Further Results on Robust Semiglobal Stabilization with Dynamic Input Uncertainties

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Abstract

This paper presents a dynamic state feedback approach to the semiglobal stabilization of nonlinear systems with minimum-phase dynamic input uncertainties. The assumption needed to get this new result is weaker than the assumption of input feedback passivity or that of nonlinear small gain considered up to now. Here we show how the result proposed in [3] can be extended to the general relative degree case. For ease of presentation, we restrict ourselves to the single input single output case.

1 Introduction

Although more and more results are made available on global stabilization of nonlinear systems, it is known (see [10] for instance) that this property may be lost in the presence of some dynamic input uncertainties. Nevertheless, it can be made robust to input strictly passive dynamic uncertainties (see [9, 5]) or those satisfying a small gain condition (see [4, 2]). Such dynamic uncertainties are minimum phase and with zero relative degree.

In this paper, we prove that the minimum-phase property only is already sufficient. This fact is known for linear systems. It has been extended to the nonlinear case in [3] for the relative degree zero case. We state it here for a general relative degree. The price to be paid in this generalization is that we will only achieve a semiglobal (practical) stabilization and that the dominant part of the "high-frequency" gain as well as the relative degree are needed to be known.

Section 2 describes the class of systems to be controlled and states the needed assumptions on the plant. Section 3 proposes an observer/controller mixed scheme and formulates the main results. Section 4 gives the proof of the main theorem. Concluding remarks are contained in Section 5.

2 Systems and Problem Statement

We consider systems of the form

$$\begin{aligned} \dot{x} &= f(x) + g(x)y \\ \dot{z} &= q(z,x,u) \\ y &= h(z,x,u) \end{aligned} \qquad (1)$$

where x in $\mathbb{R}^n$ denotes the state of the certain part, z in $\mathbb{R}^p$ is the state of the uncertain part (i.e. not available for feedback design), u in $\mathbb{R}$ is the control input, y in $\mathbb{R}$ is the output of the uncertain z-subsystem and the input of the certain x-subsystem.

The goal of this paper is to address the following control problems for system (1).

Semiglobal Practical Stabilization Problem. Find a family of dynamic feedback laws, indexed by $\lambda \in \mathbb{R}^\ell$,

$$\dot{\sigma} = \nu_\lambda(\sigma, x) \ , \quad u = \mu_\lambda(\sigma, x) \qquad (2)$$

in such a way that, for any given compact neighborhoods Ω_1, Ω_2 of $(x,z) = 0$ with the property that $\Omega_1 \subset \Omega_2$, there exist a parameter vector λ and a compact set Ω_σ such that all the solutions of the closed-loop system (1), (2) starting from $\Omega_2 \times \Omega_\sigma$ reach $\Omega_1 \times \Omega_\sigma$ in finite time.

Semiglobal Asymptotic Stabilization Problem. For any given compact set Ω of $(x,z) = 0$, design a dynamic feedback law of the form (2) so that the origin of the closed-loop system is asymptotically stable with basin of attraction containing $\Omega \times \Omega_\sigma$ for some compact set Ω_σ.

The above problems will be solved on the basis of the following assumptions.

Assumption 1 *(Uniform relative degree)* There exist a nonnegative integer $m \geq 0$ and a global diffeomorphism Ψ such that $\Psi(z) = z$ if $m = 0$ and $\Psi(z) = (y, y_1, \dots, y_{m-1}, \zeta)$ if $m > 0$ and that the z-subsystem of (1) is rewritten as

$$\begin{array}{rcl} \dot{y} &=& y_1 \\ &\vdots& \\ \dot{y}_{m-2} &=& y_{m-1} \\ \dot{y}_{m-1} &=& u - h_1(y, y_1, \dots, y_{m-1}, \zeta, x, u) \\ \dot{\zeta} &=& q_\zeta(y, y_1, \dots, y_{m-1}, \zeta, x, u) \end{array} \tag{3}$$

where h_1 and q_ζ are C^2 and there is a real number $\varepsilon \in (0, 1]$ so that, for all $(y, y_1, \dots, y_{m-1}, \zeta, x, u)$

$$\left| \frac{\partial h_1}{\partial u}(y, y_1, \dots, y_{m-1}, \zeta, x, u) \right| \leq 1 - \varepsilon . \tag{4}$$

The constraint (4) in this Assumption means that the dominant part of the "high frequency" gain is known (and normalized to 1).

