Proceedings of the 2nd ECC, Vol. 1, Pages: 63-67. June 1993

Tools for robust semi-global stabilization

Andrew Teel
Dept. of Electrical Engineering
University of Minnesota
4-174 EE/CS Building
200 Union St. SE
Minneapolis, MN 55455
USA

Laurent Praly
Centre Automatique et Systèmes
École des Mines de Paris
35 rue St Honoré
77305 Fontainebleau cédex
FRANCE

Keywords: Semi-global stabilization, Lyapunov functions, small gain property.

Notation:

- $d(t)$ is a continuous time-varying signal contained in a compact set $D \subset \mathbb{R}^d$.
- $\dot{V}_{(0)}$ denotes the function $\frac{\partial V}{\partial x}(x)f(x,d) : \mathbb{R}^l \times D \to \mathbb{R}$ and the subscript (0) refers to equation number (0) of the differential equation:

$$\dot{x} = f(x, d(t)). \tag{0}$$

- $|\cdot|$ denotes the Euclidean norm.
- A function $f : A \to \mathbb{R}_+$ where A is a neighborhood of 0 in $\mathbb{R}^p$ is said to be *proper on* A if :

$$\lim_{x \to \partial A} f(x) = \infty \tag{1}$$

  where ∂A denotes the boundary of the set A. Note that if f is proper on A then, with $0 \leq c_1 \leq c_2$, $\{x : c_1 \leq f(x) \leq c_2\}$ is a compact subset of A.
- A function $f : A \to \mathbb{R}_+$ is said to be *positive (negative) definite on* B, a subset of A if $f(x)$ is strictly positive (negative) for all x in B.

1 Introduction

An equilibrium of a finite dimensional nonlinear control system is semi-globally stabilizable if it can be made locally asymptotically stable and be given an arbitrarily large basin of attraction. This problem has recently been addressed in [12, 2, 1]. In [1], Bacciotti provided a proof for this problem that separated the issue of boundedness from the issue of convergence. Inspired by his proof, we present tools for semi-global stabilization that generalize previous results. The proofs for the lemmas that follow can be found in [14].

2 Boundedness tools

Our first two tools will consider the problem of adding integrators to an already stabilized system. Namely, we will consider integrators added to the following system

$$\dot{z} = h(z, u, d(t)) \tag{2}$$

where $z \in \mathbb{R}^m$ and the following property is satisfied:

Assumption MMP (Modified Minimum Phase)
Assume for the system (2), with a zero input u, there exists a neighborhood A_1 of the origin in $\mathbb{R}^m$ and a C^1 function $V : A_1 \to \mathbb{R}_+$ which is positive definite on $A_1 \backslash \{0\}$ and proper on A_1 and satisfies when u is zero,

$$\dot{V}_{(2)} \leq -\Phi_1(z) \tag{3}$$

where $\Phi_1(z)$ is continuous on A_1 and positive definite on $\{z : \vartheta < V(z) \leq c+1\}$ for some nonnegative real number $\vartheta < 1$ and some real number $c \geq 1$.

This assumption is satisfied, for example, if the system (2) does not depend on d and, when u is zero, the origin is a locally asymptotically stable equilibrium with domain of attraction A_1 (see [5, Theorem 7]).

From now on the positive real number c is given by this assumption and the parameter μ, encoutered later, is to be computed from the given compact set of initial conditions.

Lemma 2.1 (Semi-global 'backstepping' I)
Consider the C^1 nonlinear control system

$$\begin{aligned} \dot{z} &= h(z, x, d(t)) \\ \dot{x} &= f(z, x, d(t)) + g(z, x, d(t))u \end{aligned} \tag{4}$$

where $x \in \mathbb{R}$,$z \in \mathbb{R}^m$, assumption MMP is satisfied and

$$|g(z, x_1, \ldots, x_r, d)| \geq b \qquad \forall (z, x, d) \in \mathbb{R}^m \times \mathbb{R}^r \times D. \tag{5}$$

