Copyright © IFAC Nonlinear Control Systems Design,
Enschede, The Netherlands, 1998

SUFFICIENT CONDITIONS FOR A DYNAMICAL SYSTEM TO POSSESS AN UNBOUNDED SOLUTION

Robert Orsi *,[1] **Laurent Praly** ** **Iven Mareels** *

* *Department of Electrical and Electronic Engineering, University of Melbourne, Parkville VIC 3052, Australia. e-mail: r.orsi@ee.unimelb.edu.au, i.mareels@unimelb.edu.au*
** *Centre Automatique et Systèmes, École des Mines de Paris, 77305 Fontainebleau Cédex, France. e-mail: praly@cas.ensmp.fr*

Abstract: Presented in this paper are readily verifiable conditions under which a dynamical system of the form $\dot{x} = f(x)$ possesses an unbounded solution. This result is illustrated by showing it can be used to infer results about lack of global stabilizability for nonlinear control systems. The key observation in the paper is that behaviour at infinity can be studied using local methods applied to an auxiliary system.
Copyright © 1998 IFAC

Keywords: Unbounded solutions, global stabilizability.

1. INTRODUCTION

Given a dynamical system of the form

$$\dot{x} = f(x) \tag{1}$$

it is often of interest to know if all possible solutions of the system are bounded or if the system possesses an unbounded solution. Presented in this paper are sufficient conditions for a system of the form (1) to possess an unbounded solution.

When considering whether a system possesses an unbounded solution, one is asking how the system behaves arbitrarily far away from the origin, that is, how it behaves near "infinity". The key observation in the paper is that behaviour at infinity can be studied using local methods. It will be shown that the existence of appropriate auxiliary functions and variables allows us to construct a new system of dimension one greater than the original from which local methods can be used to infer the existence of an unbounded solution of the original system.

As mentioned above, the result given in the paper relies on finding appropriate auxiliary functions and variables. In this manner it is similar to Lyapunov's stability theorem which requires one to find an appropriate auxiliary function, namely a Lyapunov function, in order to give a positive result about system stability.

The paper is structured as follows. Sufficient conditions for a system to possess an unbounded solution are presented in Section 2. Section 3 contains some examples that illustrate how the results of Section 2 can be used to infer results about lack of global stabilizability for nonlinear control systems. A partial converse theorem to the main result of Section 2 is presented in Section 4 along with some additional comments. The paper ends with some concluding remarks. (Note that a

[1] The authors wish to acknowledge the funding of the activities of the Cooperative Research Centre for Sensor Signal and Information Processing by the Australian Commonwealth Government under the Cooperative Research Centre Program. The first author would also like to acknowledge the support of Telstra under the TRL Postgraduate Fellowship scheme.

technical lemma used in Sections 2 and 4 has been placed at the end of the paper in an appendix.)

2. MAIN RESULT

In this section sufficient conditions for a system to possess an unbounded solution are given.

Let $\mathbb{Z}$ denote the integers, $\mathbb{R}$ the real numbers, $\mathbb{R}_+$ the set $\{z \in \mathbb{R} \mid z > 0\}$ and C^1 the class of continuously differentiable functions. If $\varphi(z,x)$ is a map from $\mathbb{R} \times \mathbb{R}^n$ to $\mathbb{R}^n$ which is differentiable at $(z,x) = (a,b)$, using the notation of (Dieudonne, 1960), let $D_1\varphi(a,b)$ denote

$$\frac{\partial \varphi}{\partial z}(z,x)\bigg|_{(z,x)=(a,b)},$$

and $D_2\varphi(a,b)$ denote

$$\frac{\partial \varphi}{\partial x}(z,x)\bigg|_{(z,x)=(a,b)}.$$

Definition 2.1. A *stability preserving extension* is defined to be a C^1 function $\phi : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ for which there exists a point $\bar{y} \in \mathbb{R}^n$ such that ϕ has properties

P1. $\phi(z_1, \phi(z_2, x)) = \phi(z_1 z_2, x)$ for all $z_1, z_2 \in \mathbb{R}_+$ and $x \in \mathbb{R}^n$,
P2. $\phi(1,x) = x$ for all $x \in \mathbb{R}^n$,
P3. $\psi : \mathbb{R}^n \to \mathbb{R}^n$, $y \mapsto D_1\phi(1,y)$ is C^1 in a neighbourhood of $\bar{y}$,
P4. $\lim_{(z,y)\to(0^+,\bar{y})} |\phi(\frac{1}{z}, y)| = \infty$.

