t; x_0)$ or simply $x(t)$. We say that the system $\dot{x} = f(x)$, is globally asymptotically stable (in short GAS) if the trivial solution $x(t; x_0) = 0$ is GAS. A continuous function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{K}$, $\alpha \in \mathcal{K}$, if $\alpha(x)$ is strictly increasing and $\alpha(0) = 0$; $\alpha \in \mathcal{K}_\infty$ if in addition $\alpha(x) \to \infty$ as $x \to \infty$. A continuous function $\beta(t, x) : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if $\beta(t, \cdot) \in \mathcal{K}$ for each fixed $t \geq 0$ and $\beta(t, x) \to 0$ as $t \to \infty$ for each $x \geq 0$. Unless otherwise specified we use in general the letter c to denote a positive constant. Functions and constants may also carry a subindex to distinguish them from each other when necessary. $\|\cdot\|$ denotes the Euclidean norm. $\dot{V}_{(\#)}(x)$ is the time derivative of Lyapunov function $V(x)$ along the trajectories represented by the differential equation ($\#$). In occasions we use the compact notation $V(x(t)) = V(t)$.

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1 Introduction

1.1 Motivation

We consider in this paper systems with a so-called cascaded structure:

$$\Sigma_1' : \dot{x}_1 = f_1(x_1, x_2) \tag{1}$$
$$\Sigma_2' : \dot{x}_2 = f_2(x_2, u) \tag{2}$$

where $x_1 \in \mathbb{R}^n$, $x_2 \in \mathbb{R}^m$, $u \in \mathbb{R}^m$ and the functions $f_1(\cdot)$, $f_2(\cdot)$ are continuously differentiable in their arguments (or particular cases of these). These systems may appear in practical applications, for instance when one has two (closed loop) systems which separately, have some stability properties. For instance, one may think about the coupling of two different mechanical systems as it is the case of the global trajectory tracking problem of robot manipulators driven by AC motors analyzed in [5].

Other interesting practical examples that are worth mentioning concern the orientation stabilization of an spacecraft with only two controls [10] and the controlled synchronization problem of two oscillating pendula [2]. In both cases the authors of the respective references showed that a suitable change of coordinates transforms the original dynamic systems into cascaded systems like Σ_1', Σ_2'. Then the stabilization task is considerably simplified to design a control u for the system Σ_2'.

The practical problems exposed above show that the global stabilization of (1), (2) may be (sometimes) achieved by addressing two simpler problems: to ensure GAS for both subsystems separately. The only question remaining is then to know whether the stability properties of both subsystems separately, will remain valid under a cascaded interconnection as (1), (2). The latter motivates us to study the following stability *analysis* problem: to find sufficient conditions under which the cascaded system (1), (2) is GAS under the assumption that the zero-input dynamics of the *perturbed* system Σ_1', i.e.,

$$\dot{x}_1 = f_1(x_1, 0) \tag{3}$$

is GAS, and that there exists $u = u(x_2)$ such that the *perturbing* system Σ_2' is also GAS. We briefly mention below some important results in this direction.

1.2 Literature review

The stability analysis problem was addressed for instance in [9] where the author used the "Converging Input - Bounded State" (CIBS) property:

CIBS: For each input $x_2(\cdot)$ on $[0, \infty)$ such that $\lim_{t\to\infty} x_2(t) = 0$ and for each initial state x_{1_0}, the solution of (1) with $x_1(0) = x_{1_0}$ exists for all $t \geq 0$ and it is bounded

to prove that the cascade system is GAS if both subsystems are GAS and CIBS holds. Also, based on Krasovskii-LaSalle's invariance principle, the authors of [7] showed that the composite system is GAS assuming that all solutions are bounded (in short, BS) and that both subsystems (2) and (3) are GAS, in short we have [7]:

Fact 1: $\boxed{\text{GAS} + \text{GAS} + \text{BS} \Rightarrow \text{GAS.}}$

For *autonomous* systems the Fact 1 is a fundamental result which is now well known and has been used by many authors to prove GAS of the cascade (1), (2). The natural essential question which arises then is "how to guarantee boundedness of the solutions?".