Given $\mu \geq 1$, we define the function and the set

$$W(z, x) = c\frac{V(z)}{c+1-V(z)} + \mu\frac{x^2}{\mu+1-x^2} \tag{6}$$

$$A_2 = \{z : V(z) < c+1\} \times \{x : x^2 < \mu+1\}. \tag{7}$$

Then $W(z, x) : A_2 \to \mathbb{R}_+$ is positive definite on $A_2 \backslash \{0\}$ and proper on A_2. Further, if

$$u = -Ksgn(g)x \tag{8}$$

then, for each $\rho > 0$, there exists $K_* > 0$, such that, for each $K \geq K_*$, W satisfies

$$\dot{W}_{(4)} \leq -\Phi_2(z,x) \qquad (9)$$

where $\Phi_2(z,x)$ is positive definite on $\{(z,x) : \vartheta + \rho \leq W(z,x) \leq c^2 + \mu^2 + 1\}$.

Remark 2.1 Note that u in (8) does not depend on z. Observe that eventual bounds on (z,x) can be determined from the fact that

$$\begin{aligned} & c \geq 1\ ,\ \mu \geq 1\ ,\ W(z,x) \leq \vartheta + \rho \\ & \qquad \Longrightarrow \quad V(z) < 2(\vartheta+\rho)\ ,\ x^2 < 2(\vartheta + \rho)\ . \end{aligned} \qquad (10)$$

Also note that the number "1" in (6) and (7) is arbitrary and could be replaced by any strictly positive real number.

Example 2.1 Lemma 2.1 allows a semi-global solution to the almost disturbance decoupling problem as described in [7] for minimum phase systems in an upper triangular (strict feedback) form. An illustration is to achieve an utlimately arbitrarily small bound on the output y, from an arbrarily large set of initial conditions, for the system:

$$\begin{array}{rcl} \dot{x}_1 & = & x_2 + d_1(t) \\ \dot{x}_2 & = & x_2^3 d_2(t) + d_3(t) + u \\ y & = & x_1 \end{array} \qquad (11)$$

where $d_1(t), d_2(t), d_3(t)$ are unmeasured disturbances with known bounds. This example was previously used in [4] and [7] to show that a naive high gain design would lead to a vanishing region of attraction. $\diamond$

When the system has the structure described in the following lemma, it is possible to handle a block of integrators in one step, instead of iterating lemma 2.1.

Lemma 2.2 (Semi-global 'backstepping' II) *Consider the C^1 nonlinear control system*

$$\begin{array}{rcl} \dot{z} & = & h(z, x_1, d(t)) \\ \dot{x}_1 & = & x_2 + f_1(z, x_1, d(t)) \\ & \vdots & \\ \dot{x}_{j-1} & = & x_j + f_{j-1}(z, x_1, d(t)) \\ \dot{x}_j & = & u + f_j(z, x_1, d(t)) \end{array} \qquad (12)$$

where $x = (x_1, \ldots, x_j)^T \in \mathbb{R}^j$, $z \in \mathbb{R}^m$. Suppose assumption MMP is satisfied. Let the polynomial

$$p(s) = s^j + a_j s^{j-1} + \ldots + a_1 \qquad (13)$$

be Hurwitz and let A be the controllable canonical form matrix corresponding to $p(s)$. Also let P solve the matrix equation $A^T P + PA = -I$. For $K \geq 1$ to be specified, define the variables

$$\xi_i = \frac{x_i}{K^{i-1}} \qquad i = 1, \ldots, j. \qquad (14)$$

Then given $\mu \geq 1$, define the function and the set

$$W(z,\xi) = c\frac{V}{c+1-V} + \mu\frac{\xi^T P \xi}{\mu + 1 - \xi^T P \xi} \qquad (15)$$

$$A_2 = \{z : V(z) < c+1\} \times \{\xi : \xi^T P \xi < \mu + 1\}. \quad (16)$$

Then, $W(z,\xi) : A_2 \to \mathbb{R}_+$ is positive definite on $A_2 \backslash \{0\}$ and proper on A_2. Also, if

$$u = -K^j(a_1\xi_1 + \ldots + a_j\xi_j)\ , \qquad (17)$$

then, for each $\rho > 0$, there exists $K_* \geq 1$ such that, $\forall K \geq K_*$, W satisfies

$$\dot{W}_{(4)} \leq -\Phi_2(z,\xi) \qquad (18)$$

where $\Phi_2(z,\xi)$ is positive definite on $\{(z,\xi) : \vartheta + \rho \leq W(z,\xi) \leq c^2 + \mu^2 + 1\}$.

