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On assigning the derivative of a disturbance attenuation clf

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Abstract

This paper[2] considers feedback design for nonlinear, multi-input affine control systems with disturbances. It studies the problem of assigning, by choice of feedback, a desirable upper bound to a given control Lyapunov function (clf) candidate's derivative along closed-loop trajectories. Specific choices for the upper bound are motivated by $\mathcal{L}_2$ and $\mathcal{L}_\infty$ disturbance attenuation problems. The main result leads to corollaries on "backstepping" locally Lipschitz disturbance attenuation control laws that are perhaps implicitly defined through a locally Lipschitz equation. The results emphasize that only rough information about the clf is needed to synthesize a suitable controller.

1 Introduction

One of the main analysis tools for verifying stability and/or disturbance attenuation properties for closed-loop control systems is the Lyapunov function - a smooth, positive definite, radially unbounded function. If the derivative of the Lyapunov function can be bounded appropriately then the resulting differential inequality may be integrated to establish desired closed-loop properties, e.g., $\mathcal{L}_2$ or $\mathcal{L}_\infty$ disturbance attenuation. The control synthesis problem can then be seen as the problem of finding a Lyapunov function that can be assigned a desirable derivative by appropriate choice of feedback.

A (global) control Lyapunov function (clf) for a smooth control system of the form $\dot{x} = f(x) + g(x)u$ has been defined in [13] to be a smooth, positive definite, radially unbounded function whose derivative along the parameterized vector field $f(x) + g(x)u$ can be made negative for each $x \neq 0$ by an appropriate choice of the control parameter u. When a clf for $\dot{x} = f(x) + g(x)u$ is given, a smooth function $\psi : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^m$ can be constructed from the clf and its derivatives along $f(x)$ and $g(x)$ so that the derivative of the clf along the vector field $f(x) + g(x)\psi(x)$ is negative whenever $x \neq 0$. Thus, there is an intimate connection between the existence of this type of clf for $\dot{x} = f(x) + g(x)u$ and the construction of a feedback law that renders the origin GAS. See [1] and [13] for more details.

Control Lyapunov functions have also been characterized for control systems with disturbances. In [6, section 4], a robust control Lyapunov function (rclf) for a system $\dot{x} = f(x,d) + g(x,d)u$ was defined to be a smooth, positive definite, radially unbounded function whose derivative along $f(x,d) + g(x,d)u$ can, for each $x \neq 0$, be made negative uniformly in d belonging to a compact set depending on x by an appropriate choice of u. A similar notion is used in [15]. In [6, section 6], it was pointed out how the notion of an rclf encompasses the $\mathcal{L}_\infty$ disturbance attenuation property. The rclf's discussed in [6] don't address other disturbance attenuation properties directly but the required modifications to the definition of the rclf are not difficult. Part of the contribution here is in that direction.

In this paper, we consider feedback design for nonlinear, multi-input control systems with disturbances of the form $\dot{x} = f(x,d) + g(x)u$ where $x \in \mathbb{R}^n$, $d \in \mathbb{R}^p$ and $u \in \mathbb{R}^m$. In section 2 we motivate, via certain disturbance attenuation problems, assigning an upper bound to the derivative of a clf for such systems. In section 3.1 we will develop a general notion of a clf for these systems. The definition of a clf will be in terms of a desired upper bound, denoted $\widetilde{\alpha}(x,d)$, for the clf's derivative and also in terms of a preliminary feedback $\pi(x)$ which is more general than, but can be thought of as being like, the derivative of the clf along the matrix field $g(x)$. In section 3.2 we will characterize a class of upper bounds $\widetilde{\alpha}(x,d)$ that can be assigned to the derivative of the clf based on a bound for the derivative of the clf when the derivative of the clf along the vector field $g(x)\pi(x)$ is zero. In section 4 we will apply our results to $\mathcal{L}_\infty$ and $\mathcal{L}_2$ disturbance attenuation problems, including such problems when adding perturbed integrators. These problems have also been discussed in, e.g., [8] and [7, section 9.5].

2 Preliminaries

Motivated by certain disturbance attenuation problems discussed below, we will develop sufficient con-

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[2]Proofs and additional motivation for this work can be found in the full-length version of this paper, available from the authors.

ditions for synthesizing a continuous state feedback $u = k(x)$ that assigns a given upper bound $\widetilde{\alpha}(x(t), d(t))$ to the time derivative of a locally Lipschitz clf candidate $V(x(t))$ along closed-loop trajectories; i.e., we are looking for a continuous function $k(x)$ so that

$$\dot{V}(x(t)) \leq \widetilde{\alpha}(x(t), d(t)) \qquad \text{for almost all t} \qquad (1)$$

where $x(t)$ is an absolutely continuous function s.t.

$$\dot{x}(t) = f(x(t), d(t)) + g(x(t))k(x(t)) \qquad \text{for almost all t} . \qquad (2)$$

