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Control of Underactuated Mechanical Systems Using Switching and Saturation*

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Abstract

In this paper we present some ideas on the control of underactuated mechanical systems using switching and saturation. We focus on the swingup control problem for a class of "gymnast" robots and also for the classical cart–pole system. The design methodology is based on partial feedback linearization in a first stage to linearize the actuated degrees of freedom followed by the control of the transfer of energy from the actuated to the unactuated degrees of freedom in a second stage. In a typical swingup control the desired equilibrium is unstable in the closed loop system as a consequence of the non-minimum phase behavior of the system. For this reason it is necessary to switch controllers at the appropriate time to a controller which renders the equilibrium stable. The successful implementation of the switching control has proved to be nontrivial, both in simulation and in experiment. We discuss both local and global design methods and present some simulation results.

1 Introduction

Underactuated mechanical systems are mechanical systems with fewer actuators than degrees–of–freedom and arise in several ways, from intentional design as in the brachiation robot of Fukuda [1] or the Acrobot [2], in mobile robot systems when a manipulator arm is attached to a mobile platform, a space platform, or an undersea vehicle, or because of the mathematical model used for

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control design as when joint flexibility is included in the model [3]. In the latter sense, then, all mechanical systems are underactuated if one wishes to control flexible modes that are not directly actuated (the noncollocation problem), or even to include such things as actuator dynamics in the model description.

We consider an n-degree-of-freedom system with generalized coordinates $q^1, \ldots, q^n$, and $m < n$ actuators, each of which directly actuates a single degree of freedom. We partition the vector $q \in R^n$ of generalized coordinates as $q_1 \in R^\ell$ and $q_2 \in R^m$, where $q_1 \in R^\ell$ represents the unactuated (passive) joints and $q_2 \in R^m$ represents the actuated (active) joints. The Euler-Lagrange equations of motion of such a system are then given by [4]

$$M_{11}\ddot{q}_1 + M_{12}\ddot{q}_2 + h_1 + \phi_1 = 0 \tag{1}$$

$$M_{21}\ddot{q}_1 + M_{22}\ddot{q}_2 + h_2 + \phi_2 = \tau \tag{2}$$

where

$$M(q) = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \tag{3}$$

is the symmetric, positive definite inertia matrix, the vector functions $h_1(q, \dot{q}) \in R^\ell$ and $h_2(q, \dot{q}) \in R^m$ contain Coriolis and centrifugal terms, the vector functions $\phi_1(q) \in R^\ell$ and $\phi_2(q) \in R^m$ contain gravitational terms, and $\tau \in R^m$ represents the input generalized force produced by the m actuators at the active joints. For notational simplicity we will henceforth not write the explicit dependence on q of these coefficients.

2 Partial Feedback Linearization

Unlike fully actuated systems, which are always feedback linearizable, the system (1)-(2) is not linearizable in the q-coordinates, although in some cases, the system is linearizable after a nonlinear coordinate transformation. However, we may still linearize a portion of the system in the original q-coordinates. To see this, consider the first equation (1)

$$M_{11}\ddot{q}_1 + M_{12}\ddot{q}_2 + h_1 + \phi_1 = 0 \tag{4}$$

The term M_{11} is an invertible $\ell \times \ell$ matrix as a consequence of the uniform positive definiteness of the robot inertia matrix M in (3). Therefore we may solve for $\ddot{q}_1$ in equation (4) as

$$\ddot{q}_1 = -M_{11}^{-1}(M_{12}\ddot{q}_2 + h_1 + \phi_1) \tag{5}$$

and substitute the resulting expression (5) into (2) to obtain

$$\bar{M}_{22}\ddot{q}_2 + \bar{h}_2 + \bar{\phi}_2 = \tau \tag{6}$$

where the terms $\bar{M}_{22}$, $\bar{h}_2$, $\bar{\phi}_2$ are given by

$$\begin{aligned} \bar{M}_{22} &= M_{22} - M_{21}M_{11}^{-1}M_{12} \\ \bar{h}_2 &= h_2 - M_{21}M_{11}^{-1}h_1 \\ \bar{\phi}_2 &= \phi_2 - M_{21}M_{11}^{-1}\phi_1 \end{aligned}$$