Remark 2.2 The effect of large K in (14) and (17), likewise in (8), is to introduce an exponential dichotomy between the z and x subsystems. This implies the existence of a center-stable manifold $x = H(z,K)$ and the fact that the motion can be decomposed into two stages: "convergence" to this manifold and "sliding" along this manifold. This decomposition is the standard tool used to analyze (12), (14), (17) with K large. (See [12, 2] for example.) Here instead, like in [1] but now with robustness, we completely ignore this decomposition and use a Lyapunov argument showing the decrease of an energy function outside a neighborhood of the origin. More precisely what is implicitly used here is the fact that as K tends to ∞, the manifold tends to the set $\{(z,x) : x = 0\}$. So a Lyapunov function which is simply the sum of the energy functions of x and z separately should be sufficient and indeed it is. Note also that the availability of a Lyapunov function makes explicit the ultimate bound on trajectories as well as the domain of attraction without the formalism of invariant manifolds.

Example 2.2 We have used lemma 2.2 as a tool in [15] to design a semi-globally stabilizing output feedback for the following class of systems:

$$\begin{array}{rcl} \dot{z} & = & h(z,y) \\ y_{(r)} & = & u + \sum_{i=1}^r f_i(z,y)^{(r-i)} \end{array} \qquad (19)$$

where $\cdot^{(i)}$ denotes the ith time derivative, when the equilibrium $z = 0$ of $\dot{z} = h(z,0)$ is globally asymptotically stable. Our algorithm, which is inspired by the global results in [8, 9], yields a semi-global solution but requires fewer assumptions. We begin by building the dynamic compensator

$$\begin{array}{rcl} \dot{\hat{x}}_1 & = & \hat{x}_2 + \ell_1(x_1 - \hat{x}_1) \\ & \vdots & \\ \dot{\hat{x}}_r & = & u + \ell_r(x_1 - \hat{x}_1) \end{array} \qquad (20)$$

where the coefficients ℓ_i are the coefficients of a Hurwitz polynomial. If we define $e_i = x_i - \hat{x}_i$ we get the error dynamics

$$\dot{e} = Ae + F(z, x_1)\ . \qquad (21)$$

We remark that the equilibrium $(z,e) = (0,0)$ of the augmented zero dynamics

$$\dot{z} = h(z,0)\ ,\ \dot{e} = Ae + F(z,0) \tag{22}$$

is globally asymptotically stable. This follows from the cascade structure and since the state e is input-to-state stable with respect to the input z. (See [11] or lemma 2.4.) Now we consider the complete system

$$\begin{array}{rcl} \dot{z} &=& h(z,x_1) \\ \dot{e} &=& Ae + F(z,x_1) \\ \dot{x}_1 &=& \hat{x}_2 + e_2 + f_1(z,x_1) \\ \dot{\hat{x}}_2 &=& \hat{x}_3 + \ell_2 e_1 \\ &\vdots& \\ \dot{\hat{x}}_r &=& u + \ell_r e_1 . \end{array} \tag{23}$$

It is in the form (12), and we can apply lemma 2.2 to construct a controller depending on $x_1, \hat{x}_2, \ldots, \hat{x}_r$, achieving bounded trajectories from a compact set of initial conditions. To prove convergence to the equilibrium $(z, e, x_1, \hat{x}_2, \ldots, \hat{x}_r) = (0, \ldots, 0)$ we will need the result of lemma 2.4. $\diamond$

The output feedback problem for a more general class of systems can not be solved in this manner. A possible solution is to construct a controller using the output and a sufficient number of its derivatives which are estimated by a high gain "approximate observer". However high gain observers may exhibit a destabilizing peaking effect. In [3] it was shown that, for a special class of systems, saturating the controller outside the domain of interest could overcome this effect. The success of this modification was demonstrated using a singular perturbation approach. The result is now generalized. In the following, the e subsystem represents the observer error, the initial conditions of which depend on the high gain L. When $e = 0$, the z subsystem is appropriately stabilized. The interconnection conditions (26) is satisfied by saturating the controller.

