Proceedings of the 33rd
Conference on Decision and Control
Lake Buena Vista, FL - December 1994

WA-5 10:00

Adding an integration and Global asymptotic stabilization of feedforward systems

F. Mazenc and L. Praly
Centre Automatique et Systèmes
École des Mines de Paris
35 rue St Honoré
77305 Fontainebleau cédex
FRANCE

Abstract : We are concerned with systems which generalize the form $\dot{x} = h(y,u)$, $\dot{y} = f(y,u)$, where the state components x integrates functions of the others components y and the inputs u. We give sufficient conditions under which global asymptotic stabilizability of the y-subsystem (resp. by saturated control) implies global asymptotic stabilizability of the overall system (resp. by saturated control). This is established by an explicit Lyapunov design of the control law. And we show how it serves as a basic tool to be used, may be recurrently, to deal with more complex systems. In particular the stabilization problem of the so called feedforward systems is solved this way.

1 Problem statement and main results

1.1 Problem statement

The technique of adding one integrator, as introduced by Tsinias [16] or Byrnes and Isidori [2], has become one of the basic tools invoked today to design stabilizing controllers. It concerns the problem of knowing when asymptotic stabilizability for the system :

$$\dot{y} = f(y,u) \tag{1}$$

implies asymptotic stabilizability for the system :

$$\dot{y} = f(y,x) \quad , \quad \dot{x} = h(x,y,u) . \tag{2}$$

From the solution of this problem, control designs for systems admitting the following recurrent structure, called feedback form can readily be obtained :

$$\begin{cases} \dot{x}_1 = f(x_1, x_2) , \\ \quad \vdots \\ \dot{x}_n = f(x_1, \dots, x_n, u) . \end{cases} \tag{3}$$

In this paper, we propose a solution to the problem of knowing when global asymptotic stabilizability for :

$$\dot{y} = f(y,u) \tag{4}$$

implies global asymptotic stabilizability for :

$$\dot{x} = h(y,u) \quad , \quad \dot{y} = f(y,u) . \tag{5}$$

The knowledge of a solution for this problem, called "adding one integration", allows us to deal with another recurrent structure, called feedforward form[1]

$$\begin{cases} \dot{x}_n = f_n(x_1, \dots, x_n, u) , \\ \quad \vdots \\ \dot{x}_1 = f_1(x_1, u) . \end{cases} \tag{6}$$

[1]Note that, on the contrary of the feedback form (3), systems in the form (6) are "generically" not feedback linearizable.

For this feedforward form, a seminal result has already been obtained by A. Teel in [13]. The usefulness of this result has been demonstrated by Teel in [12] and Sussmann et al. in [11] to prove the stabilizability of null controllable linear systems via saturated control and by Teel in [14] and Lin and Saberi in [8] to prove asymptotic stabilizability for some partially linear composite systems (see Corollary 1.1). The technique introduced in [13] is based on the robustness of local exponential stability together with the use of comparison theorems. It takes advantage of the property shared by some systems that small or convergent inputs lead to bounded state and eventually small state.

Our intent here is to make the tool of adding one integration more efficient and, for this, to propose a Lyapunov design counterpart to the approach of Teel. Among other things, this will allow us to slightly relax some assumptions of [13] and to reach a class of control laws larger than the one considered by Teel.

1.2 Notations and Basic definitions

- Regularity plays very little role here. So this aspect will be considered only when really needed.
- Throughout the paper, the symbol c is used to denote generically a strictly positive real number.
- For any matrix Φ we denote by $\lambda_{\Phi i}$ one of its eigenvalues.
- By $< h(x,y) \, , \, y >$, we denote a matrix whose (i,j) entry $(< h(x,y) \, , \, y >)_{ij}$ is:
$$< h(x,y) \, , \, y >_{ij} = \sum_k h(x,y)_{(k,i,j)} \, y_k \, .$$
- A function $f(x)$ on $\mathbb{R}^l$ is said of order p if :
$$\limsup_{|x|\to 0} \left\{ \frac{|f(x)|}{|x|^p} \right\} < +\infty \, .$$
- For any positive definite symmetric matrix Q, we denote : $|x|_Q = \sqrt{x^\top Q x}$.
- $\dot{V}_{(7)}$ denotes the function $\frac{\partial V}{\partial x}(x) f(x) : \mathbb{R}^n \to \mathbb{R}$ and the subscript (7) refers to equation number (7) of the differential equation:
$$\dot{x} = f(x) \, . \tag{7}$$
- For a C^1 function $k(t)$ of the real variable, we denote by $k'(t)$ its first derivative.