It is easily shown that the $m \times m$ matrix $\bar{M}_{22}$ is itself symmetric and positive definite. A partial feedback linearizing controller can therefore be defined for equation (6) according to

$$\tau = \bar{M}_{22}u + \bar{h}_2 + \bar{\phi}_2 \tag{7}$$

where $u \in R^m$ is an additional control input yet to be defined. The complete system up to this point may be written as

$$\begin{aligned} M_{11}\ddot{q}_1 + h_1 + \phi_1 &= -M_{12}u && (8)\\ \ddot{q}_2 &= u && (9) \end{aligned}$$

Setting

$$u = -k_1 q_2 - k_2 \dot{q}_2 + \bar{u} \tag{10}$$

and defining state variables

$$\begin{array}{ll} z_1 = q_2 & z_2 = \dot{q}_2 \\ \eta_1 = q_1 & \eta_2 = \dot{q}_1 \end{array} \tag{11}$$

we may write the system in state space as

$$\begin{aligned} \dot{z} &= Az + B\bar{u} && (12)\\ \dot{\eta} &= w(z, \eta, \bar{u}) && (13) \end{aligned}$$

where $z^T = (z_1^T, z_2^T)$, $\eta^T = (\eta_1^T, \eta_2^T)$, and the matrix A is Hurwitz. We see from (12) and (13) that $z = 0$, $\bar{u} = 0$ defines an invariant manifold in state space. Since A is Hurwitz for positive values of gains in the matrices k_p and k_d this manifold is attractive. The dynamics on the manifold are given by

$$\dot{\eta} = w(0, \eta) \tag{14}$$

We now take as our starting point the problem of designing the control input u to stabilize the system (12)–(13). This class of systems falls into the class of feedforward systems considered by Teel [5], Mazenc and Praly [6], and Janković, et. al. [7].

3 Swingup Control: Case Studies

3.1 The Acrobot

The Acrobot is a two–link planar robot with a single actuator at the elbow. The equations of motion are given by [4]

$$\begin{aligned} m_{11}\ddot{q}_1 + m_{12}\ddot{q}_2 + h_1 + \phi_1 &= 0 && (15)\\ m_{21}\ddot{q}_1 + m_{22}\ddot{q}_2 + h_2 + \phi_2 &= \tau && (16) \end{aligned}$$

where

$$\begin{aligned}
m_{11} &= m_1\ell_{c1}^2 + m_2(\ell_1^2 + \ell_{c2}^2 + 2\ell_1\ell_{c2}\cos(q_2)) + I_1 + I_2 \\
m_{22} &= m_2\ell_{c2}^2 + I_2 \\
m_{12} &= m_{21} = m_2(\ell_{c2}^2 + \ell_1\ell_{c2}\cos(q_2)) + I_2 \\
h_1 &= -m_2\ell_1\ell_{c2}\sin(q_2)\dot{q}_2^2 - 2m_2\ell_1\ell_{c2}\sin(q_2)\dot{q}_2\dot{q}_1 \\
h_2 &= m_2\ell_1\ell_{c2}\sin(q_2)\dot{q}_1^2 \\
\phi_1 &= (m_1\ell_{c1} + m_2\ell_1)g\cos(q_1) + m_2\ell_{c2}g\cos(q_1+q_2) \\
\phi_2 &= m_2\ell_{c2}g\cos(q_1+q_2)
\end{aligned}$$

The parameters m_i, ℓ_i, ℓ_{ci}, and I_i are masses, link lengths, centers of masses, and moments of inertia, respectively. The zero configuration, $q_i = 0$, in this model corresponds to the arm extended horizontally. Therefore, the swing up task is move the robot from the vertically downward configuration $q_1 = -\pi/2$, $q_2 = 0$ to the inverted configuration $q_1 = +\pi/2$, $q_2 = 0$. Our strategy is as follows: We first apply the partial feedback linearization control (7) with the outer loop term given by (10). The resulting system can be written as

$$m_{11}\ddot{q}_1 + h_1 + \phi_1 = -m_{12}(\bar{u} - k_2\dot{q}_2 - k_1q_2) \tag{17}$$