One way is to use the now well known property of Input-to-State stability (ISS) introduced in [8] by Sontag. Unfortunately, proving the ISS property as a condition to imply CIBS may appear in some cases very restrictive, for instance consider the one-dimensional system

$$\dot{x}_1 = -x_1 + x_1 x_2 \tag{4}$$

which is not ISS with respect to input $x_2 \in \mathbb{R}$.

Concerned by the control design problem, i.e. to stabilize the cascaded system Σ_1', Σ_2' by using feedback of the state x_2 only, the authors of [6] studied the particular case when Σ_2' is a linear controllable system. Under the assumption that $f_1(x_1, x_2)$ in (1) is continuously differentiable in its arguments, (1) can be rewritten as

$$\dot{x}_1 = F_1(x_1) + g(x_1, x_2)x_2, \tag{5}$$

in [6] the authors introduced the following *linear* growth condition

$$\|g(x_1, x_2)x_2\| \leq \theta(\|x_2\|)\,\|x_1\| \tag{6}$$

where θ is $\mathcal{C}^1$, $\theta(0) = 0$, together with other conditions to prove boundedness of the solutions. Using such a condition one can deal with systems which are not ISS such as (4).

From these examples one may conjecture that, in order to prove CIBS for system (5) with decaying input $x_2(t)$, some growth restrictions should be imposed on functions $F_1(x_1)$ and $g(x)$. More precisely, for the NL system (5) one may use for instance a linear growth condition such as (6) or the ISS property, which seemingly "needs" that function $F_1(x_1)$ grow faster than $g(x_1, x_2)$ uniformly in x_2 (that is, when considered as a function of x_1 only).

More recent results are [3] and [1] where the authors addressed the problem of global stabilizability of the related so called feedforward systems, by a systematic recursive design procedure which leads to the construction of a Lyapunov function for the complete system. Even though the design procedures differ in both references, as mentioned before a common starting point (implicitly or explicitly) is the stability analysis of cascaded systems as a particular case of systems with feedforward structure. In this respect, it is worth mentioning that the authors of [1] used the linear growth restriction

$$\|g(x_1, x_2)x_2\| \leq \theta_1(\|x_2\|)\,\|x_1\| + \theta_2(\|x_2\|) \tag{7}$$

where $\theta_1(\cdot)$, $\theta_2(\cdot)$ are $\mathcal{C}^1$ and $\theta_i(0) = 0$, together with the growth rate condition on the Lyapunov function $V(x_1)$ for the zero-dynamics (3), $\left\|\frac{\partial V}{\partial x_1}\right\| \|x_1\| \leq cV$ for $\|x_1\| \geq c_2$ (which holds e.g. for all polynomial $V(x_1)$) to prove that all solutions remain bounded under the cascaded interconnection. In contrast to this, in [3] the authors introduced the assumption on the existence of continuous nonnegative functions $\rho(V)$, and $\kappa : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$, $\kappa_2 \in \mathcal{L}_1$, such that $\left|\frac{\partial V}{\partial x_1} \cdot g(x)x_2\right| \leq \kappa(x_2)[1 + \rho(V)]$ and $\frac{1}{1+\rho(V)} \notin \mathcal{L}_1$. The latter condition being less restrictive than those used in [1].

1.3 Problem formulation

We consider the stability *analysis* problem for the time-invariant system

$$\begin{aligned} \Sigma_1 : \dot{x}_1 &= f_1(x_1) + g(x)x_2 \quad (8)\\ \Sigma_2 : \dot{x}_2 &= f_2(x_2) \quad (9) \end{aligned}$$

where $x_1 \in \mathbb{R}^n$, $x_2 \in \mathbb{R}^m$, $x \triangleq \text{col}[x_1, x_2]$. The function $f_1(x_1)$ is continuously differentiable in (x_1) and $f_2(x_2)$, $g(x)$ are continuous in their arguments, and locally Lipschitz.