The zero relative degree case (i.e. $m = 0$) was considered in [3]. In the rest of this paper, we consider the nonzero relative degree case, i.e. $m > 0$.

Assumption 2 *(Minimum-phase)* There exist a C^1 positive definite, radially unbounded function W, a class-$\mathcal{K}_\infty$ function α and a class-$\mathcal{K}$ function γ such that, for all $(y, y_1, \dots, y_{m-1}, \zeta, x, u)$,

$$\begin{array}{l} \frac{\partial W}{\partial \zeta}(\zeta) q_\zeta(y, y_1, \dots, y_{m-1}, \zeta, x, u) \\ \qquad \leq -\alpha(|\zeta|) + \gamma(|(x, y, \dots, y_{m-1})|) . \end{array} \tag{5}$$

This assumption implies that, whatever the input u may be, if x and y and its derivative are bounded so is z and if these signals converge to 0, so does z (see [7] for a more detailed analysis).

Assumption 3 *(Stabilizability)* There is a C^m function ϑ such that the origin is a globally asymptotically stable equilibrium point of $\dot{x} = f(x) + g(x)\vartheta(x)$.

This Assumption says that we know how to stabilize the system (1) whenever there is no uncertainty, i.e. the x-subsystem with $y = u$. Applying Theorem 1 in Sontag [6], while preserving its differentiability properties, the control law ϑ can be modified to ensure the input-to-state stability of the system $\dot{x} = f(x) + g(x)(\vartheta(x) + v)$ with respect to the input v. In the rest of the paper, we assume that ϑ possesses this property.

With this stabilizability information in mind, it is natural to choose the controller as

$$u = \vartheta(x) + v \tag{6}$$

where v remains to be designed to counteract the effect of input uncertainty. As demonstrated in our previous work [3] in the zero-relative-degree case, the synthesis of the extra control term v can be based on some observer output that approximates the uncertain nonlinearity. We show in the sequel that this idea can be extended to the higher relative degree case.

As in [3], a rectifiability-like condition is required to design such an observer. Namely,

Assumption 4 *(Rectifiability)* We know a C^1 mapping $l : \mathbb{R}^n \to \mathbb{R}$ such that

$$\frac{\partial l}{\partial x}(x) g(x) = 1 \tag{7}$$

This Assumption is meaning in effect that y is observable from x.

3 Control Design and Main Results

3.1 Controller/observer synthesis

Motivated by [3] and the type of control laws (6) we are looking for, we first introduce a change of coordinates in the unmodeled dynamics space and then design a suitable state observer.

Let

$$\chi_0 = y - \vartheta(x) . \tag{8}$$

From (3) in Assumption 1, we have

$$\dot{\chi}_0 = y_1 - \frac{\partial \vartheta}{\partial x}(f(x) + g(x)(\vartheta(x) + \chi_0)) \tag{9}$$

Then, we introduce

$$\chi_1 = y_1 - \frac{\partial \vartheta}{\partial x}(f(x) + g(x)(\vartheta(x) + \chi_0)) \tag{10}$$

which implies

$$\dot{\chi}_0 = \chi_1 \tag{11}$$

By induction, we can obtain the following

$$\chi_{i+1} = y_{i+1} + \varphi_i(x, \chi_0, \dots, \chi_i) \tag{12}$$

$$\dot{\chi}_i = \chi_{i+1} , \quad \forall\, 1 \leq i \leq m-2 \tag{13}$$

Finally, we have

$$\dot{\chi}_{m-1} = u + \delta(\chi_0, \dots, \chi_{m-1}, \zeta, x, u) \tag{14}$$

where

$$\delta(\chi_0,\ldots,\chi_{m-1},\zeta,x,u) \;=\; \tag{15}$$
$$\varphi_{m-1}(x,\chi_0,\ldots,\chi_{m-1}) - h_1(y,y_1,\ldots,y_{m-1},\zeta,x,u)$$

Note that, by Assumption 1, the function δ satisfies, for all $(\chi_0,\ldots,\chi_{m-1},\zeta,x,u)$

$$\left|\frac{\partial\delta}{\partial u}(\chi_0,\ldots,\chi_{m-1},\zeta,x,u)\right| \;\leq\; 1 \;-\; \varepsilon\;. \tag{16}$$