Lemma 2.3 (Robust Observer [3]) *Consider*

$$\dot{z} = h(z,e,d(t))\ ,\ \dot{e} = LAe + g(z,e,d(t)) \tag{24}$$

where $z \in \mathbb{R}^m$, $e \in \mathbb{R}^n$ and L is a strictly positive real number. Suppose assumption MMP is satisfied and let

$$\Gamma = \{z : V(z) \leq c+1\}\ . \tag{25}$$

Also assume the matrix A is Hurwitz, and assume there exist positive real numbers β and ν and a bounded function γ with $\gamma(0) = 0$ satisfying, for all (z,e,d) in $\Gamma \times \mathbb{R}^n \times D$:

$$\left.\begin{array}{rcl} |h(z,e,d) - h(z,0,d)| &\leq& \gamma(|e|) \\ |g(z,e,d)| &\leq& \beta + \nu|e| \end{array}\right\} \tag{26}$$

Let $\mu(L)$ be a class-K_∞ function satisfying

$$\lim_{L\to\infty} \frac{L}{\mu^4(L)} \to \infty. \tag{27}$$

Let P solve the matrix equation $A^T P + PA = -I$. Define the function and the set

$$W(z,e) = c\frac{V(z)}{c+1-V(z)} + \mu(L)\frac{\ln(1+e^TPe)}{\mu(L)+1-\ln(1+e^TPe)}$$
$$A_2 = \{z : V(z) < c+1\} \times \{e : \ln(1+e^TPe) < \mu(L)+1\}.$$

Then, for each $L > 0$, $W(z,e) : A_2 \to \mathbb{R}_+$ is positive definite on $A_2\backslash\{0\}$ and proper on A_2. Also, for each $\rho > 0$, there exists $L_* > 0$ such that, for all $L \geq L_*$, W satisfies

$$\dot{W}_{(24)} \leq -\Phi_2(z,e) \tag{28}$$

where $\Phi_2(z,e)$ is positive definite on $\{(z,e) : \vartheta + \rho \leq W(z,e) \leq c^2 + \mu^2(L) + 1\}$.

Remark 2.3 The motivation for allowing μ to depend on L, in contrast to the previous two lemmas, is to allow for the initial conditions of e to possibly depend on L. If the initial conditions of e can be bounded independent of L then:

1- the bounds in (26) are not needed,

2- μ can be chosen independent of L and the function $\ln(1 + e^T Pe)$ can be replaced by $e^T Pe$.

The motivation for the choice of the function 'ln' is that we wish to allow initial conditions e that are of order L^{r-1}. If we disregard the issue of ultimate convergence, this will recover the result of [3, Theorem 2]. This requires that we choose a Lyapunov function $U(e)$ and a function $\mu(L)$ satisfying the limit (27) and such that, given a strictly positive real number λ_1,

$$U(e) \leq \mu(L) \Longrightarrow |e| \leq \lambda_1 L^{r-1} \tag{29}$$

For instance, if we choose $\mu(L) = \ln(1 + \lambda_2 L^{2(r-1)})$ with λ_2 any strictly positive real number, then the limit (27) is satisfied since

$$\lim_{s\to\infty} \frac{s}{\ln(1+\lambda_2 s^{\alpha_1})^{\alpha_2}} = \infty \qquad \forall \lambda_2, \alpha_1, \alpha_2 > 0. \tag{30}$$

Then, with the appropriate choice of λ_2, (29) is satisfied if we choose $U(e) = \ln(1 + e^T Pe)$. The choice of 'ln' in turn requires the special form of the bounds imposed in (26).