An example of such a transformation is $\phi(z,x) = (z^{\alpha_1}x_1, \ldots, z^{\alpha_n}x_n)$ where $\alpha_i \in \mathbb{R}$, $i = 1, \ldots, n$, with at least one α_i strictly positive. (This particular transformation will be considered further in Corollary 2.5.)

Theorem 2.2. Given a function $f : \mathbb{R}^n \to \mathbb{R}^n$ and an arbitrary continuous function $F : \mathbb{R}^n \to \mathbb{R}_+$, define

$$g(x) := \frac{f(x)}{F(x)}.$$

If there exists $\lambda \in \mathbb{R}_+$ and a stability preserving extension ϕ such that

(1) there exists a function $h : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ which is C^1 in a neighbourhood N of $(0,\bar{y})$ and which equals

$$[D_2\phi(\frac{1}{z}, y)]^{-1} g(\phi(\frac{1}{z}, y))$$

for all $(z,y) \in N \cap (\mathbb{R}_+ \times \mathbb{R}^n)$,
(2) $h(0,\bar{y}) - \lambda D_1\phi(1,\bar{y}) = 0$, and
(3) there exists $v_1 \neq 0 \in \mathbb{R}$ and $v_2 \in \mathbb{R}^n$ such that

$$-\lambda v = Av,$$

where

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$
$$A = \begin{pmatrix} -\lambda & 0 \\ D_1h(0,\bar{y}) & A_{22} \end{pmatrix},$$
$$A_{22} = D_2h(0,\bar{y}) - \lambda[\frac{d}{dy}D_1h(1,y)]_{y=\bar{y}},$$

then the system

$$\dot{x} = f(x) \tag{2}$$

has an unbounded solution.

Proof. Consider the system

$$\begin{aligned} \dot{z} &= -\lambda z \\ \dot{y} &= h(z,y) - \lambda D_1\phi(1,y). \end{aligned} \tag{3}$$

By assumption, $h(z,y)$ is C^1 in a neighbourhood of $(0,\bar{y})$. In addition, from property $(P3)$, $\psi(y) = D_1\phi(1,y)$ is C^1 in a neighbourhood of $\bar{y}$ and it follows that the vector field of (3) is C^1 in a neighbourhood of $(0,\bar{y})$.

The point $(z,y) = (0,\bar{y})$ is an equilibrium point of system (3). Linearization of (3) at this point gives

$$\begin{pmatrix} \dot{\tilde{z}} \\ \dot{\tilde{y}} \end{pmatrix} = A \begin{pmatrix} \tilde{z} \\ \tilde{y} \end{pmatrix}.$$

As $-\lambda$ is a negative eigenvalue of A with an associated eigenvector v whose z component, v_1, is non-zero, the Center Manifold Theorem (Guckenheimer and Holmes, 1997) implies the existence of a solution to (3) passing through a point (z^0, y^0), $z^0 > 0$, and converging to $(0,\bar{y})$.

Define

$$x(t) := \phi(\frac{1}{z(t)}, y(t)) \tag{4}$$

where $(z(t), y(t))$ denotes the solution of (3) with initial point (z^0, y^0). Note that $z(t) > 0$ for $t \in [0,\infty)$. Differentiating (4) with respect to t gives,

$$\dot{x} = D_1\phi(\frac{1}{z}, y)(-\frac{\dot{z}}{z^2}) + D_2\phi(\frac{1}{z}, y)\,\dot{y}. \tag{5}$$

Substituting (3) into (5) and then using condition (1) of the theorem statement and simplifying gives,

$$\begin{aligned} \dot{x} = \frac{\lambda}{z} D_1\phi(\frac{1}{z}, y) + g(\phi(\frac{1}{z}, y)) \\ - \lambda D_2\phi(\frac{1}{z}, y) D_1\phi(1,y). \end{aligned}$$

It now follows from equations (A.1) and (4) that $x(t)$ satisfies

$$\dot{x} = g(x). \qquad (6)$$

As $z(t) \to 0^+$ and $y(t) \to \bar{y}$, property $(P4)$ implies that $x(t)$ is an unbounded solution of (6).