Roughly speaking, we analyze three cases: (i) The function $f_1(x_1)$ grows faster than $g(x)$ in x_1, (ii) functions $f_1(x_1)$ and $g(x)$ grow at similar rate with respect to x_1 and finally, (iii) function $g(x)$ grows faster than $f_1(x_1)$ as functions of x_1. For each case, we give sufficient conditions to guarantee that a GAS nonlinear system

$$\dot{x}_1 = f_1(x_1) \tag{10}$$

remains GAS when it is perturbed by the output of another GAS system of the form Σ_2.

Our main results relate the conditions described above which appear to be only sufficient to imply boundedness, such as the concept of ISS, and the linear growth conditions on $g(x)$. Thus, in this note we consider systems for which the ISS, or a linear growth condition on $g(x)$ or none of these conditions can be verified.

The rest of this paper is organized as follows: in the next section we enunciate our main results and in section 3, we provide the proofs of our claims. We conclude with some remarks in section 4.

2 Main results

2.1 Preliminary assumptions

Since we consider here cascades with zero-input dynamics (10) to be GAS, from the converse Lyapunov theorems we know that there exists for this system, a Lyapunov function $V(x_1)$. Our first general assumption has to do with the growth rate of this function.

A 1 System $\dot{x}_1 = f_1(x_1)$ is GAS with a Lyapunov function $V(x_1)$, $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ which satisfies the following: there exist some class $\mathcal{K}$ function $\alpha_3(\cdot)$, functions $\alpha_1(\cdot)$, $\alpha_2(\cdot) \in \mathcal{K}_\infty$ and a continuous non-decreasing function $\alpha_4(\cdot)$ such that

$$\alpha_1(\|x_1\|) \leq V(x_1) \leq \alpha_2(\|x_1\|) \tag{11}$$

$$\dot{V}_{(10)}(x_1) \leq -\alpha_3(\|x_1\|) \tag{12}$$

$$\left\| \frac{\partial V}{\partial x_1} \right\| \leq \alpha_4(\|x_1\|). \tag{13}$$

It is worth mentioning that the first part of Assumption **A1**, i.e., the existence of $V(x_1)$ such that (11) and (12) are satisfied, follows from the assumption that (10) is GAS [8].

A 2 System Σ_2 is GAS.

We find it convenient to this point to stress some direct consequences of Assumption **A2** in order to introduce some notation we will use in the sequel. First, the fact that Σ_2 is GAS means that there exists a function β of class $\mathcal{KL}$ such that

$$\|x_2(t; x_{2_0})\| \leq \beta(t, \|x_{2_0}\|), \quad \forall\, t \geq 0, \tag{14}$$

and since (14) holds for all t and $\beta(t, \cdot)$ is decreasing then

$$\|x_2(t; x_{2_0})\| \leq c \triangleq \beta(0, \|x_{2_0}\|), \tag{15}$$

Then under Assumption **A2** it makes sense to say that there exist continuous non-decreasing functions $\gamma : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$ and $\gamma_2 : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$ such that for each bounded signal $x_2(t)$

$$\|g(x)\| \leq \gamma(\|x_2\|)\alpha_5(\|x_1\|) \leq c_g \alpha_5(\|x_1\|). \tag{16}$$

where the constant $c_g = c_g(x_{2_0})$. Under these assumptions we consider three different cases according to the growth rates of $f_1(x_1)$ and $g(x)$ taken as functions of x_1 only, that is, uniformly in x_2. Hence in the sequel, when speaking about growth rate of $f_1(x_1)$ and $g(x)$, it shall be understood in that sense.

For the sake of clarity we cite below a definition of *growth order* borrowed from [4]. It is worth remarking that for the purposes of this paper we have modified the original definition by considering continuous functions instead of piecewise continuous.

Definition 1 (small 'o'). *Let $f(x)$, $g(x)$ be continuous. We denote $g(x) = o(f(x))$ if there exists a continuous function $\lambda : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $\|g(x)\| = \lambda(\|x\|)\,\|f(x)\|$ for all $x \in \mathbb{R}^n$ and $\lim_{\|x\| \to \infty} \lambda(\|x\|) = 0$.*

A direct consequence of the definition above is that

$$\lim_{\|x\| \to \infty} \frac{\|g(x)\|}{\|f(x)\|} = 0.$$

In the sequel we will say that a function $f(x)$ "grows faster than" $g(x)$ or $g(x)$ is "small order of" $f(x)$ if $g(x) = o(f(x))$.