Thanks to the rectifiability condition in Assumption 4, it holds

$$\chi_0 \;=\; \dot{l} \;-\; \frac{\partial l}{\partial x}(f(x)+g(x)\vartheta(x)) \tag{17}$$

This allows us to introduce the following observer:

$$\begin{aligned}
\dot{\widehat{\chi}}_0 &= \widehat{\chi}_1 + Lp_1\left(\dot{l} - \frac{\partial l}{\partial x}(f(x)+g(x)\vartheta(x)) - \widehat{\chi}_0\right)\\
\dot{\widehat{\chi}}_1 &= \widehat{\chi}_2 + L^2p_2\left(\dot{l} - \frac{\partial l}{\partial x}(f(x)+g(x)\vartheta(x)) - \widehat{\chi}_0\right)\\
&\vdots\\
\dot{\widehat{\chi}}_{m-1} &= u+\widehat{\delta} + L^mp_m\left(\dot{l} - \frac{\partial l}{\partial x}(f(x)+g(x)\vartheta(x)) - \widehat{\chi}_0\right)\\
\dot{\widehat{\delta}} &= L^{m+1}p_{m+1}\left(\dot{l} - \frac{\partial l}{\partial x}(f(x)+g(x)\vartheta(x)) - \widehat{\chi}_0\right)
\end{aligned} \tag{18}$$

where $L>0$ and the constants p_i's are design parameters.

The above observer can be realized with the states:

$$\begin{aligned}
\sigma_i &= \widehat{\chi}_i - L^{i+1}p_{i+1}l(x) \quad \forall\, 0\leq i\leq m-1\\
\sigma_m &= \widehat{\delta} - L^{m+1}p_{m+1}l(x)
\end{aligned} \tag{19}$$

The control is then chosen as:

$$u \;=\; -\mathtt{sat}\left(\sum_{i=0}^{m-1} c_i\widehat{\chi}_i + \widehat{\delta}\right) \tag{20}$$

where sat is a function to be made precise later on, either as the identity function or as a saturation function and where the c_i's are such that the polynomial $s^m + c_{m-1}s^{m-1} + \cdots + c_1 s + c_0$ has all its roots in the open left-half complex plane.

So our controller is made of (18) (realized with (19)) and (20).

3.2 Main results

Theorem 1 *Under Assumptions 1 to 4, for any compact set Ω in $\mathbb{R}^{n+p}$, there exist a compact set Ω_σ in $\mathbb{R}^{(m+1)}$, a function* sat *and design parameters p_i $(1\leq i\leq m+1)$ such that, for all sufficiently large L, the closed-loop system (1), (18) and (20) admits the origin as a practically stable equilibrium point with basin of attraction containing $\Omega\times\Omega_\sigma$.*

Under extra stability conditions around the origin on the system (1), semiglobal asymptotic stabilization can be obtained.

Theorem 2 *Under the conditions of Theorem 1, if the matrices*

$$\begin{gathered}
\frac{\partial f}{\partial x}(0) + g(0)\frac{\partial\vartheta}{\partial x}(0)\\
\frac{\partial q_\zeta}{\partial\zeta}(0) + \frac{\partial q_\zeta}{\partial u}(0)\left(1-\frac{\partial h_1}{\partial u}(0)\right)^{-1}\frac{\partial h_1}{\partial\zeta}(0)
\end{gathered} \tag{21}$$

are asymptotically stable, then the system (1) is semiglobally asymptotically stabilized by the observer-based controller (20).

Note that the eigen-values of

$$\frac{\partial q_\zeta}{\partial\zeta}(0) + \frac{\partial q_\zeta}{\partial u}(0)\left(1-\frac{\partial h_1}{\partial u}(0)\right)^{-1}\frac{\partial h_1}{\partial\zeta}(0)$$

are the zeros of the z-subsystem in (1) with u as input and y as output linearized at the origin.

4 Proof of the Main Results

Due to space limitations, we give (without all the details) only the proof of of Theorem 1. The complete proofs can be found at

`http://cas.ensmp.fr/~praly/Papers/In-Dist-Min-Phi-CDC98.ps.gz`

To prove Theorem 1, after writing the closed-loop system in appropriate coordinates, we follow the lines proposed by A. Isidori [1] for another proof of [8]. An important new ingredient here is the recursive design of the observer parameters following an algorithm which is in some sense a dual of the backstepping procedure (see Lemma 2).