Define

$$\tau(t) = \int_0^t \frac{ds}{F(x(s))}.$$

As $F(x(t))$ is continuous and strictly greater than zero for all $t \in [0, \infty)$, it follows from the Fundamental Theorem of Calculus that

$$\frac{d\tau(t)}{dt} = \frac{1}{F(x(t))}.$$

As τ is a strictly monotonically increasing function of t, it follows that t can be considered as a function of τ and that

$$\frac{dt(\tau)}{d\tau} = F(x(t(\tau))).$$

Define

$$\xi(\tau) := x(t(\tau)).$$

Then

$$\begin{aligned}\frac{d\xi(\tau)}{d\tau} &= \frac{dx}{dt}\frac{dt}{d\tau} \\ &= \frac{f(x(t(\tau)))}{F(x(t(\tau)))}F(x(t(\tau))) \\ &= f(\xi(\tau)).\end{aligned}$$

Now τ converges to some value $\bar{\tau} \in (0, \infty]$ as $t \to \infty$ and, because of the relationship between τ and t, t as a function of τ converges to infinity as $\tau \to \bar{\tau}$ and it follows that (2) has an unbounded solution. ∎

Remark 2.3. Note that the net result of dividing the vector field f by the positive function F is just a nonlinear scaling of time. The orientation of the vector field and the trajectories of the system remain unchanged.

Remark 2.4. While the matrix A will always have $-\lambda$ as an eigenvalue, it is not clear that it will have a corresponding eigenvector v with a nonzero z component, i.e., that $v_1 \neq 0$. Condition (3) of Theorem 2.2 ensures this is the case. That $v_1 \neq 0$ is central to the proof of Theorem 2.2 as it ensures that z^0 can be chosen to be positive and hence that $z(t) > 0$ for all $t \in [0, \infty)$. A sufficient condition that $v_1 \neq 0$ is that $D_1 h(0, \bar{y}) = 0$.

Corollary 2.5. Given a function $f : \mathbb{R}^n \to \mathbb{R}^n$ and an arbitrary continuous function $F : \mathbb{R}^n \to \mathbb{R}_+$, define

$$g(x) := \frac{f(x)}{F(x)}$$

and let $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, n$, denote the components of g. If there exists $\lambda \in \mathbb{R}_+$, $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ and $(\bar{y}_1, \ldots, \bar{y}_n) \in \mathbb{R}^n$ such that

(1) for each i=1,...,n, there exists a function $h_i(z, y)$ from $\mathbb{R} \times \mathbb{R}^n$ into $\mathbb{R}^n$ which is C^1 in a neighbourhood N of $(0, \bar{y}_1, \ldots, \bar{y}_n)$ and which equals

$$z^{\alpha_i} g_i\left(\frac{y_1}{z^{\alpha_1}}, \ldots, \frac{y_n}{z^{\alpha_n}}\right)$$

for all $(z, y_1, \ldots, y_n) \in N \cap (\mathbb{R}_+ \times \mathbb{R}^n)$,

(2) $h_i(0, \bar{y}_1, \ldots, \bar{y}_n) - \lambda\alpha_i\bar{y}_i = 0$, for all $i = 1, \ldots, n$,

(3) $\bar{y}_j \neq 0$, $\alpha_j > 0$, for some $j \in \{1, \ldots, n\}$, and

(4) there exists $v_1 \neq 0 \in \mathbb{R}$ and $v_2 \in \mathbb{R}^n$ such that

$$-\lambda v = Av,$$

where

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$
$$A = \begin{pmatrix} -\lambda & 0 \\ D_1 h(0, \bar{y}) & A_{22} \end{pmatrix},$$
$$A_{22} = D_2 h(0, \bar{y}) - \lambda[\frac{d}{dy} D_1 h(1, y)]_{y=\bar{y}},$$

then the system

$$\dot{x} = f(x)$$

has an unbounded solution.