Example 2.3 We consider the multi-input, multi-output nonlinear system

$$\dot{q} = r\ ,\ \dot{r} = f(q,r) + g(q,r)u \tag{31}$$

with $q \in \mathbb{R}^n$, $r \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ is the input, and f and g are C^1. This system could represent a robot model for example. We assume that there exists a (dynamic) compensator

$$\dot{v} = C(q,r,u)\ ,\ u = \alpha(q,r,v) \tag{32}$$

with $v \in \mathbb{R}^l$ such that the closed loop system

$$\begin{array}{rcl} \dot{q} & = & r \\ \dot{r} & = & f(q,r) + g(q,r)\alpha(q,r,v) \\ \dot{v} & = & C(q,r,\alpha(q,r,v)) \end{array} \tag{33}$$

which we rewrite, with $z = (q^T, r^T, v^T)^T$, as

$$\dot{z} = h(z,0) \tag{34}$$

satisfies assumption MMP for some neighborhood A_1 and some function V with $\vartheta = 0$ and c arbitrarily large. Assumption MMP is satisfied, for example, if the equilibrium $(q,r,v) = (0,0,0)$ is made (locally) asymptotically stable by the compensator (32). To implement the compensator (32) without measurement of r we build the observer

$$\dot{\hat{q}} = \hat{r} + L\ell_1(q - \hat{q}) \ , \ \dot{\hat{r}} = L^2\ell_2(q - \hat{q}) \tag{35}$$

where L is an adjustable parameter and ℓ_1, ℓ_2 are coefficients of a Hurwitz polynomial. We implement the compensator

$$\dot{v} = C(q, \Delta(\hat{r}), v, u) \ , \ u = \alpha(q, \Delta(\hat{r}), v) \tag{36}$$

where

$$\Delta(\hat{r}) = \hat{r} \min\{1, \frac{r_{max}}{|\hat{r}|}\} \tag{37}$$

and where r_{max} is the maximum value of $|r|$ on the set $\Gamma = \{(q,r,v) = z : V(z) \le c+1\}$ where $V(z)$ and c come from assumption MMP. This idea for the modification of the compensator is based on the idea in [3]. We choose to saturate the state $\hat{r}$ rather than the entire control u and compensator C because the state r has physical significance and thus determining r_{max} in the region of interest should be quite natural. If we define the error states $e_q = L(q - \hat{q})$ and $e_r = r - \hat{r}$, we have the error dynamics

$$\begin{array}{rcl} \dot{e}_q & = & Le_r - L\ell_1 e_q \\ \dot{e}_r & = & -L\ell_2 e_q + f(q,r) + g(q,r)\alpha(q, \Delta(r - e_r), v) \end{array} \tag{38}$$

and we can apply lemma 2.3. The bounds in (26) can be readily checked, and follow from the introduction of Δ in the compensator (36). Consequently, by choosing c large enough, the modified compensator (36) together with the observer (35) can be used to yield bounded trajectories from the compact set of initial conditions $U \times U_{\hat{q},\hat{r}} \subset \mathbb{R}^{2n+l} \times \mathbb{R}^{2n}$ where U is any compact subset of A_1.

As pointed out in remark 2.3, the bounds in (26) are required because the initial conditions of e grow with L. Specifically, $e_q = L(q - \hat{q})$. However, observe that it may be reasonable to initialize the value of $\hat{q}$ such that $\hat{q}(0) = q(0)$ since q is measured. In this case, the initial condition of e is $(e_q(0) = 0, e_r(0) = r(0) - \hat{r}(0))$ which is independent of L. As mentioned in remark 2.3, in this case the bounds in (26), and hence the function Δ in (36), are not needed. Nevertheless, if this initialization cannot be done exactly, then the function Δ should be retained.