1.3 Main results

We consider the controlled system :

$$\begin{cases} \dot{x} = h_0(y)\, x \; + \; h_1(y) \; + \; h_2(x,y,u)\, u \\ \dot{y} = f(y) \; + \; f_2(x,y,u)\, u \end{cases} \tag{8}$$

with y in $\mathbb{R}^n$, x in $\mathbb{R}^m$, u in $\mathbb{R}^q$ and h_1 and f are zero at 0. We denote :

$$M = h_0(0)\,,\; C = \frac{\partial h_1}{\partial y}(0)\,,\; D = h_2(0,0,0) \qquad A = \frac{\partial f}{\partial y}(0)\,,\; B = f_2(0,0,0) \tag{9}$$

and we introduce the following assumptions :

A11 : *The matrix A is asymptotically stable, the matrix M is stable and the spectra of these matrices are such that, for any $(i,j,k)^2$,*

$$i)\;\; \lambda_{Ai} + \lambda_{Mj} \neq \lambda_{Mk}\;\;,\qquad ii)\;\; \lambda_{Ai} \neq \lambda_{Mk} \tag{10}$$

A12 : *The point $y = 0$ is a globally asymptotically stable point of the y-subsystem when $u \equiv 0$.*

Assumption A11 implies the existence of a positive definite symmetric matrix Q satisfying :

$$QM + M^\top Q \;\le\; 0\,. \tag{11}$$

and of P_1 and $\mathcal{P}_2$, solutions of the linear systems (see Lemma 2.1) :

$$MP_1 - P_1 A \;=\; C \tag{12}$$

$$0 \;=\; \frac{\partial h_{0\,(i,j)}}{\partial y_k}(0) + \sum_l \left[M_{(i,l)}\mathcal{P}_{2\,(k,l,j)} - \mathcal{P}_{2\,(k,i,l)}\,M_{(l,j)} - \mathcal{P}_{2\,(l,i,j)}\,A_{(l,k)}\right]\,. \tag{13}$$

Then the following assumption makes sense with :

$$\mathcal{H}_2(x)\,u \;=\; -< \mathcal{P}_2, f_2(x,0,0)u > x \;+\; \left[h_2\left(x,0,0\right) + P_1\, f_2\left(x,0,0\right)\right] u\,. \tag{14}$$

A2 : *$x = 0$ is the only bounded solution of*[3] *:*

$$\dot{x} = M\,x\;,\;\; x^\top Q\left(\mathcal{H}_2(x,0,0)\,,\, M\,x\right) = (0,0)\,. \tag{15}$$

Our main result, proved in section 2.1, is :

Theorem 1.1 *Under assumptions A11, A12 and A2, for any $\overline{u}$ in $(0,+\infty]$, the origin can be made a globally asymptotically stable solution of the system (8) by a state feedback bounded by $\overline{u}$. Moreover, if the pair $\left(\begin{pmatrix} M & C \\ 0 & A\end{pmatrix}, \begin{pmatrix} D \\ B\end{pmatrix}\right)$ is stabilizable, the linearized closed-loop system is asymptotically stable. Finally, in the case where M is asymptotically stable, the feedback can be taken identically zero.*

In [6], we give systems which while not satisfying only one of the assumptions in this Theorem cannot be globally asymptotically stabilized.

[2] This requirement is satisfied if A is asymptotically stable and the real part of the eigenvalues of M are zero.

[3] When M has all its eigenvalues with zero real part, Assumption A2 holds if we have that, at all point x, the vectors $\mathcal{H}_2(x)$, $Ad_{M_x}\mathcal{H}_2(x)$, ..., $Ad_{M_x}^{m-1}\mathcal{H}_2(x)$) span the entire space.

1.4 Application

Consider the following system :

$$\begin{cases} \dot{x}_n = h_{0n}(y_{n-1})x_n + h_{1n}(y_{n-1}) + h_{2n}(x_n, y_{n-1}, v)v \\ \quad\vdots \\ \dot{x}_1 = h_{01}(y_0)x_1 + h_{11}(y_0) + h_{21}(x_1, y_0, v)v \\ \dot{y}_0 = f_0(y_0) + f_{20}(y_0, v)v \end{cases} \tag{16}$$

with h_{1i} and f_0 zero at 0 and :

$$y_i \;=\; \left(x_i^\top, x_{i-1}^\top, \ldots, x_1^\top, y_0^\top\right)^\top\,. \tag{17}$$

By applying Theorem 1.1 repeatedly, we have :

Theorem 1.2 ([6]) *Assume the following for (16) :*
1.2.1 : There exists a control law $v_0(y_0)$, with $v_0(0) = 0$, which globally asymptotically stabilizes the origin of the y_0-subsystem of (16) and so that the linearized closed-loop system is asymptotically stable.
1.2.2 : For all i, there exists a positive definite matrix Q_i satisfying :

$$Q_i h_{0i}(0) \;+\; h_{0i}(0)^\top Q_i \;=\; 0\,. \tag{18}$$

1.2.3 : The linearized system is stabilizable.
1.2.4 : The function h_{2i} satisfies, for all i in $\{1,\ldots,n\}$ and for all vectors (x_i, y_{i-1}, v),

$$\frac{\partial h_{2i}}{\partial x_i}(x_i,0,0) = 0\;,\quad \frac{\partial^2 h_{2i}}{\partial x_i^2}(x_i, y_{i-1}, v) = 0\,.$$