Throughout the paper when considering solutions of ordinary differential equations, generically denoted $\dot{X} = F(X, t)$, we assume that the Carathéodory conditions are satisfied, i.e., F is continuous in X, measurable in t, and for each compact set $\mathcal{C}$ of $\mathbb{R}^n$ and each interval $[a, b]$ of $\mathbb{R}_{\geq 0}$, there exists an integrable function $m : [a, b] \to \mathbb{R}_{\geq 0}$ such that $|F(X, t)| \leq m(t) \quad \forall (x, t) \in \mathcal{C} \times [a, b]$. This guarantees that, for each initial condition, at least one absolutely continuous solution of (2) exists locally in time, i.e., on $[0, T)$ for some $T > 0$. Then, since V is locally Lipschitz, $V(x(t))$ is absolutely continuous [10, Theorem 2, p. 245] and $\dot{V}(x(t))$ is well-defined for almost all $t \in [0, T)$ [10, Corollary, p. 246]. When $\widetilde{\alpha}(x(t), d(t))$ is locally $\mathcal{L}_1$,

$$\varphi(t) := V(x(t)) - V(x(0)) - \int_0^t \widetilde{\alpha}(x(s), d(s)) ds \qquad (3)$$

is absolutely continuous [10, Theorem 1, p. 252] with derivative defined almost everywhere as

$$\dot{\varphi}(t) = \dot{V}(x(t)) - \widetilde{\alpha}(x(t), d(t)) \leq 0 \qquad \text{for almost all t} . \qquad (4)$$

It follows ([16, Theorem 3.1]) that, $\forall t \in [0, T)$,

$$V(x(t)) \leq V(x(0)) + \int_0^t \widetilde{\alpha}(x(s)), d(s)) ds . \qquad (5)$$

For $\mathcal{L}_2$ disturbance attenuation, one function $\widetilde{\alpha}$ in (1) that we will use is $\widetilde{\alpha}(x, d) = \gamma^2 |d|^2$. If $d \in \mathcal{L}_2$, i.e., d is measurable and $||d||_2^2 := \int_0^\infty |d(t)|^2 dt < \infty$, then we have

$$V(x(t)) \leq V(x(0)) + \gamma^2 ||d||_2^2 . \qquad (6)$$

If $V(x)$ is positive definite and radially unbounded in x then we conclude that each solution is defined on $[0, \infty)$. In turn, if we also have that (1) is satisfied with $\widetilde{\alpha}(x, d) = -\kappa(x)|h(x)|^2 + \gamma^2 |d|^2$ where κ is continuous and positive, then denoting by V_{max} the upper bound on $V(x(t))$ from (6), we have

$$||h(x)||_2^2 \leq \left(\gamma^2 ||d||_2^2 + V(x(0))\right) \frac{1}{\min\limits_{\{x : V(x) \leq V_{max}\}} \kappa(x)} . \qquad (7)$$

When $\kappa(x) \equiv 1$ this is the standard case of $\mathcal{L}_2$ disturbance attenuation with linear gain $\gamma \cdot s$ (gain γ). When $\kappa(x) = 1/\tilde{\kappa}(V(x))$ with $\tilde{\kappa}$ nondecreasing and $V(x(0)) = 0$, we get a nonlinear $\mathcal{L}_2$ gain from

$$||h(x)||_2^2 \leq \tilde{\kappa}(\gamma^2 ||d||_2^2) \gamma^2 ||d||_2^2 . \qquad (8)$$

For $\mathcal{L}_\infty$ disturbance attenuation, we are interested in functions $\widetilde{\alpha}$ satisfying

$$V(x) \geq \max\{\gamma(|d|), \epsilon\} \Longrightarrow \widetilde{\alpha}(x, d) \leq -\kappa(V(x)) \qquad (9)$$

where $\epsilon \geq 0$, and γ and κ are functions of class-$\mathcal{K}_\infty$, i.e., they are continuous, zero at zero, strictly increasing and unbounded. If $d \in \mathcal{L}_\infty$, i.e., d is measurable and $||d||_\infty := \sup_{t \geq 0} |d(t)| < \infty$, we then have

$$V(x(t)) \geq \max\{\gamma(|d|), \epsilon\} \Longrightarrow \dot{V} \leq -\kappa(V(x(t)) . \qquad (10)$$

The differential inequality $\dot{V} \leq -\kappa(V(x(t)))$ is implicit in V but here and in other more general situations we can bound $V(x(t))$ using comparison theorems for differential inequalities satisfying the Carathéodory conditions (see, e.g., [9, Theorem 1.10.2]). Such a result is used in [14, Proof of Theorem 1] to conclude that if, in addition to (10), $V(x)$ is radially unbounded and there exists a class-$\mathcal{K}_\infty$ function δ such that $\delta(|h(x)|) \leq V(x)$ then, for all $t \geq 0$, $|h(x(t))|$ satisfies the estimate

$$|h(x(t))| \leq \max\left\{\beta(|x_\circ|, t), \delta^{-1} \circ \gamma(||d||_\infty), \delta^{-1}(\epsilon)\right\} \qquad (11)$$

where $\beta \in \mathcal{KL}$, i.e., continuous, class-$\mathcal{K}_\infty$ in its first argument and decreasing to zero in its second argument.