Step a. – Description of the closed-loop system: For the time being, let sat be a C^2 function satisfying

$$\left|\frac{d\mathtt{sat}}{ds}\right| \;\leq\; 1\;. \tag{22}$$

Let us denote

$$e_i \;=\; L^{m-i}(\widehat{\chi}_i-\chi_i)\;,\quad e_m \;=\; \widehat{\delta}-\delta \tag{23}$$
$$\chi=(\chi_0,\ldots,\chi_{m-1})\;,\quad e=(e_0,\ldots,e_m)\;. \tag{24}$$

It can be shown that the closed-loop system (1), (18) and (20) can be rewritten in the (x,χ,ζ,e) coordinates as

$$
\begin{aligned}
\dot{x} &= f(x) + g(x)(\vartheta(x) + \chi_0) \\
\dot{\chi}_i &= \chi_{i+1} \ , \quad 0 \le i \le m-2 \\
\dot{\chi}_{m-1} &= \Phi_{\mathtt{sat}}(L, x, \chi, \zeta, e) + \delta \\
\dot{\zeta} &= q_1(\zeta, \chi, x, \Phi_{\mathtt{sat}}(L, x, \chi, \zeta, e)) \\
\dot{e}_i &= L(e_{i+1} - p_{i+1} e_0) \ , \quad 0 \le i \le m-2 \\
\dot{e}_{m-1} &= L(e_m - p_m e_0) \\
\dot{e}_m &= -L p_{m+1} \left(1 - \frac{\partial h_1}{\partial u} \frac{d\mathtt{sat}}{ds}\right) e_0 \\
&\quad -\Pi_{\mathtt{sat}}(L, x, \chi, \zeta, e) \\
&\quad -\frac{\partial h_1}{\partial u} \frac{d\mathtt{sat}}{ds} \sum_{i=0}^{m-1} c_i \frac{e_{i+1} - p_{i+1} e_0}{L^{m-i-1}}
\end{aligned}
\tag{25}
$$

where $\Phi_{\mathtt{sat}}$ and $\Pi_{\mathtt{sat}}$ are C^1 functions which are :
- bounded in L when $L \ge 1$.
- bounded in e if sat is a bounded function.

These functions are denoted $\Phi_{\mathtt{Id}}$ and $\Pi_{\mathtt{Id}}$ when sat is simply the identity function. In fact, we have :

$$
u = \Phi_{\mathtt{sat}}(L, x, \chi, \zeta, e) \ . \tag{26}
$$

Also, we want to study the solutions of the closed-loop system (1), (18) and (20) with initial condition $(x(0), z(0), \sigma(0))$ in a given compact set $\Omega \times \Omega_\sigma$. From Assumption 1, (12), (15), (19), (23) and since u is bounded in e when sat is a bounded function, there exist a positive real number b, independent of L, such that, for $L \ge 1$,

$$
|(x(0), \chi(0), \zeta(0))| \le b \quad , \qquad |e(0)| \le b L^{(m+1)} \ . \tag{27}
$$

Step b. – The (x, χ, ζ)-subsystem and definition of sat: When there is no estimation error (i.e. $\widehat{\chi}_i = \chi_i$ and $\widehat{\delta} = \delta$) and when sat = Id, the closed-loop system is

$$
\begin{aligned}
\dot{x} &= f(x) + g(x)(\vartheta(x) + \chi_0) \\
\dot{\chi}_0 &= \chi_1 \\
&\vdots \\
\dot{\chi}_{m-1} &= -\sum_{i=0}^{m-1} c_i \chi_i \\
\dot{\zeta} &= q_1(\zeta, \chi, x, \Phi_{\mathtt{Id}}(L, x, \chi, \zeta, 0))
\end{aligned}
\tag{28}
$$

From Assumption 2 and (12), we have, for all (ζ, χ, x, u),

$$
\frac{\partial W}{\partial \zeta}(\zeta) q_1(\zeta, \chi, x, u) \ \le \ -\alpha(|\zeta|) + \tilde{\gamma}(|(\chi, x)|) \tag{29}
$$

for some class-$\mathcal{K}$ function $\tilde{\gamma}$. This implies that the ζ-subsystem in (28) is ISS when (x, χ) is considered as the input. On the other hand, by construction and Assumption 1, for the (x, χ)-subsystem in (28), the origin is globally asymptotically stable. Consequently, for the overall system (28), the origin is globally asymptotically stable. From Lyapunov Converse Theorem (see, e.g., [4, Prop. 13]), we deduce the existence of a C^1 positive definite and radially unbounded function V such that its time derivative along the solutions of (28) satisfies:

$$
\dot{V}_{(28)}(x, \chi, \zeta) \ \le \ -V(x, \chi, \zeta) \tag{30}
$$

This function V will be the main tool of our analysis.