If the original compensator (32) is locally exponentially stabilizing then the conditions of lemma 2.4 will be satisfied and asymptotic stability is also achieved. $\diamond$

Example 2.4 The 'ball and beam' example can be robustly semi-globally stabilized with measurement only of the ball position and beam angle using the tools we have provided. This example is studied in detail in [13]. $\diamond$

Example 2.5 Consider the non-minimum phase system on $\mathbb{R}^3$ with y as the only measured output:

$$\begin{array}{rcl} \dot{z}_1 & = & -z_1 + z_2 - z_1 y^2 \\ \dot{z}_2 & = & z_2^2 + y + z_2 z_1^2 \\ \dot{y} & = & u + z_2. \end{array} \tag{39}$$

The solution to this example, which is inspired by the recent work in [16] on non-minimum phase systems, will make very clear the following fundamental properties we use to construct semi-globally stabilizing output feedback solutions:

1. the knowledge of a stabilizing controller depending on some components of the state vector,
2. the knowledge of functions expressing these particular components in terms of the input, the output, and their derivatives (a key idea of [16]).

For the system (39), the zero dynamics:

$$\dot{z}_1 = -z_1 + z_2 \ , \ \dot{z}_2 = z_2^2 + z_2 z_1^2 \tag{40}$$

are unstable. Indeed, any initial condition satisfying $z_2(0) > 0$ exhibits finite escape time. But the assumptions of [16], full state linearizability and observability, are not satisfied. Nevertheless, we remark that, if z_2 were the measured output, the zero dynamics would reduce to

$$\dot{z}_1 = -z_1 \tag{41}$$

which satisfies assumption MMP. Although the assumptions of lemma 2.2 cannot be satisfied because of the presence of $z_1 y^2$ in the $\dot{z}_1$ equation, the result is still valid. Namely, for K_1 large enough, the control

$$u = -K_1^2(z_2 + \frac{y}{K_1}) \tag{42}$$

is semi-globally stabilizing. This can be checked by looking at the time derivative of

$$W = c\frac{z_1^2}{c + 1 - z_1^2} + \mu\frac{\frac{3}{2}z_2^2 + z_2\frac{y}{K_1} + (\frac{y}{K_1})^2}{\mu + 1 - (\frac{3}{2}z_2^2 + z_2\frac{y}{K_1} + (\frac{y}{K_1})^2)} \ . \tag{43}$$

Local convergence follows from the exponential stability of the new zero dynamics (41). See lemma 2.4. Consequently, property 1 above is satisfied. To satisfy property 2 we need to express z_2 in terms of y, u and their derivatives. But we already have

$$z_2 = \dot{y} - u. \tag{44}$$

So we get the following change of variables:

$$\begin{pmatrix} z_1 \\ z_2 \\ y \\ u \end{pmatrix} \to \begin{pmatrix} z_1 \\ \dot{y} - u \\ y \\ u \end{pmatrix} . \tag{45}$$

This exhibits that we should incorporate u as a component of the state vector. This corresponds to the dynamic extension

$$\dot{u} = v \tag{46}$$

as proposed in [16]. By applying lemma 2.1, and 2.4 for local convergence, we have that the control

$$v = -K_2(u + K_1^2(z_2 + \frac{y}{K_1})) \tag{47}$$

is semi-globally stabilizing for the system

$$\begin{array}{rcl} \dot{z}_1 & = & -z_1 + z_2 - z_1 y^2 \\ \dot{z}_2 & = & z_2^2 + y + z_2 z_1^2 \\ \dot{y} & = & u + z_2 \\ \dot{u} & = & v. \end{array} \tag{48}$$

¿From (44), (47) is implementable if $\dot{y}$ is available. Since it is not we will propose an observer for $\dot{y}$. Observe that we have

$$\begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & v + (x_2 - u)^2 + x_1 + (x_2 - u) z_1^2 \\ y & = & x_1. \end{array} \tag{49}$$

We then build the following "observer":

$$\begin{array}{rcl} \dot{\hat{x}}_1 & = & \hat{x}_2 + L(y - \hat{x}_1) \\ \dot{\hat{x}}_2 & = & v + y + L^2(y - \hat{x}_1). \end{array} \tag{50}$$