We will use the Clarke generalized directional derivative, because of its convenient properties when $V(x)$ is locally Lipschitz (see, e.g., [2, Propositions 2.1.1, 2.2.4, 2.3.3, 2.3.13]), to bound the time derivative of $V(x(t))$ along solutions. Because the Clarke generalized directional derivative agrees with the expression $\frac{\partial V}{\partial x}(x)v =: L_v V(x)$ when $V(x)$ is continuously differentiable, and since the reader may want to first digest the results for the C^1 case, we will use the Lie derivative notation throughout the paper in place of the more typical $V^\circ(x; v)$ for the Clarke generalized directional derivative unless there is some need to be precise. As further abuse of notation, when $g(x)\pi(x)$ is a vector field we will denote $V^\circ(x; g(x)\pi(x))$ by $L_{g(x)}V(x)\pi(x)$. Also, whenever we write $L_{g(x)}V(x)$ alone, we mean the row vector with ith column given by $V^\circ(x; g_i(x))$. This last bit of notation will only be used in rigorous statements when $V(x)$ is continuously differentiable.

3 Assignable upper bounds for clfs

3.1 Main result

We have motivated our desire to solve the following problem: given a locally Lipschitz function $V(x)$ and another function $\widetilde{\alpha}(x, d)$, find, if possible, $k(x)$ s.t.

$$L_{f(x,d)}V(x) + L_{g(x)}V(x)k(x) \leq \widetilde{\alpha}(x, d) . \qquad (12)$$

In fact, we will consider the more specific problem where a function $\pi : \mathbb{R}^n \to \mathbb{R}^m$ is given such that $L_{g(x)}V(x)\pi(x)$ is nonpositive and we must find a locally bounded, i.e., bounded on compact sets, function $\psi : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that

$$k(x) = \pi(x)\psi(x) \qquad (13)$$

solves our problem. For the case where there are no disturbances and $V(x)$ is C^1 and $\pi(x) = -L_{g(x)}V(x)^T$, this mirrors Sontag's "universal formula" for stabilization [13]. We consider more general functions $\pi(x)$ since we will be considering problems where $L_{g(x)}V(x)$ is not known exactly.

Definition 1 *When there exists a locally bounded function $\psi(x)$ so that, with (13), (12) holds for all (x,d) the function $\widetilde{\alpha}(x,d)$ is said to be an* assignable upper bound *for the derivative of V using π.*

If $\psi^*(x)$ establishes that $\widetilde{\alpha}(x,d)$ is an assignable upper bound using π then, since $L_{g(x)}V(x)\pi(x)$ is assumed to be nonpositive, any feedback of the form $u = \pi(x)\psi(x)$ where $\psi(x) \geq \psi^*(x)$ also assigns the upper bound $\widetilde{\alpha}(x,d)$. As a consequence, we can always take $\psi(x)$ to be locally Lipschitz or smooth.

Two properties will be used to characterize when $\widetilde{\alpha}(x,d)$ is an assignable upper bound for the derivative of V using π. Both will be expressed in terms of

$$\omega(x) := \sup_d \left\{ L_{f(x,d)}V(x) - \widetilde{\alpha}(x,d) \right\} \ . \qquad (14)$$

The first property, a 'clf' property, parallels the 'clf' and 'rclf' definitions in [13] and [6, Definition 4.1].

Definition 2 *The locally Lipschitz function $V(x)$ is a* control Lyapunov function (clf) *for the pair $(\pi, \widetilde{\alpha})$ if $L_{g(x)}V(x)\pi(x)$ is nonpositive, $\omega(x)$ in (14) is well-defined,* $\max\{0,\omega(x)\}$ *is locally bounded and for $x \neq 0$*

$$\limsup_{z\to x} \ L_{g(z)}V(z)\pi(z) = 0 \Longrightarrow \limsup_{z\to x} \ \omega(z) < 0 \ . \qquad (15)$$

The next property is related to the 'small control property' found in [13] in the setting of stabilization without disturbances, and in [6] for stabilization with disturbances constrained to a state dependent set. Further connections will be made in theorem 3 below.

Definition 3 *The locally Lipschitz function $V(x)$ satisfies the* bounded control property (bcp) *for the pair $(\pi, \widetilde{\alpha})$ if there exist $\chi > 0$ and $\bar{v} \geq 0$ s.t., $\forall |x| \leq \chi$,*

$$\omega(x) + \bar{v} L_{g(x)}V(x)\pi(x) \leq 0 \qquad (16)$$

(where $\omega(x)$ was defined in (14).)