Let c be a positive real number such that

$$
\begin{aligned}
(x, z, \sigma) \in \Omega \times \Omega_\sigma \quad & (\implies |(x, \chi, \zeta)| \le b) \\
& \implies \quad V(x, \chi, \zeta) \le c \ .
\end{aligned}
\tag{31}
$$

We define the set

$$
\Delta := \{(x, \chi, \zeta, e) : \ |e| \le 1 \ , \ V(x, \chi, \zeta) \le c+2\} \tag{32}
$$

Associated with this set is defined the following positive real number:

$$
S_{\max} = \sup_{(x,\chi,\zeta,e) \in \Delta, L \ge 1} \left\{ \sum_{i=0}^{m-1} c_i \left(\chi_i + \frac{e_i}{L^{m-i}}\right) + \delta + e_m \right\} \tag{33}
$$

where the argument u of δ as defined in (15) is taken as $\Phi_{\mathtt{Id}}(L, x, \chi, \zeta, e)$.

We define the function sat such that it is C^2, bounded and satisfies the following properties:

$$
\begin{aligned}
\mathtt{sat}(s) &= s \quad \text{if} \quad |s| \le S_{\max} \\
\left| \frac{d\mathtt{sat}}{ds}(s) \right| &\le 1 \quad \forall s \in \mathbb{R}
\end{aligned}
\tag{34}
$$

Note that

$$
\begin{aligned}
\{(x, \chi, \zeta, e) \in \Delta \ , \quad L \ge 1\} \\
\implies \quad \Phi_{\mathtt{sat}}(L, x, \chi, \zeta, e) = \Phi_{\mathtt{Id}}(L, x, \chi, \zeta, e) \ .
\end{aligned}
\tag{35}
$$

Now the system (28) is obtained from the closed-loop system (25) by letting $e = 0$ and sat := Id on Δ. So, from (30), (14), (20) and (26), there exists a constant η_1, independent of L, such that, for (x, χ, ζ, e) in Δ and $L \ge 1$,

$$
\dot{V}_{(25)} \ \le \ -V \ + \ \eta_1 |e| \ . \tag{36}
$$

Also, since $\Phi_{\mathtt{sat}}$ is a bounded function of e, the derivative functions $\dot{x}$, $\dot{\chi}$ and $\dot{\zeta}$ are bounded on $V \le c+2$. Thus,

$$
V(x, \chi, \zeta) \le c+2 \quad \Rightarrow \quad \dot{V}_{(25)} \le \eta_2 \tag{37}
$$

with η_2 a constant independent of L.

Step c. – The e-subsystem and the observer parameters p_i: Our first step, in studying the e-subsystem, is to look at its dominant part when L is large. Namely we consider the system

$$\begin{array}{rcl} \dot{e}_0 & = & L\,e_1 \;-\; L\,p_1\,e_0 \\ & \vdots & \\ \dot{e}_{m-1} & = & Le_m \;-\; L\,p_m\,e_0 \\ \dot{e}_m & = & -L\,p_{m+1}\left(1-\dfrac{\partial h_1}{\partial u}\dfrac{d\mathtt{sat}}{ds}\right)e_0\ , \end{array} \qquad (38)$$

We observe that, from the properties of sat and (4) in Assumption 1:

$$\left|\frac{\partial h_1}{\partial u}\frac{d\mathtt{sat}}{ds}\right| \;\leq\; 1-\varepsilon\ . \qquad (39)$$

So, for any solution of the closed-loop system evolving in the set

$$\left\{(x,\chi,\zeta,e) : e\in\mathbb{R}^{(m+1)}\ ,\ V(x,\chi,\zeta) < c+2\right\}\ ,$$

we have

$$0 \;<\; \varepsilon \;\leq\; B(t) \;\leq\; 2-\varepsilon \qquad (40)$$

where

$$B(t) \;=\; 1-\left.\frac{\partial h_1}{\partial u}\frac{d\mathtt{sat}}{ds}\right|_{(x(t),\chi(t),\zeta(t),e(t))} \qquad (41)$$