Under these conditions, for any $\overline{u}$ in $(0,+\infty]$, the origin can be made a globally asymptotically stable solution of the system (16) by a state feedback bounded by $\overline{u} + \sup_{y_0}\{|v_0(y_0)|\}$ and the linearized closed-loop system is asymptotically stable.

A straightforward application of this result gives (see section 2.2) :

Corollary 1.1 *Consider the system:*

$$\dot{\xi} \;=\; \mathcal{A}\xi \;+\; \mathcal{B}u\;,\;\; \dot{\varsigma} \;=\; \phi(\varsigma, u)\,. \tag{19}$$

Assume:
1. The pair $(\mathcal{A},\mathcal{B})$ is stabilizable.
2. The eigenvalues of $\mathcal{A}$ have nonpositive real part.
3. The point $\varsigma = 0$ is a globally asymptotically stable equilibrium point of $\dot{\varsigma} = \phi(\varsigma, 0)$ and the linearization of this system is asymptotically stable.
Under these conditions, for any $\overline{u}$ in $(0,+\infty]$, the origin can be made a globally asymptotically stable solution of the system (19) by a state feedback bounded by $\overline{u}$ and the linearized closed-loop system is asymptotically stable.

By applying the technique of adding one integrator, Corollary 1.1 can be extended to systems in the form :

$$\begin{cases} \dot{\xi}_0 = \mathcal{A}_0\,\xi_0 \;+\; \mathcal{B}_0\mathcal{C}_1\xi_1\;,\quad \dot{\varsigma} \;=\; \phi(\varsigma, \mathcal{C}_1\xi_1)\,, \\ \dot{\xi}_1 = \mathcal{A}_1\,\xi_0 \;+\; \mathcal{B}_1 u\,. \end{cases} \tag{20}$$

where the linear system $(\mathcal{C}_1, \mathcal{A}_1, \mathcal{B}_1)$ has maximum relative degree. With this extension, Corollary 1.1 belongs to the class of results known for the so called partially linear composite systems studied for example in ([13, 14, 8, 7]). In particular, when the ς-subsystem is not present, we recover the result that null controllable linear systems can be stabilized by saturated control (see ([11]),([17])). Another closely related result is [8, Theorem 2.2] which generalizes [14]. There Lin and Saberi consider the more general case of multiple inputs, do not assume asymptotic stability of the matrix $\frac{\partial \phi}{\partial \varsigma}(0,0)$ and prove semi-global asymptotic stabilization by partial state feedback. Here we consider the single input case for the sake of simplicity in this illustration and we obtain global asymptotic stabilization by full state feedback. Such a result of global asymptotic stability was already proved in [7] by Saberi, Kokotovic and Sussmann with an assumption of stability for the matrix $\mathcal{A}_0$. The fact that full state feedback may be necessary follows from the fact that, for $n \geq 3$, there is no globally asymptotically stabilizing dynamic feedback law using only ξ as measurement for the system (apply [5, Lemma 3]).

$$\dot{\xi} = u \quad , \qquad \dot{\varsigma} = -\varsigma + \varsigma^n u^2 \,. \qquad (21)$$

So, compared with [8] or [7], the only restrictive assumption in Corollary 1.1 is the asymptotic stability of the matrix $\frac{\partial \phi}{\partial \varsigma}(0,0)$; we know from [15, Section 4] that it is superfluous.

2 Proof of our main results

2.1 Proof of Theorem 1.1

Our proof is constructive and, in our mind, is more important than the result itself. It is important to emphasize that it is strongly related to the technique of Jurdjevic and Quinn [4]

2.1.1 About the simplification of h_0 and h_1

Let us first investigate if we can find appropriate coordinates where h_1 and $h_0(y) - M$ would disappear in (8). We consider the change of variables :

$$X = \exp(-P_2(y))\,(x + P_1(y)) \;, \quad Y = y \qquad (22)$$

where the matrix function P_2 and the vector P_1 are to be chosen. The system (8) takes the same form :

$$\begin{cases} \dot{X} = H_0(Y)\,X + H_1(Y) + H_2(X,Y,u)\,u \\ \dot{Y} = F(Y) + F_2(X,Y,u)\,u \end{cases} \qquad (23)$$

with in particular :

$$H_0(Y) = \exp(-P_2(Y))\,h_0(Y)\,\exp(P_2(Y)) - < \frac{\partial P_2}{\partial Y}(Y), f(Y) > \qquad (24)$$

$$H_1(Y) = \exp(-P_2(Y)) \times \left[h_1(Y) + \frac{\partial P_1}{\partial Y}(Y) f(Y) - h_0(Y) P_1(Y)\right] \,. \qquad (25)$$