Theorem 1 *If $V(x)$ is a control Lyapunov function and satisfies the bounded control property for the pair $(\pi, \widetilde{\alpha})$ then $\widetilde{\alpha}(x,d)$ is an assignable upper bound for the derivative of V using π.*

Proof. Define the function $\psi^* : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ by

$$\psi^*(x) := \begin{cases} \dfrac{\max\{0,\omega(x)\}}{-L_{g(x)}V(x)\pi(x)} & \text{if} \quad L_{g(x)}V(x)\pi(x) \neq 0 \\ 0 & \text{if} \quad L_{g(x)}V(x)\pi(x) = 0 \ . \end{cases} \qquad (17)$$

We first establish that $\psi^*(x)$ is locally bounded. The bounded control property implies that $\psi^*(x) \leq \bar{v}$ for all $|x| \leq \chi$. For $|x| \geq \chi$, since $\max\{0,\omega(x)\}$ is assumed to be locally bounded and since $L_{g(x)}V(x)\pi(x)$ is nonpositive, we just need that $\limsup_{z\to x} \ L_{g(z)}V(z)\pi(z) = 0$ implies $\limsup_{z\to x}\omega(z) < 0$. This follows from (15) in the clf property. Next, we need to establish, $\forall x \in \mathbb{R}^n$,

$$\omega(x) + L_{g(x)}V(x)\pi(x)\psi^*(x) \leq 0 \qquad (18)$$

which, from the definition of $\omega(x)$, is equivalent to

$$L_{f(x,d)}V(x) + L_{g(x)}V(x)\pi(x)\psi^*(x) \leq \widetilde{\alpha}(x,d) \qquad (19)$$

for all $(x,d) \in \mathbb{R}^n \times \mathbb{R}^p$. If $L_{g(x)}V(x)\pi(x) \neq 0$ then

$$\omega(x) + L_{g(x)}V(x)\pi(x)\psi^*(x) = \omega(x) - \max\{0,\omega(x)\} \leq 0 \ . \qquad (20)$$

When $L_{g(x)}V(x)\pi(x) = 0$ we must show that $\omega(x) \leq 0$. From the bounded control property, when $|x| \leq \chi$ and $L_{g(x)}V(x)\pi(x) = 0$ we have $\omega(x) \leq 0$. For $|x| \geq \chi$,

$$L_{g(x)}V(x)\pi(x) = 0 \quad \Longrightarrow \limsup_{z\to x} L_{g(z)}V(z)\pi(z) = 0 \qquad (21)$$

so from (15) in the clf property we also have that $L_{g(x)}V(x)\pi(x) = 0$ and $|x| \geq \chi$ imply $\omega(x) \leq 0$. $\triangle$

The function $\psi^*(x)$ is the minimum norm value for v, as a function of x, that satisfies

$$\omega(x) + vL_{g(x)}V(x)\pi(x) \leq 0 \ . \qquad (22)$$

It is the same choice as in [6, equation (23)]. When $\omega(x)$ and $L_{g(x)}V(x)\pi(x)$ are locally Lipschitz on $\mathbb{R}^n \setminus \{0\}$ then it is not difficult to verify that $\psi^*(x)$ defined in (17) is locally Lipschitz on $\mathbb{R}^n \setminus \{0\}$. If $\pi(x)$ and $\psi^*(x)$ are locally Lipschitz on $\mathbb{R}^n \setminus \{0\}$ and $\pi(x)$ is everywhere continuous and zero at zero then the feedback we are proposing, $u = \pi(x)\psi(x)$, is locally Lipschitz on $\mathbb{R}^n \setminus \{0\}$ and everywhere continuous. This observation is useful for guaranteeing existence of solutions to the closed-loop differential equation. Recall that even when $\psi^*(x)$ is not locally Lipschitz we can always upper bound it by a locally Lipschitz (on $\mathbb{R}^n$) function and assign the same bound $\widetilde{\alpha}(x,d)$. This emphasizes that only rough information about the system and $V(x)$ is needed to synthesize our controller. For example, assuming that $V(x)$ is a clf and satisfies bcp for $(\pi, \widetilde{\alpha})$, we only need to find an upper bound for the function $\psi^*(x)$ defined in (17) to assign the bound $\widetilde{\alpha}$. Moreover, in the case where V is a clf for the pair $(-L_{g(x)}V(x)^T, \widetilde{\alpha})$, we do not need the exact magnitude or direction of $L_{g(x)}V(x)$ to find a smooth function $\pi(x)$ such that V is a clf for the pair $(\pi, \widetilde{\alpha})$. This observation is related to gain and phase margin properties of L_gV controllers made precise in [12].