We have

Lemma 1 *For the system*

$$\begin{array}{rcl} \dot{e}_0 & = & L\,e_1 \;-\; L\,p_1\,e_0 \\ & \vdots & \\ \dot{e}_{m-1} & = & Le_m \;-\; L\,p_m\,e_0 \\ \dot{e}_m & = & -L\,p_{m+1}\,B(t)e_0\ , \end{array} \qquad (42)$$

there exist real numbers p_i's and positive definite symmetric matrices P and Q such that, for any continuous function B, satisfying (40), we have

$$\overbrace{e^\top P\,e}^{\cdot} \;\leq\; -L\,e^\top Q\,e \qquad (43)$$

This Lemma can be proved by induction from the following technical result given without proof :

Lemma 2 *Assume that for the system*

$$\dot{E} \;=\; (A-K(t)C)\,E \qquad (44)$$

with A, K and C, matrices of appropriate dimensions, satisfying

$$|K(t)| \;\leq\; k \qquad (45)$$

we have positive definite symmetric matrices P and Q such that

$$\overbrace{E^\top P\,E}^{\cdot} \;\leq\; -E^\top Q\,E \qquad (46)$$

then there exists a positive real number p such that for the system

$$\begin{array}{rcl} \dot{e} & = & CE \;-\; pe \\ \dot{E} & = & AE \;-\; pK(t)e \end{array} \qquad (47)$$

we have positive definite symmetric matrices $\bar{P}$ and $\bar{Q}$ such that

$$\overbrace{(E^\top\ e)\bar{P}\begin{pmatrix}E\\ e\end{pmatrix}}^{\cdot} \;\leq\; -(E^\top\ e)\bar{Q}\begin{pmatrix}E\\ e\end{pmatrix} \qquad (48)$$

Consider now the actual e-subsystem in the closed-loop system (25). It is obtained from (38) by adding C^1 functions of (x,χ,ζ,e) which are linearly bounded in e and bounded in L for $L\geq 1$. So, from Lemma 1, there exist three positive real numbers η_3, η_4, η_5 (independent of L) such that

$$\begin{array}{l}\{V(x,\chi,\zeta)\leq c+2\ ,\ L\geq 1\} \\ \qquad\Longrightarrow\qquad \overbrace{e^\top P\,e}^{\cdot} \;\leq\; -(\eta_3 L-\eta_4)e^\top Pe \;+\; \eta_5\ .\end{array} \qquad (49)$$

Step d. – Practical stability: Consider any solution of (25) defined on the open set

$$\left\{(x,\chi,\zeta,e) : e\in\mathbb{R}^{(m+1)}\ ,\ V(x,\chi,\zeta) < c+2\right\}$$

with the initial condition satisfying (see (27) and (31))

$$V(x(0),\chi(0),\zeta(0))\leq c \quad \text{and} \quad |e(0)|\leq bL^{(m+1)}\ . \qquad (50)$$

where b and c are independent of L.

Such a solution is well defined on a right maximal interval $[0,T_c)$. We show in the sequel that $T_c = +\infty$.

If T_c were finite, (49) would imply that $|e(t)|$ cannot escape to infinity and so we would have

$$\lim_{t\to T_c} V(x(t),\chi(t),\zeta(t)) = c+2\ . \qquad (51)$$

Let us show that this is impossible when L is large enough. Denote $T_1 = 1/\eta_2$ with η_2 taken from (37). From (37) and (50), we know that

$$V(x(t),\chi(t),\zeta(t)) \;\leq\; c \;+\; 1 \quad \forall t\in[0,T_1] \qquad (52)$$

It follows that $T_c > T_1$. Using (49) and (50), we have, for t in $[0, T_c)$,

$$e(t)^T P e(t) \leq e^{-(\eta_3 L - \eta_4)t} |P| b^2 L^{2(m+1)} + \frac{\eta_5}{(\eta_3 L - \eta_4)} . \quad (53)$$

So, there exists a positive real number L_1^* such that, for $L \geq L_1^*$ and t in $[T_1, T_c)$, we have :

$$|e(t)| \ \leq \ \min\left\{1 \ , \ \frac{c+1}{\eta_1}\right\} \quad (54)$$

with η_1 involved in (36). For such values of L, the solution is in Δ for t in $[T_1, T_c)$. So it follows from (36), (52) and (54) that

$$V(t) \ \leq \ c \ + \ 1 \quad \forall t \in [T_1, T_c) \quad (55)$$

This contradicts (51). So we have established that $T_c = +\infty$ and that the closed-loop solution is bounded and remains in the open set $\{(e, x, \chi, \zeta) : e \in \mathbb{R}^{(m+1)} \ , \ V(x, \chi, \zeta) < c + 2\}$.