Finally, we implement the control:

$$v = -K_2[u + K_1^2(\Delta(\hat{x}_2 - u) + \frac{y}{K_1})] \tag{51}$$

with Δ defined as in the previous two examples, here with respect to z_2. This control is semi-globally stabilizing which can be deduced from lemma 2.3 with lemma 2.4, which follows, providing the local convergence result. $\diamond$

The tools we have presented up to this point have just focused on boundedness of solutions. However, we have constructed Lyapunov functions to guarantee that, in appropriate coordinates, the states become ultimately arbitrarily small. Now, if the linear approximation in these coordinates is exponentially stable we are effectively done. Hence the motivation for assuming exponential stability of the state feedback algorithm in [3]. If the linear approximation is not exponentially stable, then the problem reduces to studying the local stability on the center manifold whose existence is guaranteed by the exponential dichotomy introduced by the fact that K and L can be chosen arbitrarily large. Because the center manifold analysis can be quite involved, we choose to develop a sufficient condition other than exponential stability that can be checked a priori.

Our approach will be to appeal to the notion of 'small gain'. Our next lemma provides a weak form of the small gain theorem which includes explicitly the effects of initial conditions. For other purely input-output results see [6] and the references therein. The global result is included for completeness.

Lemma 2.4 (Small Gain) *Consider the two (time-varying) subsystems*

$$\dot{x}_i = h_i(x_i, u_i, t) \qquad i = 1, 2 \tag{52}$$

with $x_i \in \mathbb{R}^{n_i}$. Assume the existence of strictly positive real numbers δ_i, ϵ_i and of functions β_i and γ_i, of class-KL and class-K respectively, such that, for each $t_\circ \geq 0$,

$$|x_i(t_\circ)| \leq \delta_i \quad \text{and} \quad \sup_{t_\circ \leq \tau \leq \infty} |u_i(\tau)| \leq \epsilon_i$$

implies the existence of solutions $x_i(t)$ for $t \in [t_\circ, +\infty)$ and:

$$|x_i(t)| \leq \beta_i(|x_i(t_\circ)|, t - t_\circ) + \gamma_i(\sup_{t_\circ \leq \tau \leq \infty} |u_i(\tau)|) \quad \forall t \geq t_\circ \, . \tag{53}$$

Suppose also there exist strictly positive real numbers λ, ω such that

$$\left.\begin{array}{l} (1+\lambda)\gamma_1 \circ (1+\lambda)\gamma_2(s) \leq s \\ (1+\lambda)\gamma_2 \circ (1+\lambda)\gamma_1(s) \leq s \end{array}\right\} \quad \forall s \in [0, \omega] \, . \tag{54}$$

Then $(x_1, x_2) = (0, 0)$ is a locally asymptotically stable equilibrium of the system

$$\dot{x}_1 = h_1(x_1, x_2, t) \, , \; \dot{x}_2 = h_2(x_2, x_1, t) \tag{55}$$

with basin of attraction containing the set:

$$\max\{|x_1|, |x_2|\} \leq \kappa\left(\frac{\lambda}{1+\lambda} \min\{\delta_1, \delta_2, \epsilon_1, \epsilon_2, \omega\}\right) \tag{56}$$

where κ is a function of class-K. If inequality (53) holds for all strictly positive δ_i, ϵ_i and inequality (54) holds for all $\omega > 0$, then $(x_1, x_2) = (0, 0)$ is a globally asymptotically stable equilibrium point.

Remark 2.4 Condition (54) is a small gain requirement. For the local case, the lemma can be seen as a generalization of [2, Lemma 4.13] where, there, $\gamma_2 \equiv 0$. For the global case, systems that satisfy (53) are said to be input-to-state stable. For further details see [10]. In the global case, this lemma is a generalization of the result that the cascade of an ISS system and a GAS (globally asymptotically stable) system is GAS. The condition, analogous to (54), used in [6] is more general and could be useful if we did not restrict our analysis to linear controllers.