PROOF. The result follows in a straightforward manner from Theorem 2.2 by choosing the stability preserving extension $\phi(z, x) = (z^{\alpha_1} x_1, \ldots, z^{\alpha_n} x_n)$. ∎

Remark 2.6. The next section contains some examples that utilize Corollary 2.5. In these examples, when verifying the conditions of Corollary 2.5, no formal distinction will be made between

$$z^{\alpha_i} g_i\left(\frac{y_1}{z^{\alpha_1}}, \ldots, \frac{y_n}{z^{\alpha_n}}\right)$$

and h_i. It will follow from the context what is meant.

3. EXAMPLES

In this section the results of Section 2 are illustrated with some examples. In particular, it will be shown how the results of Section 2 can be used to infer results about lack of global stabilizability for nonlinear control systems.

The following result in Example 3.1 was originally proved in (Sepulchre *et al.*, 1997) using quite different methods.

Example 3.1. Consider the system

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2 x_3 \\ \dot{x}_2 &= -x_2 + x_1^2 x_2 \\ \dot{x}_3 &= u \end{aligned} \tag{7}$$

where $(x_1, x_2, x_3) \in \mathbb{R}^3$ denotes the state and u the input. It is now shown that there exists no C^1 partial-state feedback $u = k(x_3)$ that globally stabilizes (7).

Rather than giving a concise proof of this result, a more lengthy proof is presented in order to try to demonstrate the approach one might take in trying to prove such a result.

First, the proof is attempted using $F(x) \equiv 1$. As will be seen, this choice of F does not work out. However, in general, when searching for an appropriate function F, $F(x) \equiv 1$ is a good starting point. Even though it may not lead directly to the desired result, trying to prove a result using $F(x) \equiv 1$ usually gives good insight into what properties the correct F should have.

Let $F(x) \equiv 1$. Condition (1) of Corollary 2.5 then requires that the functions

$$\begin{aligned} h_1 &= z^{\alpha_1}\left(-\frac{y_1}{z^{\alpha_1}} + \frac{y_2 y_3}{z^{\alpha_2+\alpha_3}}\right) \\ &= -y_1 + z^{\alpha_1-\alpha_2-\alpha_3} y_2 y_3, \\ h_2 &= z^{\alpha_2}\left(-\frac{y_2}{z^{\alpha_2}} + \frac{y_1^2 y_2}{z^{2\alpha_1+\alpha_2}}\right) \\ &= -y_2 + z^{-2\alpha_1} y_1^2 y_2, \\ h_3 &= z^{\alpha_3} k\left(\frac{y_3}{z^{\alpha_3}}\right) \end{aligned} \tag{8}$$

be C^1 in a neighbourhood of some point $(0, \bar{y}_1, \bar{y}_2, \bar{y}_3)$ (which is still to be determined). As $k(x_3)$ is allowed to be any C^1 function, a necessary and sufficient condition that this be true for h_3 is that $\alpha_3 = 0$. The equations for h_1 and h_2 then imply that $\alpha_1 - \alpha_2 \geq 0$ and $-2\alpha_1 \geq 0$, that is, $\alpha_2 \leq \alpha_1 \leq 0$. This however violates condition (3) of Corollary 2.5 that requires at least one of the exponents α_1, α_2 or α_3 to be positive.

The only possible way to overcome this problem is to try a different function F. Leaving $\alpha_3 = 0$, let α_1 be positive. F is now chosen so that h_2 is locally C^1. A suitable choice is $F = 1 + x_1^2$, which upon substitution of $x_1 = y_1/z^{\alpha_1}$ gives

$$F = \frac{z^{2\alpha_1} + y_1^2}{z^{2\alpha_1}}.$$

The h_i equations in (8) now become

$$\begin{aligned} h_1 &= \frac{-z^{2\alpha_1} y_1 + z^{3\alpha_1-\alpha_2} y_2 y_3}{z^{2\alpha_1} + y_1^2}, \\ h_2 &= \frac{-z^{2\alpha_1} y_2 + y_1^2 y_2}{z^{2\alpha_1} + y_1^2}, \\ h_3 &= \frac{z^{2\alpha_1} k(y_3)}{z^{2\alpha_1} + y_1^2}. \end{aligned}$$

Conditions (1) and (3) of Corollary 2.5 are now satisfied if $\alpha_1 > 0 \in \mathbb{Z}$, $3\alpha_1 - \alpha_2 \geq 0 \in \mathbb{Z}$ and $\bar{y}_1 \neq 0$.