We introduce one more order relation between functions $f(x)$ and $g(x)$.

Definition 2 *Let $f(x)$, $g(x)$ be continuous. We say that function $f(x)$ majorates function $g(x)$ if*

$$\overline{\lim_{\|x\| \to \infty}} \frac{\|g(x)\|}{\|f(x)\|} < +\infty.$$

Notice that as a consequence of the definition above, it holds true that there exist finite positive constants η and λ such that

$$\|x\| \geq \eta \;\Rightarrow\; \frac{\|g(x)\|}{\|f(x)\|} \leq \lambda. \tag{17}$$

2.2 Case 1: "Function $f_1(x_1)$ grows faster than $g(x)$"

This is the simplest case, as it will become clear later the class of systems which fall in this category are ISS.

Theorem 1 . *If Assumptions* **A1** *and* **A2** *hold and*

A3 the function $g(x)$ satisfies (uniformly in x_2)

$$\|g(x)\| = o\left(\frac{\alpha_3(\|x_1\|)}{\alpha_4(\|x_1\|)}\right), \text{ as } \|x_1\| \to \infty \quad (18)$$

where $\alpha_3(\|x_1\|)$ and $\alpha_4(\|x_1\|)$ are defined in **A1**,

then the cascade (8), (9) is GAS. □

Remark 1 If for a particular system we have $\alpha_3(\|x_1\|) = \left\|\frac{\partial V}{\partial x_1}\right\| \|f_1(x_1)\|$ and $\alpha_4 = \left\|\frac{\partial V}{\partial x_1}\right\|$ then condition (18) reads simply $g(x) = o(f_1(x_1))$ however, it must be understood that in general, such relation of order between functions $f_1(x_1)$ and $g(x)$ is not implied by condition (18). This motivates the use of "brackets" in the phrase "Function $f_1(x_1)$ grows faster than $g(x)$".

Remark 2 It is important to notice that the functions $\alpha_3(\|x_1\|)$ and $\alpha_4(\|x_1\|)$ depend on the choice of the Lyapunov function $V(x_1)$ for system (10). However, note that if V is replaced by $\rho(V)$, $\rho \in \mathcal{K}_\infty$, then the ratio $\alpha_3(\|x_1\|)/\alpha_4(\|x_1\|)$ does not change. This proves that as far as V is concerned, we have an assumption on the shape of the level set not on its value

The theorem above allows us to deal with systems which are ISS but which do not necessarily satisfy a linearity condition as considered in some of the references mentioned in the introduction. In other words, if $\|f_1(x_1)\|$ grows faster than linearly, then Assumption **A3** allows to deal with interconnection terms which grow faster than linearly in the variable x_1.

Example 1 [10] In this reference Sontag showed that by a suitable change of coordinates, the orientation dynamics of a rigid spacecraft with only two controls has a cascaded structure and moreover, is an ISS system. It is easy to see as well that this example fits into the class of systems considered in this first case.

2.3 Case 2: "Function $f_1(x_1)$ majorates $g(x)$"

In order to present our main result for this case we need to introduce an additional assumption on the growth rates of $V(x_1)$ and $g(x)$.

A4 There exists a continuous non-decreasing function $\alpha_6 : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$, such that

$$\alpha_6(V) \geq \alpha_4(\alpha_1^{-1}(V))\gamma_2(\alpha_1^{-1}(V)) \quad (19)$$

where γ_2 is defined in (16) and

$$\int_a^\infty \frac{dV}{\alpha_6(V)} = \infty \quad (20)$$

for some $a > 0$.

Notice that Assumption **A4** imposes a particular relation between the growth rates of functions $V(x_1)$ and $g(x)$ with respect to the variable x_1. As it will become clear from the proof, the condition established by equality (20) guarantees (considering that the "inputs" $x_2(t)$ are continuous on $[0, \infty)$) that the solutions of the overall cascaded system $x(t, x_0)$ do not escape in finite time. This assumption is equivalent to the hypothesis of [3] mentioned in the Introduction, on the existence of a nonnegative function $\rho(V)$, such that $\frac{1}{1+\rho(V)} \notin \mathcal{L}_1$.