We have :

Lemma 2.1 *If the spectra of A and M satisfy (10), then there exist smooth functions P_1 and P_2 which give $H_0 - M$, in (24), and H_1, in (25), of order 2.*

Proof : Arguments similar to those used in [1, Proof of Lemma 1.1] allow us to prove, with (10.i), the existence of a unique solution $\mathcal{P}_2$ to (13). Then by letting :

$$(\exp(P_2(Y)) - I)_{(i,j)} = \frac{\sum_k \mathcal{P}_{2\,(k,i,j)} Y_k}{1 + \sum_{i,j} |\sum_k \mathcal{P}_{2\,(k,i,j)} Y_k|} \,,$$

we obtain that the function $H_0(Y) - M$ is of order 2.

Similarly, H_1 in (25) is of order 2 if we choose :

$$P_1(Y) = P_1 Y \qquad (26)$$

where P_1 is the solution of the linear system (12) which exists if (10.ii) holds (see [3, Section 8.1]). □

2.1.2 $h_0 - M$ and h_1 are of higher order

In view of Lemma 2.1, we consider the system :

$$\begin{cases} \dot{X} = M\,X + H_1(Y) + H_2(X,Y,u)\,u + H_3(Y)X \\ \dot{Y} = F(Y) + F_2(X,Y,u)\,u \end{cases} \qquad (27)$$

We first assume conditions related to A11, A12, A2 :

B1 : *There exist a positive definite and proper C^2 function V and a positive definite matrix Q so that :*

$$QM + M^\top Q = -R \leq 0 \,, \qquad (28)$$

$$\frac{\partial V}{\partial Y}(Y) F(Y) = -W(Y) < 0 \qquad \forall Y \neq 0 \,. \qquad (29)$$

B2 : *$X = 0$ is the only bounded solution of :*

$$\dot{X} = M\,X \,, \; X^\top Q\,(H_2(X,0,0)\,,\, M\,X) = (0,0) \qquad (30)$$

But we need also the following extra assumption :

$$\limsup_{|Y| \to 0} \frac{|H_1(Y)| + |H_3(Y)|}{W(Y)} \leq c \,. \qquad (31)$$

We have :

Proposition 2.1 *Assume the system (27) satisfies Assumptions B1, B2 and (31). Under these conditions, for any $\overline{u}$ in $(0, +\infty]$ the origin can be made a globally asymptotically stable solution by a state feedback bounded by $\overline{u}$.*

Proof : Our candidate Lyapunov function for (27) is (see [6] for a motivation) :

$$U(X,Y) = \mathcal{V}(Y) + \mathcal{Q}(X) + k_2(V(Y))(1 + \mathcal{Q}(X)) \quad (32)$$

where we have introduced the following notations :

$$\begin{array}{l} \mathcal{V}(Y) = k_1(V(Y)) \,, \; \mathcal{W}(Y) = k_1'(V(Y))\,W(Y) \\ \mathcal{Q}(X) = \ell(|X|_Q^2) \quad, \; \mathcal{R}(X) = \ell'(|X|_Q^2)\,X^\top R X \,, \end{array} \qquad (33)$$

where k_1, k_2 and ℓ are three positive, definite and proper C^1 functions[4] satisfying :

$$k_1'(t) > 0 \,, \forall t \geq 0 \,, \quad \ell'(t) > 0 \,, \forall t > 0 \,. \qquad (34)$$

and, for all X and Y,

$$\frac{1}{2}\frac{k_2'(V(Y)) W(Y)}{1 + k_2(V(Y))} \geq c\,[|H_1(Y)| + |H_3(Y)|] \,, \quad (35)$$

$$\left|\frac{\partial \mathcal{Q}}{\partial X}(X)\right| (1 + |X|) \leq c\,(1 + \mathcal{Q}(X)) \,. \qquad (36)$$

In Lemma B.2, we show that (35) is possible under assumption (31). And the condition (36) is met by choosing ℓ as any polynomial function $\ell(t) = t^k$ or as :

$$\ell(t) = \int_0^{\sqrt{t}} \sigma(s)ds \,. \qquad (37)$$

[4]In fact, it is sufficient that the functions $k_1'(V(Y))\frac{\partial V}{\partial Y}(Y)$ and $\ell'(|X|_Q^2)|X|_Q^2$ be continuous.