In fact, for any $\epsilon > 0$, there exist $T_2 > 0$ and $L_2^* > 0$ such that, for $L \geq L_2^*$ and t in $[T_2, +\infty)$, we have

$$\max\{V(x(t), \chi(t), \zeta(t)) \ , \ |e(t)|\} \ \leq \ \epsilon \ . \quad (56)$$

Indeed, let

$$\epsilon^* \ = \ \min\left\{\epsilon \ , \ \frac{\epsilon}{2\eta_1}\right\} \quad (57)$$

$$L_2^* \ = \ \max\left\{L_1^* \ , \ \frac{\eta_4 + (2\eta_5 |P^{-1}|)/\epsilon^{*\,2}}{\eta_3}\right\} \quad (58)$$

From (53), it is seen that, for all $L \geq L_2^*$, there exists $T_3 > 0$ such that

$$|e(t)| \ \leq \ \epsilon^* \quad \forall\, t \geq T_3 \quad (59)$$

Hence, in view of (36) and (52), we deduce the existence of some time instant $T_2 \geq T_3$ such that

$$V(x(t), \chi(t), \zeta(t)) \ \leq \ \epsilon \quad \forall\, t \geq T_2 \quad (60)$$

The proof of Theorem 1 is completed.

5 Concluding Remarks

We addressed the robust stabilization problem for nonlinear systems in the presence of minimum-phase dynamic input uncertainties. The zero relative degree case studied in our previous work [3] has been extended to the case of dynamic input uncertainty with *arbitrary* relative degree, provided that the a priori knowledge of this relative degree as well as of the "high-frequency gain" is available. We proposed a dynamic feedback semiglobal method to achieve semiglobal practical stabilization and, under additional conditions, semiglobal asymptotic stabilization. The present framework can be seen as an extension of recent work on the basis of passivity and nonlinear small-gain arguments.

In this paper the z-dynamics are treated as uncertain. However our results hold even if these dynamics are known. From a practical viewpoint, the measurement of z may be corrupted or the dynamics may be too complicated to be taken into account explicitly in the control law.

References

[1] A. Isidori. Private communication.

[2] Z. P. Jiang and I. Mareels, "A small gain control method for nonlinear cascaded systems with dynamic uncertainties", *IEEE Trans. Automat. Control*, Vol. 42, No. 3, pp. 292-308, 1997.

[3] L. Praly and Z.P. Jiang, "Semiglobal stabilization in the presence of minimum-phase dynamic input uncertainties". NOLCOS'98. Vol. 2 pp. 325-330, July 1998

[4] L. Praly and Y. Wang, "Stabilization in spite of matched unmodeled dynamics and an equivalent definition of input-to-state stability", *Math. Control Signals Systems*, Vol. 9, pp. 1-33, 1996.

[5] R. Sepulchre, M. Jankovic and P. V. Kokotović, *Constructive Nonlinear Control.* Springer-Verlag, 1997.

[6] E. D. Sontag, "Smooth stabilization implies coprime factorization", *IEEE Trans. Automat. Control*, **34** (1989), 435-443.

[7] E. D. Sontag and Y. Wang, "Output-to-state stability and detectability of nonlinear systems", *Systems & Control Letters*, Vol. 29, pp. 279-290, 1997.

[8] A. Teel and L. Praly, "Global stabilizability and observability imply semi-global stabilizability by output feedback", *Systems & Control Letters.*, 22 (1994), pp. 313-325.

[9] J. N. Tsitsiklis and M. Athans, "Guaranteed robustness properties of multivariable nonlinear stochastic optimal regulators", *IEEE Trans. Automat. Contr.*, Vol. 29, No. 8, pp. 690-696, 1984.

[10] Y. Zhang and P. A. Ioannou, "Robustness of nonlinear control systems with respect to unmodeled dynamics", *Proc. 35th IEEE Conf. Dec. Control*, pp. 315-320, Kobe, Japan, 1996.