If $V(x)$ and $L_{g(x)}V(x)$ are known and $V(x)$ is a clf and satisfies bcp for the pair $(-L_{g(x)}V(x)^T, \widetilde{\alpha}(x))$: there exist $\chi > 0$ and $\bar{v} \geq 0$ such that for all $|x| \leq \chi$,

$$\sup_d \left\{ L_{f(x,d)}V(x) - \widetilde{\alpha}(x,d) \right\} \leq \bar{v} |L_{g(x)}V(x)|^2 \ , \qquad (23)$$

then the control can be taken as

$$\begin{aligned} k(x) &= -(L_{g(x)}V(x))^T\psi_v(V(x)) \\ &= -L_{g(x)}\left(\int_0^{V(x)}\psi_v(s)ds\right)^T \end{aligned} \qquad (24)$$

with ψ_v continuous s.t. $\psi_v(V(x)) \geq \psi^*(x)$. So, maybe after reassigning the values of the level sets of V to obtain a new clf $\hat{V}$, we can assign the upper bound $\psi_v(V(x))\widetilde{\alpha}(x,d)$ to the derivative of $\hat{V}$ using an '$L_g\hat{V}$' controller (cf. [12, section 3.4.3] and [4]).

3.2 Sufficient conditions for clf

The next two results provide sufficient conditions for V to be a clf for the pair $(\pi, \widetilde{\alpha})$. The advantage of these results is that they can be used to guarantee that (15) holds without actually having to compute $\omega(x)$. The sufficient conditions are given in terms of a relationship between $\widetilde{\alpha}(x,d)$ and a bound on $L_{f(x,d)}V(x)$ when $L_{g(x)}V(x)\pi(x) = 0$ (the quantity $L_{g(x)}V(x)\pi(x)$ is assumed to be continuous in these results.)

Theorem 2 *Let $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be locally Lipschitz and let $\pi : \mathbb{R}^n \to \mathbb{R}^m$ be such that $L_{g(x)}V(x)\pi(x)$ is continuous and nonpositive. Let $\alpha(x,d)$ be a function such that $L_{g(x)}V(x)\pi(x) = 0$ implies $L_{f(x,d)}V(x) \leq \alpha(x,d)$. Then $V(x)$ is a clf for the pair $(\pi, \widetilde{\alpha})$ for any $\widetilde{\alpha}$ satisfying all of the following:*

1. *$\sup_d \left\{L_{f(x,d)}V(x) - \widetilde{\alpha}(x,d)\right\}$ is well-defined and locally bounded,*

2. *$\widetilde{\alpha}(x,d)$ is lower semi-continuous on the set $L_{g(x)}V(x)\pi(x) = 0$, i.e.,*

$$L_{g(x)}V(x)\pi(x) = 0 \Longrightarrow \liminf_{(y,e)\to(x,d)} \widetilde{\alpha}(y,e) = \widetilde{\alpha}(x,d),$$

3. *$\exists\ \rho_1(x)$ (continuous and nonnegative) and $\rho_2(x)$ (continuous and positive definite) s.t.*

$$|d| \geq \rho_1(x) \Longrightarrow L_{f(x,d)}V(x) - \widetilde{\alpha}(x,d) \leq -\rho_2(x),$$

4. *$\widetilde{\alpha}(x,d) - \alpha(x,d) \geq \rho(x)$ for some continuous, positive definite function ρ.*

Corollary 1 *Let $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be locally Lipschitz and let $\pi : \mathbb{R}^n \to \mathbb{R}^m$ be such that $L_{g(x)}V(x)\pi(x)$ is continuous and nonpositive. Let $\alpha(x,d)$ be a function such that $L_{g(x)}V(x)\pi(x) = 0$ implies $L_{f(x,d)}V(x) \leq \alpha(x,d)$ and such that the quantity $\sup_d \left\{L_{f(x,d)}V(x) - \alpha(x,d)\right\}$ is well-defined and locally bounded. Then $V(x)$ is a clf for the pair $(\pi, \widetilde{\alpha})$ for any $\widetilde{\alpha}$ satisfying*

1. *$\widetilde{\alpha}$ is lower semi-continuous on the set $L_{g(x)}V(x)\pi(x) = 0$,*

2. *$\exists\ \rho_1(x)$ (cont., pos. def.) and $\rho_2(d)$ (cont., pos. def., radially unbounded) s.t.*

$$\widetilde{\alpha}(x,d) - \alpha(x,d) \geq \max\left\{\rho_1(x), \rho_2(d)\right\} . \qquad (25)$$

3.3 Sufficient conditions for bcp

Throughout this section we assume that V is continuously differentiable and we address the relationship between the bounded control property (definition 3) and the more familiar small control property used in [13] and [6]. We will show that the bounded control property holds if a small control property holds (see definition 5) and the function $\pi(x)$ is strong enough. The latter is made precise by the following definition:

Definition 4 *Given a C^1 function $V(x)$ and a function $\lambda : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, the function π is said to locally dominate λ if there exist $\chi > 0$ and $\mu > 0$ s.t., $\forall |x| \leq \chi$,*

$$-L_{g(x)}V(x)\pi(x) \geq \mu |L_{g(x)}V(x)| \lambda(|L_{g(x)}V(x)|)$$

Definition 5 *The C^1 function V satisfies* the small control property (scp) for $\widetilde{\alpha}$ *if there exists a continuous, positive definite function $p(x)$ satisfying: for each $\varepsilon > 0$, there exists $\nu > 0$ such that $|x| \leq \nu$ implies the existence of u such that $|u| \leq \varepsilon$ and*

$$\sup_d \left\{L_{f(x,d)}V(x) - \widetilde{\alpha}(x,d)\right\} + L_{g(x)}V(x)u + p(x) \leq 0 .$$

Theorem 3 *If the C^1 function V satisfies the small control property for $\widetilde{\alpha}$ then there exists $\lambda : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ smooth on $(0,\infty)$, continuous everywhere and zero at zero, such that if π locally dominates λ then $V(x)$ satisfies the bounded control property for $(\pi, \widetilde{\alpha})$.*

There are simple examples illustrating a tradeoff between smoothness of the control at the origin and the inherent gain and phase margin robustness of L_gV-type controllers. However, it is possible to get smoothness near the origin while still retaining L_gV-type controller properties away from the origin (see the full-length version for more details).

4 Disturbance attenuation clf's

This section includes a study of assigning $\mathcal{L}_2$ or $\mathcal{L}_\infty$ input-output gain when adding integrators. In particular, we will study the system

$$\begin{aligned} \dot{x} &= \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} q(x,d) \\ u + \phi(x,d) \end{pmatrix} = f(x,d) + g(x)u \\ y &= h(x) , \end{aligned} \qquad (26)$$

where the output function $h(x)$ is continuous, and we will discuss what can be said about gain assignment for the full system based on what can be said for the x_1 subsystem with x_2 thought of as control. For other results on "backstepping" locally Lipschitz control laws, see [5, Section 5.4] and [3].

Recall that we say $d \in \mathcal{L}_\infty$ if d is essentially bounded, i.e., $||d||_\infty :=$ ess. $\sup_{t\geq 0} |d(t)| < \infty$. We say that $d \in \mathcal{L}_2$ if d is square integrable, i.e., $||d||_2^2 := \int_0^\infty |d(t)|^2 dt < \infty$.

4.1 ISS clf's

For the $\mathcal{L}_\infty$ (ISS/IOS) case, we have motivated in section 2 that we want to assign to the derivative of V a bound $\widetilde{\alpha}(x,d)$ that satisfies

$$V(x) \geq \max\{\gamma(|d|), \epsilon\} \quad \Longrightarrow \quad \widetilde{\alpha}(x,d) \leq -\kappa(V(x)) \tag{27}$$

for some $\epsilon \geq 0$ and functions γ and κ of class-$\mathcal{K}_\infty$ (continuous, zero at zero, strictly increasing and unbounded). The following assumption on V and π will make V a clf for the pair $(\pi, \widetilde{\alpha})$ with $\widetilde{\alpha}$ satisfying (27).

Assumption 1 *The locally Lipschitz, positive definite, radially unbounded function $V(x)$, the function $\pi(x)$, the class-$\mathcal{K}_\infty$ functions δ, γ and κ satisfy*

1. $\delta(|h(x)|) \leq V(x)$,
2. $L_{g(x)}V(x)\pi(x)$ *is continuous and nonpositive,*
3. $\{L_{g(x)}V(x)\pi(x) = 0\,,\ V(x) \geq \gamma(|d|)\} \Longrightarrow L_{f(x,d)}V(x) \leq -\kappa(V(x))$.

Remark 4.1 From [6, Definition 4.1], when V is C^1 and if $L_{g(x)}V(x)\pi(x) = 0$ only when $L_{g(x)}V(x) = 0$ then assumption 1 makes V a rclf. △

Corollary 2 *If assumption 1 holds then $\exists\widetilde{\alpha}(x,d)$ s.t.*

$$V(x) \geq \gamma(|d|) \quad \Longrightarrow \quad \widetilde{\alpha}(x,d) \leq -0.5\kappa(V(x)) \tag{28}$$

such that $V(x)$ is a clf for the pair $(\pi, \widetilde{\alpha})$. Moreover, $V(x)$ satisfies bcp for $(\pi, \widetilde{\alpha})$ if there exist $\chi > 0$ and $\bar{v} \geq 0$ such that

$$\sup_{\{d: V(x) \geq \gamma(|d|)\}} L_{f(x,d)}V(x) + 0.5\kappa(V(x)) \leq -\bar{v}L_{g(x)}V(x)\pi(x) \qquad \forall |x| \leq \chi\ . \tag{29}$$

The next corollary says that if an $\mathcal{L}_\infty$ gain with an arbitrarily small offset at the origin is allowed, i.e., $\epsilon > 0$ in (27), then bcp will hold as long as $f(0,0) = 0$.