It must now only be ensured that conditions (2) and (4) of Corollary 2.5 are satisfied. Noting that $h_3 = 0$ when $z = 0$ and that $\alpha_3 = 0$, the only additional requirements needed to satisfy condition (2) of Corollary 2.5 are that

$$\frac{z^{3\alpha_1-\alpha_2}\bar{y}_2\bar{y}_3}{\bar{y}_1^2} - \lambda\alpha_1\bar{y}_1 = 0 \quad \text{and}$$
$$\bar{y}_2 - \lambda\alpha_2\bar{y}_2 = 0$$

when $z = 0$, and that $\lambda > 0$. In order that $\bar{y}_1 \neq 0$ it is required that $3\alpha_1 = \alpha_2$, $\bar{y}_2 \neq 0$, $\bar{y}_3 \neq 0$ and $\lambda = 1/\alpha_2$. Hence conditions (1), (2) and (3) of Corollary 2.5 are satisfied if for example $\alpha_1 = 1$, $\alpha_2 = 3$, $\alpha_3 = 0$, $\lambda = 1/3$ and $(\bar{y}_1, \bar{y}_2, \bar{y}_3) = (3, 3, 3)$. It is easily verified that for this choice of α_i's, λ and $\bar{y}$, that $D_1 h(0, \bar{y}) = 0$. Remark 2.4 now implies condition (4) of Corollary 2.5 is also satisfied. This proves the desired result. □

One possibility that freedom to choose F allows is the ability to introduce a multiplicative factor z^γ, $\gamma > 0$, into the vector field components $f_1, \ldots, f_n$. This technique was demonstrated in Example 3.1 where, by appropriate choice of F, a factor of $z^{2\alpha_1}$ was introduced to make h_2 locally continuously differentiable. Note that $\gamma > 0$ cannot be chosen too large as this would result in the functions $h_1, \ldots, h_n$ all being zero when $z = 0$ and lead to an inability to simultaneously satisfy conditions (2) and (3) of Corollary 2.5.

Example 3.2. It is now shown that system (7) cannot be globally stabilized via linear full state feedback. That is, it will be shown that there does not exist a feedback of the form

$$u = b_1 x_1 + b_2 x_2 + b_3 x_3,$$

$(b_1, b_2, b_3) \in \mathbb{R}^3$, which globally stabilizes (7).

Let $F = 1+x_1^2$. Then it is straightforward to verify that

$$h_1 = \frac{-z^{2\alpha_1}y_1 + z^{3\alpha_1-\alpha_2-\alpha_3}y_2y_3}{z^{2\alpha_1}+y_1^2},$$
$$h_2 = \frac{-z^{2\alpha_1}y_2 + y_1^2y_2}{z^{2\alpha_1}+y_1^2},$$
$$h_3 = \frac{z^{\alpha_1+\alpha_3}b_1y_1 + z^{2\alpha_1-\alpha_2+\alpha_3}b_2y_2 + z^{2\alpha_1}b_3y_3}{z^{2\alpha_1}+y_1^2}.$$

Three sub-cases are considered.

Case 1: $b_2 \neq 0$. Let $\alpha_1 = 2$, $\alpha_2 = 5$, $\alpha_3 = 1$, $\lambda = 1/5$ and $(\bar{y}_1, \bar{y}_2, \bar{y}_3) = ((25b_2/2)^{\frac{1}{5}}, 1, 2\bar{y}_1^3/5)$. It is left to the reader to verify that all the conditions of Corollary 2.5 are satisfied. (Condition (4) can be seen to hold by verifying that $D_1h(0,\bar{y}) = 0$.) Note that $b_2 \neq 0$ ensures $\bar{y}_1 \neq 0$ which is required in order to satisfy condition (1) of Corollary 2.5.