where σ is any continuous function satisfying :

$$\begin{cases} \sigma(0) = 0\,, \quad 0 < t\sigma(t) \le c|t| \quad \forall t \ne 0 \\ \liminf_{t \to +\infty} \sigma(t) > 0 \end{cases} \tag{38}$$

We denote :

$$\begin{aligned} \mathcal{G}(X,Y,u) = & \left[1 + k_2(V(Y))\right] \frac{\partial \mathsf{q}}{\partial X}(X) H_2(X,Y,u) \\ & + \left[k_1'(V(Y)) + k_2'(V(Y))(1 + \mathsf{q}(X))\right] \\ & \times \frac{\partial V}{\partial Y}(Y) F_2(X,Y,u) \end{aligned} \tag{39}$$

We get :

$$\begin{aligned} \dot{\overline{U(X,Y)}}_{(27)} = & -W(Y) - k_2'(V(Y))W(Y)(1 + \mathsf{q}(X)) \\ & - \left[1 + k_2(V(Y))\right] \mathcal{R}(X) + \mathcal{G}(X,Y,u)\,u \\ & + \left[1 + k_2(V(Y))\right] \frac{\partial \mathsf{q}}{\partial X}(X) \left[H_1(Y) + H_3(Y)X\right] . \end{aligned} \tag{40}$$

$$\begin{aligned} & \le -W(Y) - \tfrac{1}{2}k_2'(V(Y))W(Y)(1 + \mathsf{q}(X)) \\ & - \left[1 + k_2(V(Y))\right] \mathcal{R}(X) + \mathcal{G}(X,Y,u)\,u\,. \end{aligned} \tag{41}$$

The conclusion follows by applying LaSalle's invariance principle and Lemma A.1 in appendix A. □

Remark 1 :

1. We remark that if $H_3 \equiv 0$, then by choosing ℓ as in (37), we can take $k_2 \equiv 0$ and k_1 satisfying :
$$\frac{1}{2}W(Y) \ \ge\ c\,|H_1(Y)|\,. \tag{42}$$
2. The function $\dot{\overline{U(X,Y)}}_{(27)}$ is made negative definite by the feedback u_1, in appendix A, if, for all $X \ne 0$,
$$\left[X^\top Q M X\right]^2 + \left[X^\top Q\, H_2(X,0,0)\right]^2 \ne 0\,.$$
This is always satisfied if X is of dimension 1.

2.1.3 Explicit expressions for the control laws

The control law obtained from Lemma A.1 involves explicitly the functions V, k_1 and k_2 that we may not want to evaluate. In the following Proposition we give another expression for the control law which depends only on data from the system except for a single parameter μ which has to be tuned.

To state this result, we choose R and $\overline{u}$ as two strictly positive real numbers and we introduce two functions independent of V :

1. Let φ_R be a smooth positive function onto $[0,1]$ such that :
$$\varphi_R(0) = 1\,, \quad \varphi_R(|Y|^2) = 0 \quad \forall Y \ge R\,. \tag{43}$$
2. Let $\lambda_{R,\overline{u}}$ be a smooth function satisfying :

$$\lambda^F_{R,\overline{u}}(X) = \sup_{\substack{|u| \le \overline{u} \\ |Y| \le R}} \left\{ \frac{|F_2(X,Y,u) - F_2(X,Y,0)|}{|u|} \right\}$$

$$\lambda^H_{R,\overline{u}}(X) = \sup_{\substack{|u| \le \overline{u} \\ |Y| \le R}} \left\{ \frac{|H_2(X,Y,u) - H_2(X,Y,0)|}{|u|} \right\}$$

$$\lambda_{R,\overline{u}}(X) \ge \max\left\{1\,,\ \lambda^F_{R,\overline{u}}(X) + \lambda^H_{R,\overline{u}}(X)\right\} \tag{44}$$

Proposition 2.2 ([6]) *Assume (27) satisfies Assumptions (31), B1 and B2. Under these conditions, if :*

$$\liminf_{Y \to 0} \frac{W(Y)}{\left|\frac{\partial V}{\partial Y}(Y)\right|^2} \ >\ 0\,, \tag{45}$$

then, for any $\overline{u}$ in $(0,+\infty)$, there exists a positive real number μ^* in $(0,\overline{u}]$ so that the origin can be made a globally asymptotically stable solution by a state feedback bounded by $\overline{u}$ and of the form :

$$u(X,Y) \ =\ -\beta(X,Y) \frac{\partial \mathsf{q}}{\partial X}(X)\, H_2(X,Y,0) \tag{46}$$

$$\beta(X,Y) = \frac{\mu\,\varphi_R(|Y|^2)\left(1 + \left|\frac{\partial \mathsf{q}}{\partial X}(X) H_2(X,Y,0)\right|^2\right)^{-1}}{\lambda_{R,\overline{u}}(X)\left(1 + \mathsf{q}(X) + |F_2(X,Y,0)|^2\right)} \tag{47}$$

where μ is any real number in $(0,\mu^*]$.