Corollary 3 *If assumption 1 holds and $f(0,0) = 0$ and $V(0) = 0$ then, for each $\epsilon > 0$, $\exists\widetilde{\alpha}_\epsilon(x,d)$ s.t.*

$$V(x) \geq \max\{\gamma(|d|), \epsilon\} \Longrightarrow \widetilde{\alpha}_\epsilon(x,d) \leq -0.5\kappa(V(x)) \tag{30}$$

s.t. $V(x)$ is a clf and satisfies bcp for the pair $(\pi, \widetilde{\alpha})$.

The next corollary says that if V satisfies bcp for some continuous, negative definite function $\alpha(x)$ when $d \equiv 0$ then the $\mathcal{L}_\infty$ gain γ can be modified locally so that bcp holds for $(\pi, \widetilde{\alpha})$ where $\widetilde{\alpha}(x,d)$ satisfies (27) with $\epsilon = 0$.

Corollary 4 *If assumption 1 holds and, for the system $\dot{x} = f(x,0) + g(x)u$, $V(x)$ satisfies the bounded control property for the pair $(\pi, -\tilde{\kappa}(V(x)))$ then, for each $\nu > 0$ there exist class-$\mathcal{K}_\infty$ functions γ_ν and κ_ν and a function $\widetilde{\alpha}_\nu(x,d)$ satisfying*

$$\begin{aligned} s \geq \nu \quad &\Longrightarrow \quad \gamma_\nu(s) = \gamma(s)\ ,\ \kappa_\nu(s) = \kappa(s) \\ V(x) \geq \gamma_\nu(|d|) \quad &\Longrightarrow \quad \widetilde{\alpha}_\nu(x,d) \leq -0.5\kappa_\nu(V(x)) \end{aligned} \tag{31}$$

s.t. $V(x)$ is a clf and satisfies bcp for the pair $(\pi, \widetilde{\alpha}_\nu)$.

The next result, which applies to the system (26) where a perturbed integrator is added, is similar to what is reported in [8] and [11].

Proposition 1 *For the system (26), if there exist a locally Lipschitz function $\theta_1(x_1)$, a locally Lipschitz function $\pi(x)$, a positive definite, radially unbounded, locally Lipschitz function $V_1(x_1)$ and three class-$\mathcal{K}_\infty$ functions δ, γ and κ satisfying:*

1. $\delta(|h(x)|) \leq V_1(x_1)$
2. $V_1(x_1) \geq \gamma(|d|)\ ,\ x_2 = \theta_1(x_1) \Longrightarrow L_{q(x,d)}V_1(x_1) \leq -\kappa(V_1(x_1))$,
3. $(x_2 - \theta_1(x_1))^T\pi(x)$ *is nonpositive and zero only when $x_2 = \theta_1(x_1)$*

then, for each $\mu > 0$, the functions

$$V(x) = V_1(x_1) + \mu|x_2 - \theta_1(x_1)|^2 \tag{32}$$

$\pi(x)$ and δ, γ and κ satisfy assumption 1.

Combining the proposition with corollary 2, we have conditions under which $V(x)$ defined in (32) is a clf for the pair $(\pi, \widetilde{\alpha})$ with $\widetilde{\alpha}$ satisfying (28). If $V(x)$ also satisfies (29) then combining corollary 2 with theorem 1, we have a new locally Lipschitz feedback $\theta_2(x) = \pi(x)\psi(x)$ and a new positive definite radially unbounded locally Lipschitz function $V_2(x) = V(x)$ that can be used for another application of the proposition if another perturbed integrator is added. In the process, γ, which characterizes the IOS gain, remains unchanged, and κ becomes 0.5κ. By iterating this process for a chain of n perturbed integrators (which is possible if, at each step, a bounded control property holds), we get a control law of the form $u = \psi_{n-1}(x)\pi_{n-1}(x)$ where $\pi_{i-1}(x) = -(x_i - \pi_{i-2}(x)\psi_{i-2}(x))$ and $\psi_i(x)$ comes from the ith application of corollary 2 with theorem 1. The form of this control law is very similar to what is used in [17] for semi-global stabilization with partial state feedback. The only difference is that, there, the functions $\psi_i(x)$ are (sufficiently large) constants.

4.2 $\mathcal{L}_2$ clf's

For $\mathcal{L}_2$ disturbance attenuation problems, we have motivated in section 2 that we want to assign to the derivative of V a bound $\widetilde{\alpha}(x,d)$ of the form

$$\widetilde{\alpha}(x,d) = -\kappa(x)|h(x)|^2 + \gamma^2|d|^2 \tag{33}$$

where κ is a continuous, positive-valued function. We will see that the following assumption on V and π will make V a clf for the pair $(\pi, \widetilde{\alpha})$ with $\widetilde{\alpha}$ of the form given in (33).