Case 2: $b_1 \neq 0$, $b_2 = 0$. Let $\alpha_1 = 1$, $\alpha_2 = 4$, $\alpha_3 = -1$, $\lambda = 1/4$ and $(\bar{y}_1, \bar{y}_2, \bar{y}_3) = (2|b_1|^{\frac{1}{4}}, -b_1/|b_1|, -4b_1/\bar{y}_1)$. Again it is left to the reader to verify that all the conditions of Corollary 2.5 are satisfied.

Case 3: $b_1 = 0$, $b_2 = 0$. In this case $u = b_3x_3$ and the result follows from Example 3.1.

This proves the desired result. □

4. TOWARDS A CONVERSE THEOREM

In this section a partial converse theorem to Theorem 2.2 is presented. Some additional comments are also made.

Theorem 4.1. Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is C^1 and that the system

$$\dot{x} = f(x) \tag{9}$$

has an unbounded solution. Then there exists a C^1 function $\phi : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ and a point $\bar{y} \in \mathbb{R}^n$ such that ϕ satisfies properties $(P1)$, $(P2)$, $(P3)$, and

P4′. $\limsup_{z\to 0^+} |\phi(\frac{1}{z}, \bar{y})| = \infty.$

Furthermore, there exists a continuous function $F : \mathbb{R}^n \to \mathbb{R}_+$ and a scalar $\lambda \in \mathbb{R}_+$ that together with ϕ satisfy conditions (1), (2) and (3) of Theorem 2.2.

Proof. Define

$$F(x) := 1 + |f(x)|^2$$

and

$$g(x) := \frac{f(x)}{F(x)}.$$

Let $\xi(t,x)$ denote the solution of

$$\dot{w} = g(w), \quad w(0) = x. \tag{10}$$

As g is C^1 and $|g(w)| < 1$ for all $w \in \mathbb{R}^n$, it follows that $\xi(t,x)$ is uniquely defined for all $(t,x) \in \mathbb{R}\times\mathbb{R}^n$ and furthermore that $\xi(t,x)$ is C^1 (Hale, 1980). Define

$$\phi(z,x) := \xi(\log(z), x). \tag{11}$$

Note that $\phi(z,x)$ is a C^1 function on $\mathbb{R}_+ \times \mathbb{R}^n$. General properties of ordinary differential equations imply that ϕ satisfies property $(P1)$. That ϕ satisfies property $(P2)$ follows from the fact that $\xi(t,x)$ satisfies (10).

Differentiating (11) with respect to z gives

$$D_1\phi(z,x) = \frac{1}{z}D_1\xi(\log(z),x). \tag{12}$$

Substituting $z = 1$ into (12) gives

$$\begin{aligned} D_1\phi(1,x) &= D_1\xi(0,x) \\ &= g(\xi(0,x)) \\ &= g(x), \end{aligned} \tag{13}$$

where the last two equalities follow from the fact that $\xi(t,x)$ satisfies (10). Property $(P3)$ now follows as g is a C^1 function.

As (9) has an unbounded solution, it follows from Remark 2.3 that the system $\dot{w} = g(w)$ has an unbounded solution. Hence there exists $\bar{x} \in \mathbb{R}^n$ such that $\limsup_{t\to\infty} |\xi(t,\bar{x})| = \infty$. This implies

$$\limsup_{z\to 0^+} |\phi(\frac{1}{z}, \bar{y})| = \infty$$

where $\bar{y} = \bar{x}$ and hence ϕ satisfies property $(P4')$.