Remark 2 : If the linear approximation of the Y-subsystem of (27), is asymptotically stable, then V can be chosen so that $\frac{\partial V}{\partial Y}$ is of first order and W is lower bounded by a positive definite quadratic form on a neighborhood of 0. In this case, (45) holds.

2.1.4 Proof of Theorem 1.1

Global asymptotic stability : To prove the first point of Theorem 1.1, we show that Assumptions A11, A12 and A2 imply that Proposition 2.1 applies.

1. With (10) and Lemma 2.1 we can find P_1 and P_2 so that by using the change of variables (22), the system (8) can be rewritten in the form (27) with H_1 and H_3 of order 2.
2. From a converse Lyapunov theorem, Assumptions A11 and A12 imply assumption B1 holds with :
$$|Y| \le c \ \Rightarrow\ V(Y) \ge c|Y|^2\,,\ W(Y) \ge c|Y|^2\,. \tag{48}$$
3. From points 1 and 2, Assumption (31) holds.
4. Assumption A2, (11), imply that B2 holds.

So from Proposition 2.1, the first statement of Theorem 1.1 holds with a control law $u(X,Y)$ obtained from Lemma A.1 or from (46).

Local exponential stability :

Claim 2.1 *If the pair (M,D) is stabilizable and there exists a positive definite matrix Q satisfying (11), then $X = 0$ is the only bounded solution of :*

$$\dot{X} = M\,X\,, \quad X^\top Q\,(D\,,\,M\,X) = (0,0)\,. \tag{49}$$

To get asymptotic stability of the linearized closed-loop system, we choose ℓ in (33) so that $\ell'(0) = 1$. Then, we write, with (9), the linearization of (27) :

$$\dot{X} \ =\ MX + \mathcal{D}u\,, \quad \dot{Y} \ =\ AY + Bu\,, \tag{50}$$

where, with $\mathcal{H}_2$ given by (14),

$$\mathcal{D} \ =\ \mathcal{H}_2(0)\,. \tag{51}$$

To prove that the linearization of the control u_N given by Lemma A.1 or by (46) is stabilizing this linear system, we proceed in two steps :

1. We apply Proposition 2.1 to obtain a linear controller u_L for this linear system (50).
2. We check that this linear controller u_L is nothing but the linearization at 0 of u_N.

Step 1 : We first remark that the system (50) is in the form (8). Then assumption A11 implies that B1 holds. Also, the assumed stabilizability of the pair $\left(\begin{pmatrix} M & C \\ 0 & A\end{pmatrix}, \begin{pmatrix} D \\ B\end{pmatrix}\right)$ implies the stabilizability of the pair $(M, \mathcal{D})$. This fact with Claim 2.1 implies B2 holds. It follows that the following linear control stabilizes (50) (see [6]) :

$$u_L(X,Y) = -\beta_0 \left(\gamma_0\, \alpha_0\, Y^\top \frac{\partial^2 V}{\partial y^2}(0)\, B + X^\top Q\, \mathcal{D} \right) \tag{52}$$

where :

- V is the Lyapunov function satisfying (29) and (48) and therefore :

$$\frac{\partial^2 V}{\partial y^2}(0) A \,+\, A^\top \frac{\partial^2 V}{\partial y^2}(0) \,<\, 0\,, \tag{53}$$

- Q is the positive definite matrix satisfying (11),
- γ_0 is any strictly positive real number,
- β_0, a strictly positive real number, and α_0, a real number, satisfy :

$$-\frac{\lambda_{\min}\left\{\frac{\partial^2 V}{\partial y^2}(0)A + A^\top \frac{\partial^2 V}{\partial y^2}(0)\right\}}{\lambda_{\max}\left\{\frac{\partial^2 V}{\partial y^2}(0)BB^\top \frac{\partial^2 V}{\partial y^2}(0)\right\}} \geq \frac{1}{2}\beta_0[\alpha_0 - 1]^2\gamma_0 \tag{54}$$

Step 2 : Either, using the definitions (39), (9), we take $\alpha_0 = 1$ and $\beta_0 = \beta(0,0)$, where β is the function defined in the proof of Lemma A.1. Then (54) is satisfied. In this case, (52) is the linear approximation at zero of the control law given in Lemma A.1.
Or we choose $\alpha_0 = 0$ and $\beta_0 = \beta(0,0)$, where β is the function defined in (47). Then with μ small enough (54) is satisfied. In this case, (52) is the linear approximation at zero of (46).