Theorem 2 . *If Assumptions* **A1**, **A2** *and* **A4** *hold and*

A5 function $g(x)$ is majorated by the function $\frac{\alpha_3(\|x_1\|)}{\alpha_4(\|x_1\|)}$, that is, there exist positive constants λ and η s.t.

$$\|g(x)\| \frac{\alpha_4(\|x_1\|)}{\alpha_3(\|x_1\|)} \leq \lambda \text{ for } \|x_1\| \geq \eta \quad (21)$$

where $\alpha_3(\|x_1\|)$ and $\alpha_4(\|x_1\|)$ are defined in **A1**.

then the cascade (8), (9) is GAS. □

Example 2 System (4) clearly satisfies Assumptions **A1** and **A5** with a quadratic Lyapunov function $V = \frac{1}{2}x_1^2$, $\alpha_3 = x_1^2$ and $\alpha_4 = |x_1|$. Assumption **A4** is also satisfied with $\gamma_2 = |x_1|$, $\alpha_1 = \frac{1}{2}x_1^2$ and from (19), with $\alpha_6(V) = 4V$.

It is also worth remarking that the practical problems of tracking control of robot manipulators with induction motors [5] and controlled synchronization of two pendula [2] which were mentioned in the introduction fit into the class of systems considered in Theorem 2. It shall be mentioned however that the systems considered in the latter references are *non-autonomous*.

2.4 Case 3: "Function $f_1(x_1)$ grows slower than $g(x)$"

Theorem 3 . *If Assumptions* **A1**, **A2** *and* **A4** *hold and*

A6 the input $x_2(t)$ satisfies

$$\int_0^\infty \|x_2(t; x_{2_0})\| \, dt \leq c_1 \quad (22)$$

then the cascade (8), (9) is GAS. □

The example below illustrates the kind of systems which one can deal with using Theorem 3. From the proof it may be clear that, roughly speaking on one hand function $g(x)$ may grow slightly faster than linearly and on the other hand function $f_1(x_1)$ can grow slower than linearly, as a matter of fact it can be uniformly bounded or even decreasing.

Example 3 Let us define the saturation function $\text{sat}(\zeta) : \mathbb{R} \to \mathbb{R}$ as a $\mathcal{C}^2$ non-decreasing function that satisfies $\text{sat}(0) = 0$, $\text{sat}(\zeta)\zeta > 0$ for all $\zeta \neq 0$ and $|\text{sat}(\zeta)| < 1$. For instance, we can take $\text{sat}(\zeta) \triangleq \tanh(\omega\zeta)$, $\omega > 0$, or $\text{sat}(\zeta) = \frac{\zeta}{1+\zeta^p}$ with p any multiple of 2. Then consider the system

$$\begin{aligned} \dot{x}_1 &= -\text{sat}(x_1) + x_1 \ln(|x_1| + 1)x_2 && (23) \\ \dot{x}_2 &= f_2(x_2) && (24) \end{aligned}$$

where $x_1 \in \mathbb{R}$ and the system $\dot{x}_2 = f_2(x_2)$ is GAS and satisfies **A6**. The zero input dynamics of (23), $\dot{x}_1 = -\text{sat}(x_1)$, is GAS with Lyapunov function $V = \frac{1}{2}x_1^2$ hence let $\alpha_1(\|x_1\|) = \frac{1}{2}x_1^2$ and $\alpha_4(\|x_1\|) = |x_1|$, while function $\alpha_5(\|x_1\|) = |x_1| \ln(|x_1| + 1)$. With $\alpha_6(V) = [\ln(\sqrt{2V} + 1) + 1](2V + \sqrt{2V})$ it is easy to verify that condition (20) holds.

It is worth remarking in the example above that, even though the coupling term $g(x)$ grows faster than linearly in x_1 and the zero input dynamics is *saturated*, GAS for the cascade can still be ensured.