$$\ddot{q}_2 + k_2\dot{q}_2 + k_1q_2 = \bar{u} \tag{18}$$

We then choose the additional control $\bar{u}$ to swing the second link "in phase" with the motion of the first link in such a way that the amplitude of the swing of the first link increases with each swing. This can be accomplished with either a switching control or a saturating control. It is shown in [8] that the simple choice of $\bar{u}$ given by

$$\bar{u} = k_3 \text{ sat}(\dot{q}_1) \tag{19}$$

where sat() is the saturation function, increases the energy and swings up the Acrobot. In the closed loop system, the open loop stable equilibrium configuration, $q_1 = -\pi/2$, $q_2 = 0$, becomes unstable and the trajectory is driven towards the inverted configuration. The final step is to switch to a "balancing controller" when the "swingup controller" $\bar{u}$ brings the state into the basin of attraction of the balancing controller. We have investigated various methods for designing balancing controllers, chiefly pseudo–linearization [2] and Linear–Quadratic methods. Figure 1 shows a swing up motion using this approach.

The difficult part of this strategy is to design the gains k_i above so that the state enters into the basin of attraction of the balancing controller. This involves searching for robust balancing controllers with large basins of attraction and proper tuning of the gains, both of which are nontrivial problems.

3.2 Three–Link Gymnast Robot

Next we apply the partial feedback linearization control to execute a so–called *giant swing* maneuver and balance for a three–link planar gymnast robot with

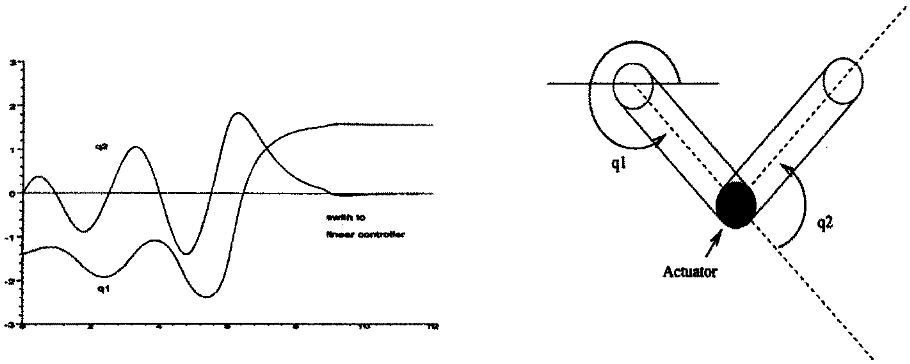

Figure 1: Swingup and Balance of The Acrobot

two actuators. The equations of motion are of the form

$$
\begin{aligned}
m_{11}\ddot{q}_1 + m_{12}\ddot{q}_2 + m_{13}\ddot{q}_3 + h_1 + \phi_1 &= 0 && (20)\\
m_{21}\ddot{q}_1 + m_{22}\ddot{q}_2 + m_{23}\ddot{q}_3 + h_2 + \phi_2 &= \tau_2 && (21)\\
m_{31}\ddot{q}_1 + m_{32}\ddot{q}_2 + m_{33}\ddot{q}_3 + h_3 + \phi_3 &= \tau_3 && (22)
\end{aligned}
$$

In this case we can linearize two of the three degrees of freedom to obtain

$$
\begin{aligned}
m_{11}\ddot{q}_1 + h_1 + \phi_1 &= -m_{12}u_2 + m_{13}u_3 && (23)\\
\ddot{q}_2 &= u_2 && (24)\\
\ddot{q}_3 &= u_3 && (25)
\end{aligned}
$$

Using our insight gained from the Acrobot, we specify the outer loop controls according to

$$
\begin{aligned}
u_2 &= -k_{21}q_2 - k_{22}\dot{q}_2 + \bar{u}_2 && (26)\\
u_3 &= -k_{32}q_3 - k_{32}\dot{q}_3 + \bar{u}_3 && (27)
\end{aligned}
$$

with

$$
\begin{aligned}
\bar{u}_2 &= k_{23}\,\mathrm{sat}(\dot{q}_1) && (28)\\
\bar{u}_3 &= k_{33}\,\mathrm{sat}(\dot{q}_2) && (29)
\end{aligned}
$$

Figure 2 shows a plot of the resulting giant swing maneuver including a switch to a linear controller at the end to balance the robot in the inverted position.