Example 2.6 Let us consider the system

$$\dot{z} = -z^3 + y \, , \; \dot{y} = u - z|z|^j \tag{57}$$

where j is some non-negative real number. We can apply lemma 2.1 to deduce that the point $(0,0)$ is semi-globally practically stabilizable by the output feedback

$$u = -Ky \quad K \geq K_* \tag{58}$$

where K_* is some positive real number. For lemma 2.4, we consider the subsystems:

$$\dot{x}_1 = -x_1^3 + u_1 \ , \ \dot{x}_2 = -Kx_2 + u_2|u_2|^j \ . \ . \tag{59}$$

Following [10, p.441], we have:

$$\gamma_1(s) = 2s^{\frac{1}{3}} \ . \ , \ \gamma_2(s) = \frac{2}{K}|s|^{j+1} \ . \tag{60}$$

Therefore by choosing K large enough we can meet the constraint (54) for some λ strictly positive and $\omega = 1$ if j is larger than 2. In this condition, we know that the equilibrium $(0,0)$ of (57)-(58) is asymptotically stable. In fact this constraint on j is not necessary. Indeed, asymptotic stability holds for $j \geq 0$ as can be seen by using the Lyapunov function $\frac{|z|^{j+2}}{j+2} + \frac{y^2}{2}$ $\diamond$

References

[1] A. Bacciotti. Linear feedback: the local and potentially global stabilization of cascade systems. In *Proceedings of IFAC Nonlinear Control Systems Design Symposium, Bordeaux*, pages 21–25, June 1992.

[2] C.I. Byrnes and A. Isidori. Asymptotic stabilization of minimum phase nonlinear systems. *IEEE Trans. on Automatic Control*, 36(10):1122–1137, 1991.

[3] F. Esfandiari and H. K. Khalil. Output feedback stabilization of fully linearizable systems. *International Journal of Control*, 1992.

[4] P. V. Kokotovic and R. Marino. On vanishing stability regions in nonlinear systems with high-gain feedback. *IEEE Transactions on Automatic Control*, AC-31:967–970, 1986.

[5] J. Kurzweil. On the inversion of lyapunov's second theorem on stability of motion. *Am. math. Soc. Transl., Ser. 2*, 24:19–77, 1956.

[6] I. M. Y. Mareels and D. J. Hill. Monotone stability of nonlinear feedback systems. *Journal of Mathematical Systems, Estimation and Control*, 2(3):275–291, 1992.

[7] R. Marino, W. Respondek, and A.J. van der Schaft. Almost disturbance decoupling for single-input single-output nonlinear systems. *IEEE Transactions on Automatic Control*, 34:1013–1017, 1989.

[8] R. Marino and P. Tomei. Robust output feedback stabilization of single input output nonlinear systems. In *Proceedings of the 30th Conference on Decision and Control*, pages 2503–2508, December 1991.

[9] L. Praly and Z. P. Jiang. Stabilization by output feedback for systems with iss inverse dynamics. *Systems and Control Letters*, 20(5), 1993.

[10] E. D. Sontag. Smooth stabilization implies coprime factorization. *IEEE Transactions on Automatic Control*, AC-34:435–443, 1989.

[11] E.D. Sontag. Further facts about input to state stabilization. *IEEE Trans. on Automatic Control*, 35(4):473–476, 1990.

[12] H.J. Sussmann and P.V. Kokotovic. The peaking phenomenon and the global stabilization of nonlinear systems. *IEEE Trans. on Automatic Control*, 36(4):424–439, 1991.

[13] A. R. Teel. Semi-global stabilization of the 'ball and beam' using 'output' feedback. In *Proceedings of the 1993 American Control Conference*, June.

[14] A. R. Teel and L. Praly. Tools for semi-global stabilization by partial state and output feedback. Submitted for publication in SIAM J. of Control and Optimization., November 1992.

[15] A.R. Teel and L. Praly. Semi-global stabilization by linear, dynamic output feedback for siso minimum phase nonlinear systems. In *Proceedings of the 12th IFAC World Congress, Sydney, Australia*, July 1993.

[16] A. Tornambè. Output feeback stabilization of a class of non-minimum phase nonlinear systems. *Systems and Control Letters*, 19:193–204, 1992.