Remark 3 For the sake of clarity, we assumed so far that the state x_2 can be factored out of the interconnection term in (8) as $g(x)x_2$. Even though this is not restrictive under smoothness assumptions, in some particular cases it may appear more reasonable to consider a dynamic system of the form $\dot{x}_1 = f_1(x_1) + g(x)$ (see for instance [2]). Hence redefining with an abuse of notation $g(x)x_2 = g(x)$ in (8) and replacing $g(x)$ by $\alpha_5(\|x_1\|)$ in (18) and (21) the proofs of Theorems 1 and 2 follow *mutatis mutandi* as shown below. Concerning Theorem 3, by imposing the integrability assumption (22) on $\gamma(\|x_2(t)\|)$ a similar result follows.

3 Proofs of main results

The proofs of our main results follow by showing that the assumptions of our theorems are sufficient to imply global boundedness of the solutions thus, under Assumptions **A1** and **A2** our claims follow from Fact 1.

3.1 Proof of Theorem 1

The time derivative of $V(x_1)$ along the trajectories of (8) is given by

$$\dot{V}_{(8)}(x) = \frac{\partial V}{\partial x_1} \cdot f_1(x_1) + \frac{\partial V}{\partial x_1} \cdot g(x)x_2 \qquad (25)$$

where the first term on the right hand side corresponds to $\dot{V}_{(10)}(x_1) \leq -\alpha_3(\|x_1\|) \leq 0$, hence

$$\dot{V}_{(8)}(x) \leq -\alpha_3(\|x_1\|) + \left\| \frac{\partial V}{\partial x_1} \right\| \|g(x)\| \, \|x_2\|, \qquad (26)$$

assuming $\|x_1\| > 0$ and using (12) and (13) we obtain

$$\dot{V}_{(8)}(x) \leq -\alpha_3(\|x_1\|) \left[1 - \frac{\alpha_4(\|x_1\|)}{\alpha_3(\|x_1\|)} \|g(x)\| \, \|x_2\| \right]. \qquad (27)$$

On the other hand, from Assumption **A3** it holds that for any $\varepsilon > 0$ there exists an $\eta = \eta(\varepsilon) > 0$ such that

$$\|x_1\| \geq \eta \;\Rightarrow\; \left\{ \frac{\alpha_4(\|x_1\|)}{\alpha_3(\|x_1\|)} \|g(x)\| < \varepsilon \right\}. \qquad (28)$$

Since ε can be arbitrarily small let $\varepsilon < 1/c$ where c is defined in (15) and pick $\eta^* > 0$ for this ε such that (28) holds. From this and using (27) we obtain that $\dot{V}_{(8)}(x) \leq -c_1\alpha_3(\|x_1\|)$ for all $\|x_1\| \geq \eta^*$ and with some $0 < c_1 < 1$ hence we conclude that $x_1(t; x_0)$ is globally bounded, that is there exists a positive constant $c_2 = c_2(x_0)$ such that $\|x_1(t)\| \leq c_2$ for all $t \geq 0$ (see [11]). GAS follows from Assumptions **A1**, **A2** and Fact 1. ■

3.2 Proof of Theorem 2

We start our proof by showing that the system is complete, i.e., we prove that the solutions $x_1(t; x_0)$ exist and are well defined for all $t \geq 0$. The time derivative of $V(x_1)$ along the trajectories of (8) is bounded by (26) then using (13) and (16) we can write

$$\dot{V}_{(8)}(x) \leq \alpha_4(\|x_1\|)\alpha_5(\|x_1\|)\gamma(\|x_2\|) \, \|x_2\|. \qquad (29)$$

Denoting $\bar{\gamma}(t, x_{2_0}) = \gamma(\beta(t, x_{2_0}))\beta(t, x_{2_0})$ we obtain that

$$\dot{V}_{(8)} \leq \bar{\gamma}(t)\alpha_6(V). \qquad (30)$$

Consider now the function

$$V_{new}(x_1) = \int_a^{\max\{V(x_1), a\}} \frac{dv}{\alpha_6(v)}$$

with $a > 0$ defined in Assumption **A4**, then $V_{new}(x_1)$ is radially unbounded. Using (30) we obtain that

$$V_{new}(x_1(t)) - V_{new}(x_1(0)) \leq \int_0^t \bar{\gamma}(s)ds$$

that is, $V_{new}(x_1(t))$ is bounded for all bounded t hence the system is complete.