Assumption 2 *The locally Lipschitz, positive definite, radially unbounded function V, a continuous, positive definite function $\alpha : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, a strictly positive real number γ, a continuous function $\kappa : \mathbb{R}^n \to \mathbb{R}_{>0}$ and a function π satisfy:*

1. *$L_{g(x)}V(x)\pi(x)$ is continuous and nonpositive,*

2. $L_{g(x)}V(x)\pi(x) = 0 \implies L_{f(x,d)}V(x) \leq -\alpha(x) - \kappa(x)|h(x)|^2 + \gamma^2|d|^2$,

3. *there exists $\rho_1(x)$ (continuous, nonnegative) and $\rho_2(x)$ (continuous, positive definite) such that*

$$|d| \geq \rho_1(x) \implies$$
$$L_{f(x,d)}V(x) + 0.5\alpha(x) + \kappa(x)|h(x)|^2 - \gamma^2|d|^2 \leq -\rho_2(x) \ .$$

Corollary 5 *If assumption 2 is satisfied then $V(x)$ is a control Lyapunov function for the pair $(\pi, \widetilde{\alpha})$ where*

$$\widetilde{\alpha}(x,d) = -0.5\alpha(x) - \kappa(x)|h(x)|^2 + \gamma^2|d|^2 \ . \quad (34)$$

If $f(x,d)$ is affine in d, i.e., $f(x,d) = f_o(x) + f_1(x)d$, then the third condition of assumption 2 is automatically satisfied. For this affine case and when V is C^1 and $L_{g(x)}V(x)\pi(x) = 0$ only when $L_{g(x)}V(x) = 0$ and $\kappa(x) \equiv 1$, this type of result has already been established in [7, Lemma 9.5.2, Lemma 9.5.3]. Those results produce an everywhere smooth controller but the controllers are not of the L_gV-type - the controllers $u = k(x)$ are not such that $L_{g(x)}V(x)k(x)$ is nonpositive. (The use of L_gV-type controllers for achieving the result we have presented is discussed at the end of [7, section 9.5].) Thus, for the affine case, if the $V(x)$ does not satisfy bcp for the pair $(\pi, \widetilde{\alpha})$, where $\widetilde{\alpha}$ is defined in (34) with $\kappa(x) \equiv 1$, the results of [7] and the discussion at the end of section 3 may be used to get a controller that is smooth at the origin but with the $L_{g(x)}V(x)$ structure away from the origin.

We again consider the system (26) with the added assumption that d enters in an affine manner. The result here is essentially the same as those in [7, Theorem 9.5.4, Corollary 9.5.6].

Proposition 2 *For the system (26) under the assumption that d enters in an affine manner, if there exist a locally Lipschitz function $\theta_1(x_1)$, a locally Lipschitz function $\pi(x)$, a positive definite, radially unbounded, locally Lipschitz function $V_1(x_1)$, a continuous, positive definite function α_1, a continuous, positive-valued function $\kappa_1(x_1)$ and a strictly positive real number γ such that*

1. $x_2 = \theta_1(x_1) \implies L_{q(x,d)}V_1(x_1) \leq -\alpha_1(x_1) - \kappa_1(x_1)|h(x)|^2 + \gamma^2|d|^2$,

2. *$(x_2 - \theta_1(x_1))^T\pi(x)$ is nonpositive and zero only when $x_2 = \theta_1(x_1)$*

then, for each $\mu > 0$ and each continuous, positive definite function $\alpha(x)$ satisfying

$$x_2 = \theta_1(x_1) \implies \alpha(x) \leq \alpha_1(x_1) \ , \quad (35)$$

the functions

$$V(x) = V_1(x_1) + \mu|x_2 - \theta_1(x_1)|^2 \quad (36)$$

$\pi(x)$, $\alpha(x)$, $\kappa(x) = \kappa_1(x_1)$ and γ satisfy assumption 2.

Combining the proposition with corollary 5, we have conditions under which $V(x)$ defined in (36) is a clf for the pair $(\pi, \widetilde{\alpha})$ with $\widetilde{\alpha}$ satisfying (34). If $V(x)$ also satisfies bcp for this pair then, with theorem 1, we have a new locally Lipschitz feedback $\theta_2(x) = \pi(x)\psi(x)$ and a new pos. def., radially unbounded locally Lipschitz function $V_2(x) = V(x)$ that can be used for another application of the proposition if another perturbed integrator is added. In the process, γ and κ, which characterize the $\mathcal{L}_2$ gain, remain unchanged. This procedure can be repeated for a chain of perturbed integrators of length n as long as at each step bcp is satisfied.

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