Substituting $x = \phi(1/z, y)$ into (13) gives

$$g(\phi(\frac{1}{z},y)) = D_1\phi(1, \phi(\frac{1}{z},y)). \tag{14}$$

Combining (14) and (A.1) now gives

$$g(\phi(\frac{1}{z},y)) = D_2\phi(\frac{1}{z},y)D_1\phi(1,y). \tag{15}$$

Lemma A.1 shows that $[D_2\phi(1/z,y)]^{-1}$ exists. Hence equation (15) gives

$$\begin{aligned} h(z,y) &= [D_2\phi(\frac{1}{z},y)]^{-1}g(\phi(\frac{1}{z},y)) \\ &= D_1\phi(1,y) \end{aligned} \tag{16}$$

and it follows that $h(z,y)$ is a C^1 function. Hence condition (1) of Theorem 2.2 is satisfied and so is condition (2) by letting $\lambda = 1$. Equation (16) implies $D_1 h(0, \bar{y}) = 0$. Remark 2.4 now implies that condition (3) of Theorem 2.2 is also satisfied. ■

Note that the construction of ϕ in the proof of Theorem 4.1 presumes knowledge of the solutions of (10). Let A be a real $n \times n$ matrix with eigenvalue $\tilde{\lambda}$, $\mathrm{Re}(\tilde{\lambda}) > 0$, and consider the linear system

$$\dot{x} = Ax. \tag{17}$$

Presuming knowledge of the solutions of (17), one can easily find appropriate ϕ, F and λ to satisfy Theorem 2.2. What can be said in this case without using knowledge of the solutions? Taking $\alpha_i = 1, i = 1, \ldots, n$, and $F = 1$, it is easily verified that all the conditions of Corollary 2.5 are satisfied if there exists $\bar{y} \neq 0 \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}_+$ such that

$$A\bar{y} = \lambda \bar{y}.$$

Hence, if $\tilde{\lambda}$ is real and positive, Corollary 2.5 implies (17) has an unbounded solution. If $\tilde{\lambda}$ is complex, the choice of α_i's and F used above fails to conclude that the system has an unbounded solution.

This and the gap that exists between properties ($P4$) and ($P4'$) suggests that there may be value in using multidimensional extensions. This is a path the authors are actively pursuing.

5. CONCLUDING REMARKS

In this paper a start was made at exploring the use of local methods to analyze behaviour at infinity. Presented were sufficient conditions for a dynamical system to possess an unbounded solution and it was shown that these results can be used to infer results about lack of global stabilizability for nonlinear control systems.

6. REFERENCES

Dieudonne, J. (1960). *Foundations of Modern Analysis*. Academic Press. New York.

Guckenheimer, J. and P. Holmes (1997). *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields*. Springer-Verlag. New York.

Hale, J. K. (1980). *Ordinary Differential Equations*. Krieger. Malabar, Florida.

Sepulchre, R., M. Janković and P. Kokotović (1997). *Constructive Nonlinear Control*. Springer Verlag. London.

Appendix A.

The following lemma contains results used in the proofs of Theorem 2.2 and Theorem 4.1.

Lemma A.1. If $\phi : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 function that satisfies properties ($P1$) and ($P2$) then

$$\begin{aligned} D_2\phi(\frac{1}{z}, y) D_1\phi(1, y) &= \frac{1}{z} D_1\phi(\frac{1}{z}, y) \\ &= D_1\phi(1, \phi(\frac{1}{z}, y)) \end{aligned} \tag{A.1}$$

and

$$[D_2\phi(\frac{1}{z}, y)]^{-1} = D_2\phi(z, \phi(\frac{1}{z}, y)) \tag{A.2}$$

for all $z \in \mathbb{R}_+$ and $y \in \mathbb{R}^n$.

Proof. Differentiating the identity given in ($P1$) with respect to z_2 gives

$$D_2\phi(z_1, \phi(z_2, x)) D_1\phi(z_2, x) = z_1 D_1\phi(z_1 z_2, x).$$

The first equality of (A.1) now follows by choosing $z_1 = 1/z$, $z_2 = 1$, $x = y$ and applying ($P2$).

Differentiating the identity given in ($P1$) with respect to z_1 gives

$$D_1\phi(z_1, \phi(z_2, x)) = z_2 D_1\phi(z_1 z_2, x).$$

Setting $z_1 = 1$, $z_2 = 1/z$ and $x = y$ gives the second equality of (A.1).

From properties ($P1$) and ($P2$),

$$\phi(z, \phi(\frac{1}{z}, y)) = y.$$

Differentiating with respect to y gives

$$D_2\phi(z, \phi(\frac{1}{z}, y)) D_2\phi(\frac{1}{z}, y) = I.$$

This proves (A.2). ■