M **is asymptotically stable :** Finally to prove that the origin of (8) with $u = 0$ is globally asymptotically stable if the matrix M is asymptotically stable, we simply remark that :

$$\overline{\mathcal{V}(Y) + \mathcal{q}(X)}_{(8)} \,\leq\, -\frac{1}{2}\mathcal{W}(Y) \,-\, |X|^2 \frac{\sigma(|X|_Q)}{|X|_Q}\,. \tag{55}$$

with ℓ given by (37) and Q solution of :

$$QM + M^\top Q \,=\, -I\,. \tag{56}$$

□

2.2 Proof of Corollary 1.1

There exist coordinates $(x_n, \ldots, x_1, \varsigma_1)$ so that the ξ-subsystem can be rewritten as :

$$\begin{cases} \dot{x}_n = M_n\, x_n + C_n\, x_{n-1} + D_n u \\ \quad \vdots \\ \dot{x}_1 = M_1\, x_1 + D_1\, u \\ \dot{\varsigma}_1 \,= \mathcal{A}_1\, \varsigma_1 \end{cases} \tag{57}$$

where $\mathcal{A}_1$ is asymptotically stable, the M_i's are such that their eigen-values have zero real part, are simple and of multiplicity i in $\mathcal{A}$ in (19) and the $(x_1, \ldots, x_n)$ subsystem is controllable. Then the y_0-subsystem in (16) is :

$$\dot{\varsigma}_1 \,=\, \mathcal{A}_1\, \varsigma_1\,, \quad \dot{\varsigma} \,=\, \phi(\varsigma, u)\,. \tag{58}$$

All the assumptions of Theorem 1.2 being met, the conclusion follows readily. □

3 Concluding Remarks

We have proposed a Lyapunov design for deriving a state feedback law for systems in the form $\dot{x} = h(x,y,u)$, $\dot{y} = f(y,u)$, assuming global asymptotic stabilizability for the y-subsystem. We have shown that, if a saturated control is sufficient for this subsystem, the same holds for the overall. Our technique is called *adding integration*, since the assumptions on the x-subsystem are mainly that the x components are integrating functions of y and u.

This key technical tool can be used in combination with others. In particular, the availability of a Lyapunov function makes it very well suited for association with the technique of adding one integrator or for the design of adaptive controllers.

We have applied this tool repeatedly to prove global asymptotic stabilizability for systems having a special recurrent structure called feedforward form.

Unfortunately, our design may not be efficient enough for practice. The Lyapunov function we get for the overall system has only a non positive time derivative whereas the design starts from the knowledge of a strict Lyapunov function for the y-subsystem. The consequence is that an explicit expression of a Lyapunov function for the closed-loop system is unknown in general. Nevertheless, in Remark 1.2, we noted that, in the special case where the dimension of x is one, we do get an explicit expression. Also, we have shown that, even if the Lyapunov function is not known, a control law can be explicitly written. It involves in this case the tuning of a single strictly positive real number.

References

[1] Y. Bibikov : *Local Theory of Nonlinear Analytic Ordinary Differential Equations.* Springer-Verlag Berlin Heidelberg New York 1979.

[2] C. Byrnes, A. Isidori : *New results and examples in nonlinear feedback stabilization.* Systems & Control Letters. 12 (1989) 437-442

[3] F.R. Gantmacher, *Théorie des matrices. Tome 1,* Dunod, Paris 1966.

[4] V. Jurdjevic, J.P. Quinn : *Controllability and stability.* Journal of differential equations. vol. 4 (1978) pp. 381-389

[5] F. Mazenc, L. Praly, W.P. Dayawansa, *Global stabilization by output feedback : Examples and Counter-Examples.* To appear in Systems & Control Letters.

[6] F. Mazenc, L. Praly, *Adding an integration and global asymptotic stabilization of feedforward systems.* Submitted for publication in IEEE Transactions on Automatic Control. June 1994.

[7] A. Saberi, P. V. Kokotovic, H. J. Sussmann, *Global stabilization of partially linear composite systems.* Siam J. Control and optimization. Vol. 28 , No 6 , pp. 1491 – 1503 , November 1990.

[8] Z. Lin, A. Saberi : *Robust Semi-Global Stabilization of Minimum-Phase Input-Output Linearizable Systems via Partial State and Output Feedback* To appear in IEEE Transactions on Automatic Control.

[9] E.D. Sontag : *Feedback stabilization of nonlinear systems.* in *Robust control of Linear Systems and Nonlinear Control.* M.A Kaashoek, J.H. van Schuppen, A.C.M Ran, Ed.. Birkhäuser, pages 61-81, 1990.

[10] E.D. Sontag, H.J. Sussmann , *Nonlinear output feedback design for linear systems with saturating controls.* Proceeding of the 29th IEEE conference on decision and control. December 1990.