3.3 The Cart–Pole System

In this section we treat the familiar cart–pole system and show how the same design ideas as above can be applied. The Euler-Lagrange equations for this

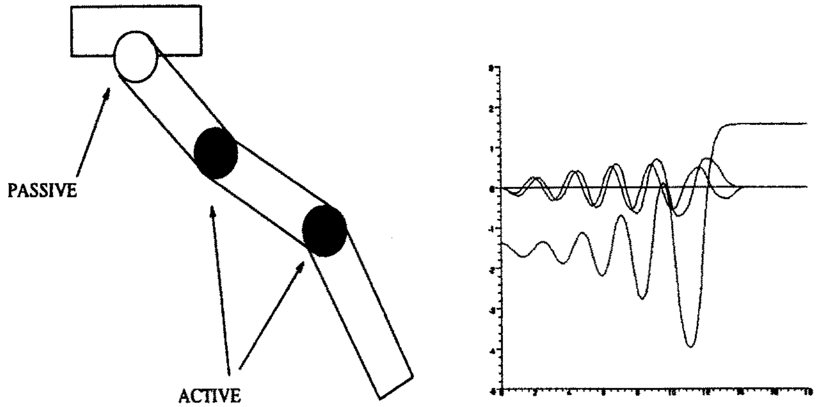

Figure 2: Three Link Gymnast Robot: Swingup and Balance

system are

$$(M+m)\ddot{x} + ml\cos(\theta)\ddot{\theta} = ml\dot{\theta}^2\sin(\theta) + F \tag{30}$$

$$\ddot{x}\cos(\theta) + l\ddot{\theta} = g\sin(\theta) \tag{31}$$

where x is the cart position and θ is the pendulum angle measured from the vertical. Applying the partial feedback linearization control, it is easy to show that the cart–pole system may be written as:

$$\dot{x} = v \tag{32}$$

$$\dot{v} = u \tag{33}$$

$$\dot{\theta} = \omega \tag{34}$$

$$\dot{\omega} = \sin(\theta) - \cos(\theta)u \tag{35}$$

where we have used the more descriptive notation, x, v, θ and ω instead of z_i and η_i to represent the cart position and velocity, and pendulum angle and angular velocity, respectively, and have normalized the parameter values. Using our above strategy, the simple control

$$u = -k_1 x - k_2 v + k_3 \operatorname{sat}(\omega) \tag{36}$$

can be used to swing up the pendulum and regulate the cart position for suitable choices of the gain parameters. However, we can improve the above controller by borrowing from section 3.4 as follows. We note that the total energy of the pendulum is given as

$$E = \frac{1}{2}\omega^2 + \cos(\theta) \tag{37}$$

and that $E = 1$ corresponds to the upright position of the pendulum. Then it can be shown that the control

$$u = -k_1 x - k_2 v + k_3 (E-1)\cos(\theta)\omega \tag{38}$$

locally guarantees that the energy converges to unity and that the cart position and velocity are regulated to zero, as shown by Chung and Hauser in [9]. We conjecture here that the control

$$\bar{u} = k_3 \,\mathrm{sat}((E-1)\cos(\theta)\omega - k_1 x - k_2 v) \tag{39}$$

renders the above result semi–global. Figure (3) shows the response of the cart–pole using the control (39).

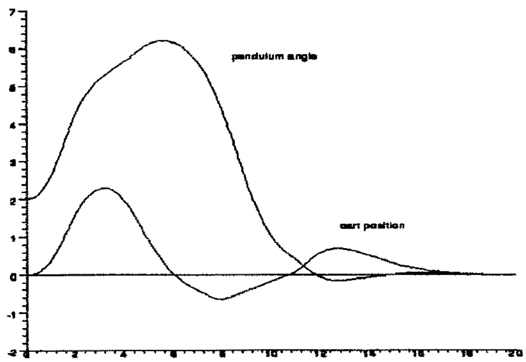

Figure 3: Cart–Pole Response Using (39)

3.4 An Almost Globally Stabilizing Controller

Whereas we expect only semi-global stabilization by using (39), an almost globally stabilizing controller can be obtained as follows : we design a first controller whose objective is to lead the pendulum to its homoclinic orbit and the cart to its desired position. As a consequence the state of the cart-pole system reaches, in finite time, a neighborhood of its desired equilibrium. Then, as in the previous section, another controller is used to stabilize this point.