We prove next that the solutions $x_1(t)$ are globally bounded. For this consider the constant η defined in Assumption **A5** and let us divide the span of time $[0, \infty)$ where the solutions $x_1(t)$ are defined, into a sequence of intervals during which either $\|x_1(t)\| \leq \eta$ or $\|x_1(t)\| > \eta$. More precisely, let $\Delta t_l \triangleq [t_1^l, t_2^l]$ with $t_2^l \geq t_1^l \geq 0$ be any interval such that $\|x_1(t)\| > \eta$ for all $t \in \Delta t_l$. Complementary, let $\Delta t_s \triangleq [t_1^s, t_2^s]$ with $t_2^s \geq t_1^s \geq 0$ be any interval such that $\|x_1(t)\| \leq \eta$ for all $t \in \Delta t_s$. Clearly, we only need to prove that $x_1(t)$ is bounded for all $t \in \Delta t_l$ and any Δt_l.

Firstly, since $\beta(t, \|x_{2_0}\|)$ as defined in (14) is of class $\mathcal{KL}$ then there exists a moment $T \geq 0$ such that $\beta(t, \|x_{2_0}\|) \leq 1/\lambda$ for all $t \geq T$ and with λ as in **A5**. Without loss of generality pick any interval $\Delta t_l^* \triangleq [t_1^{l*}, t_2^{l*}]$, where $t_1^* \geq T$ hence from Assumption **A5** we have that

$$\frac{\alpha_4(\|x_1(t)\|)}{\alpha_3(\|x_1(t)\|)} \|g(x(t))\| \leq \lambda \text{ for all } t \in \Delta t_l^*. \quad (31)$$

Then, using (27) we have that for all $t \in \Delta t_l^*$

$$\dot{V}_{(8)}(x) \leq -\alpha_3(\|x_1\|)\left[1 - \lambda\beta(t, \|x_{2_0}\|)\right] \leq 0.$$

Integrating the latter on both sides of the inequality from t_1^{l*} to t we obtain that $V(t) \leq V(t_1^{l*})$ for all $t \in \Delta t_l^*$ hence $x_1(t)$ is bounded for all $t \in \Delta t_l^*$. Since the interval Δt_l^* is arbitrary, the solutions $x(t)$ of system (8), (9) are globally bounded and using Assumption **A2** and Fact 1 GAS follows. ■

3.3 Proof of Theorem 3

Firstly, the completeness of the system can be proven exactly as in section 3.2. To prove that furthermore, solutions $x_1(t)$ are globally bounded we use the bounds (29) and (16) to write

$$\dot{V}_{(8)} \leq c_g \alpha_6(V)\|x_2(t)\| \quad (32)$$

where $\alpha_6(V) \geq \alpha_4(\alpha_1^{-1}(V))\gamma_2(\alpha_1^{-1}(V))$. Notice that, considering (14), inequality (32) is similar to (30) then using V_{new} as in the previous section we can conclude that

$$\lim_{t\to\infty} \{V_{new}(t) - V_{new}(0)\} \leq \lim_{t\to\infty} \int_0^t c_g \|x_2(\tau)\| d\tau \leq c_2,$$

hence from Assumption **A4** there exists a constant $c_4 = c_4(x_0)$ such that $\|x_1(t; x_0)\| \leq c_4$ for all $t \geq 0$. Thus from Fact 1 GAS of system (8), (9) follows. ■

4 Conclusions

Motivated by practical problems we have studied the stability analysis problem of cascaded nonlinear systems. Our contributions relate different conditions to ensure stability of cascaded nonlinear systems, some of which have been previously reported in the literature.

We have identified three classes of systems in accordance with the growth rates of the functions which define their dynamics. Each class has been illustrated with different (practical) examples and for each case we have established sufficient conditions to imply GAS of the cascade.

Even though these three classes together include a wide variety of nonlinear systems, our conditions are not yet necessary. This is an important and challenging subject of future research.

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