[11] H. Sussmann, E.D. Sontag, Y. Yang : *A General Result on the Stabilization of Linear Systems Using Bounded Controls.* SYCON - Rutgers Center for Systems and Control. Department of Mathematics, Rutgers University, New Brunswick, NJ 08903. October, 1992

[12] A. Teel : *Global stabilization and restricted tracking for multiple integrators with bounded controls.* Systems & Control Letters 18(1992) : 165–171.

[13] A. Teel : *Feedback stabilization : nonlinear solutions to inherently nonlinear problems.* Memorandum No. UCB/ERL M92/65. 12 June 1992

[14] A. Teel : *Semi-global stabilization of minimum-phase nonlinear systems in special normal forms.* Systems & Control Letters 19(1992)187 – 192

[15] A. Teel : *Additional stability results with bounded controls.* Submitted to 1994 CDC, February 26, 1994.

[16] J. Tsinias : *Sufficient Lyapunov-like conditions for stabilization.* Math. Control Signals Systems 2 (1989) 343-357

[17] Y. Yang : *Global Stabilization of Linear Systems with Bounded Feedback*, Ph. D. Thesis, Mathematics Department, Rutgers University,1993.

A A solution to $G(x,u)u \leq 0$.

Lemma A.1 *Let $G(x,u)$ be a C^1 function. For any $\overline{u}$ in $(0,+\infty]$, there exists a C^1 function u_1 such that :*

- *The function $|u_1(x)|$ is bounded by $\overline{u}$.*
- *$G(x,u_1(x))u_1(x)$ is non positive for all x and zero if and only if $G(x,0)$ is zero.*

Proof. The function G being C^1, there exists a C^1 function g and a C^0 function X such that :

$$G(x,u)^\top = g(x) + u^\top X(x,u) . \qquad (59)$$

The continuity of g and X insures the existence of a strictly positive C^1 real function β such that, for all real number u bounded by $\overline{u}$ and for all vector x :

$$\beta(x)\,|g(x)| \leq \overline{u} \quad , \quad \beta(x)\,|X(x,u)| < 1 . \qquad (60)$$

Our result holds with :

$$u_1(x) = -\beta(x)\,g(x) . \qquad (61)$$

□

B Dominating functions

Let V and W be continuous functions such that V is positive definite and proper and W is positive definite.

Lemma B.1 *Let κ be a continuous positive function satisfying :*

$$\limsup_{Y\to 0} \left\{\frac{\kappa(Y)}{W(Y)}\right\} \leq +\infty . \qquad (62)$$

Then there exists a continuous, strictly increasing, strictly positive and proper function ρ such that :

$$\frac{\kappa(Y)}{W(Y)} \leq \rho(V(Y))V(Y) \qquad \forall Y . \qquad (63)$$

Proof : The requirement (62) imposed on κ, the continuity of κ and the fact that V is a positive definite and proper function guarantee the existence of strictly positive real numbers c such that :

$$V(Y) \leq c \Longrightarrow \frac{\kappa(Y)}{W(Y)} \leq c . \qquad (64)$$

We define on $[0,+\infty[$ a function $\overline{\rho}$ as follows :

$$\overline{\rho}(v) = \sup_{\{Y:V(Y)\leq v\}} \left\{\max\left\{c, \frac{\kappa(Y)}{W(Y)}\right\}\right\} . \qquad (65)$$

It is positive, non decreasing and constantly equal to c, on a neighborhood of 0. So we may define another positive function on $(0,+\infty)$ by :

$$\rho(v) = \frac{1}{v}\int_v^{2v} \overline{\rho}(s)ds + v . \qquad (66)$$

This function is continuous, strictly increasing and proper on $]0,+\infty[$. And with the property of $\overline{\rho}$, by letting $\rho(0) = c$, we can extend the definition of ρ to $[0,+\infty[$ as a continuous, strictly increasing, strictly positive and proper function. □

Lemma B.2 *Let κ_0 and κ_1 be continuous positive real functions satisfying :*

$$\limsup_{Y\to 0} \frac{\kappa_0(Y)+\kappa_1(Y)}{W(Y)} = c < +\infty . \qquad (67)$$

Under these conditions, there exists a C^1, positive, strictly increasing and proper function k such that :

$$k'(t) > 0 \qquad \forall t \geq 0 \qquad (68)$$

and

$$k'(V(Y))W(Y) \geq c[(1+k(V(Y)))\kappa_0(Y)+\kappa_1(Y)] . \qquad (69)$$

Proof : From Lemma B.1, we can find two continuous functions ρ_0 and ρ_1 satisfying (63) for respectively κ_0 and κ_1. Then Lemma B.2 is satisfied with :

$$k(s) = \int_0^s c(\rho_0(\sigma)+\rho_1(\sigma))\left(\exp\left(\int_\sigma^s c\rho_0(\tau)d\tau\right)\right)d\sigma$$

□