The first design is achieved by applying the recursive technique of adding integration as introduced in [6]. For the cart-pole system, two stages are needed.

Consider, in a first stage, the system

$$\dot{v} = u_{s1} , \quad \dot{\theta} = \omega , \quad \dot{\omega} = \sin(\theta) - \cos(\theta)\, u_{s1} . \tag{40}$$

Defining the energy E as above and noting that

$$\dot{E} = -\cos(\theta)\omega u_{s1} \tag{41}$$

we can write, partially, the dynamics of this system (40) in terms of E and v as

$$\dot{v} = u_{s1} , \quad \dot{E} = -\cos(\theta)\,\omega\, u_{s1} , \tag{42}$$

implying that the Jurdjevic–Quinn technique applies to the (v, E)–system [11, 10]. This leads us to introduce :

$$V_{s1}(v, \theta, \omega) \;=\; \Phi_1(E-1) \;+\; \frac{k_v}{2}\, v^2 \;, \tag{43}$$

where k_v is a strictly positive real number and Φ_1 is a positive definite and proper C^2 function, defined on $[-2, +\infty)$ which satisfies :

$$\max\{|\Phi_1'(s)|\,,\,|\Phi_1''(s)|\} \;\leq\; \frac{k_E}{2\sqrt{s+2}}\;, \qquad \forall s \in (-2, +\infty)\;, \tag{44}$$

with k_E a real number in $(0, 1]$, e.g.

$$\Phi_1(s) \;=\; \frac{k_E}{8} \log(1+s^2)\;. \tag{45}$$

We get :

$$\dot{V}_{s1} \;=\; [-\Phi_1'(E-1)\,\omega\, \cos(\theta) \;+\; k_v\, v]\, u_{s1}\;. \tag{46}$$

It follows that $\dot{V}_{s1}$ can be made non positive by the following feedback law :

$$u_{s1}(v, \theta, \omega) \;=\; \frac{\Phi_1'(E-1)\,\omega\, \cos(\theta) \;-\; k_v\, v}{1+|v|+\Phi_1''(E-1)\,\omega\, \cos(\theta)\, \sin(\theta)}\;. \tag{47}$$

Note that, since :

$$|\omega| \;\leq\; \sqrt{2}\sqrt{E\,+\,1}\;, \quad \cos(\theta)\, \sin(\theta) \;\leq\; \frac{1}{2}\;, \tag{48}$$

we have :

$$|u_{s1}(v, \theta, \omega)| \;\leq\; \frac{k_E}{1-k_E/2} \;+\; k_v\;, \tag{49}$$

implying that $|u_{s1}|$ can be made arbitrarily small with an appropriate choice of k_E and k_v.

In the second stage, we consider the complete cart-pole system. To be able to apply once again Jurdjevic and Quinn technique, we follow the suggestion of [6] and introduce the following change of variables :

$$y \;=\; k_v\, x \;+\; v \;+\; \frac{v|v|}{2} \;-\; \Phi_1'(E-1)\sin(\theta)\;, \quad u \;=\; u_{s1}(v, \theta, \omega) \;+\; u_{s2}\;. \tag{50}$$

This allows us to rewrite the dynamics of the cart-pole system as

$$\dot{y} \;=\; [1+|v|+\Phi_1''\,\omega\, \cos(\theta)\, \sin(\theta)]\, u_{s2} \tag{51}$$

$$\dot{v} \;=\; u_{s1}(v, \theta, \omega) \;+\; u_{s2} \tag{52}$$

$$\dot{\theta} \;=\; \omega \tag{53}$$

$$\dot{\omega} \;=\; \sin(\theta) - \cos(\theta)\,(u_{s1}(v, \theta, \omega) + u_{s2}) \tag{54}$$

From the result of the first stage and by applying the Jurdjevic and Quinn technique, we introduce :

$$V_1(x, v, \theta, \omega) \;=\; V_{s2}(x, v, \theta, \omega) \;=\; \Phi_1(E-1) \;+\; \frac{k_v}{2}\, v^2 \;+\; \Phi_2(y)\;, \tag{55}$$

where Φ_2 is a positive definite and proper C^1 function. We have :

$$\dot{V}_1 = [1 + |v| + \Phi_1'' \omega \cos(\theta) \sin(\theta)] \left[(\Phi_2' - u_{s1}) u_{s2} - u_{s1}^2 \right] . \qquad (56)$$

It follows that the following bounded feedback law makes $\dot{V}_2$ non positive :

$$u_1(x, v, \theta, \omega) = u_{s1}(v, \theta, \omega) - \text{sat}\left(\Phi_2'(y) - u_{s1}(v, \theta, \omega)\right) . \qquad (57)$$

This feedback law makes the solution given by the cart staying at its desired position and the homoclinic orbit of the pendulum asymptotically stable with basin of attraction the whole state space minus a set of measure zero. This set is the stable manifold of the equilibrium point corresponding to the downward vertical position of the pole.

To complete the design, we consider a balancing controller of the form :

$$u_2(x, v, \omega, \theta) = a_1 x + a_2 v + a_3 \cos(\theta) \omega + a_4 \sin(\theta) , \qquad (58)$$

where the real numbers a_i's are chosen so that the closed-loop linearized at the desired equilibrium admits this point has a locally asymptotically stable point. Corresponding to this control law, given some threshold $u_{\max}$, we can find a positive definite matrix P and a positive real number p such that by letting :

$$V_2(x, v, \theta, \omega) = \left(x \;\; v \;\; \omega \;\; \sin(\tfrac{1}{2}\theta) \right) P \begin{pmatrix} x \\ v \\ \omega \\ \sin(\tfrac{1}{2}\theta)) \end{pmatrix} , \qquad (59)$$

we have, when $u = u_2(x, v, \omega, \theta)$,

$$0 < V_2(x, v, \omega, \theta) \leq p \quad \Longrightarrow \quad \begin{cases} \dot{V}_2(x, v, \omega, \theta) < 0 , \\ |u_2(x, v, \omega, \theta)| \leq u_{\max} . \end{cases} \qquad (60)$$

The feedback law is then :

$$u = u_\sigma(x, v, \theta, \omega) , \qquad (61)$$

where σ is a new state variable with values in $\{1, 2\}$ and whose dynamics are :

$$\begin{aligned} \sigma(t) &= 1 \quad \text{if} \quad \{p \leq V_2\} \quad \text{or} \quad \left\{\sigma^-(t) = 1 \text{ and } k_V\, p < V_2\right\} , \\ &= 2 \quad \text{if} \quad \{V_2 \leq k_V\, p\} \quad \text{or} \quad \left\{\sigma^-(t) = 2 \text{ and } V_2 < p\right\} , \end{aligned}$$

where k_V is a real number in $(0, 1)$ and $\sigma^-(t)$ denotes the limit of $\sigma(\tau)$ when τ tends from below to t. Figure (4) shows the response of the cart–pole to the above controller (61).

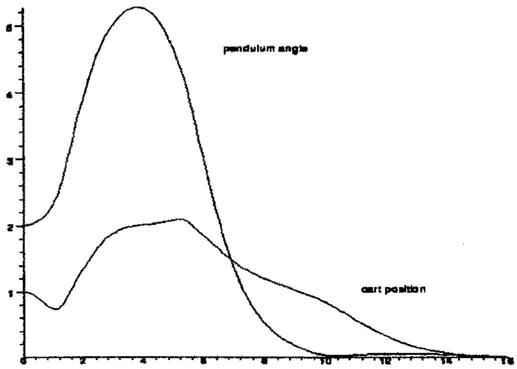

Figure 4: Cart–Pole Response Using (61)

4 Discussion

The class of underactuated mechanical systems considered here is an example of nonlinear systems in feedforward form. The design methodology presented in Section 2, takes advantage of physical insight about the dynamics of underactuated systems, and produces controllers that are simple to implement. These controllers are difficult to tune, however, and guarantee only local stability, in general. Thus, these systems are good vehicles for investigating tuning algorithms based on repetitive learning and logic based switching.

The more general procedure outline in Section 3.4, on the other hand, is more difficult to design but guarantees (almost) global stability. The design difficulty arises from the necessity to construct certain changes of variables defined as the solution of a PDE. Thus more research is needed to identify classes of systems, such as the cart–pole, for which such changes of coordinates can be readily computed.

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