LYAPUNOV DESIGN OF A DYNAMIC OUTPUT FEEDBACK FOR SYSTEMS LINEAR IN THEIR UNMEASURED STATE COMPONENTS

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Abstract: We propose a stabilizing output feedback for systems which are linear in their unmeasured state components. We assume weak linear detectability and the knowledge of a globally stabilizing state feedback and of a corresponding Lyapunov function. Our output feedback has an observer-controller structure. It is obtained by following the control Lyapunov function approach. This provides naturally correction terms compared with what would be given by the separation "principle". Global Lagrange stability is established under the extra assumption that the partial derivatives with respect to the unmeasured state components of the Lyapunov function mentioned above are bounded.

Keywords: Dynamic feedback, Output fedback, Observer, Control Lyapunov function, Nonlinear Regulation.

1 Introduction and Problem statement

We consider systems for which there exists a globally defined set of coordinates such that the unmeasured ones appear linearly in the dynamical equation, i.e.

$$\left\{ \begin{array}{l} \dot{x} = A(z,u)\,x \;+\; B(z,u) \\ \dot{z} = C(z,u)\,x \;+\; D(z,u) \end{array} \right. \tag{1}$$

where A, B, C and D are C^1 functions, x is the unmeasured state component in $\mathbb{R}^l$, z is the measured state component in $\mathbb{R}^n$ and u is a control input in $\mathbb{R}^m$. Some sufficient geometric conditions for the existence of such a set of coordinates have been proposed in the literature (see [8] and references therein for example). This question of existence will not be considered here. The problem we are concerned with here has been stated in [2] as :

Design an output feedback so that, for any initial condition, the z-component at least, if not the complete state vector (x, z) of the corresponding solution of the closed loop system converges to a desired rest point.

This problem has received some attention in the literature under various specifications :

1. For its above general form, a solution has been proposed in [2] by applying a Lyapunov design to an estimator assuming that a solution is known when x is measured. Here we will follow the same route, but thanks to a better choice of the control Lyapunov function, we will get a solution under less restrictive conditions.

2. When the objective is to have both x and z to converge to a desired rest point, this is the problem of regulation with incomplete state measurement. Whereas many authors have addressed the local regulation problem (see [16] and references therein for example), there are much less results available for the global case. With different techniques – filtered transformation in [8], iterative robust Lyapunov design in [6] – it is has been shown that stabilization can be achieved globally by a dynamic controller for minimum phase linear systems with output nonlinearities, i.e. systems which can be represented by :

$$\mathcal{A}(s)\,y \;=\; \mathcal{B}(s)\,u \;+\; \sum_{i=1}^{l+n} s^{l+n-i}\,\psi_i(y) \tag{2}$$

where $\mathcal{A}$ and $\mathcal{B}$ are polynomials in s, the derivation with respect to time, $\mathcal{B}$ is Hurwitz, y is a measured output and the ψ_i's are smooth functions. Here, we shall mention that our design applies to this kind of systems even when $\mathcal{B}$ is not Hurwitz. However, for the time being, we have no proof that any system in the form (2) can be handled. Also, we consider only the regulation problem whereas the problem of output tracking is also solved in [8, 6].

3. When the objective is to have z to converge to a desired rest point while x, given by an autonomous system (not depending on u and z), called exosystem, is acting as a disturbance, we get the Error Feedback Regulator problem as formulated in [5]. The solution in [5] assumes the output dynamic feedback stabilizability of the linearized system and is given by the sum of a nonlinear control which fixes the desired rest point for z and a linear control which stabilizes this point. Here, by dealing with a smaller class of systems – the exosystem is linear, z is completely measured – we shall obtain a global asymptotic disturbance rejection.

4. In [3], the discrete time version of our problem is considered but without assuming linearity in the unmeasured state components. Existence of a globally stabilizing output feedback provided a globally stabilizing state feedback is known (as here and in [2]) and the state vector is observable in a finite number steps. Unfortunately, this ouput feedback incorporates the inverse of a map. And, when this exact inverse is replaced by an approximated one given by a Newton's algorithm, globality may be lost.

In section 2, we write our assumptions. In section 3, we propose a dynamic output feedback obtained by applying a Lyapunov design. The properties of the solutions of the closed loop system are studied in section 4. Section 5 is devoted to an example. Finally some concluding remarks are given in section 6.

2 Assumptions

As mentioned above, the problem addressed actually in this paper is : assuming that we know how to control the system when x is measured what can be done when x is unmeasured ? What we mean by "how to control" is made precise in the following assumption :

Assumption S (Stabilizability) (3)
There exist two known functions : $u_{\mathrm{n}} : \mathbb{R}^l \times \mathbb{R}^n \longrightarrow \mathbb{R}^m$ which is C^1 and $V : \mathbb{R}^l \times \mathbb{R}^n \longrightarrow \mathbb{R}_+$ which is C^3 such that, for all (x, z) in $\mathbb{R}^l \times \mathbb{R}^n$, we have :

$$\frac{\partial V}{\partial x}\left(Ax + B\right) + \frac{\partial V}{\partial z}\left(Cx + D\right)\Big|_{(x,z,u_{\mathrm{n}}(x,z))} \quad (4)$$
$$\stackrel{\mathrm{def}}{=} -W(x, z) \leq 0 .$$

This assumption is always satisfied by systems in the form (2) when the polynomial $\mathcal{B}$ is Hurwitz. It is also invoked in [3], is equivalent to [2, Assumption A1.1] and is implied locally by [5, H1 sect. 7.2].

Since x is not measured, we cannot implement the control $u = u_{\mathrm{n}}(x, z)$. Following the well known separation "principle", one idea is to implement $u = u_{\mathrm{n}}(\widehat{x}, z)$ where $\widehat{x}$ would be an estimation of x. To derive a meaningful observer providing this estimation, some kind of observability condition is involved. Here is our assumption :

Assumption D (Detectability) (5)
There exist matrices $K_x(z, u)$, $K_z(z, u)$, Q, R, S, with K_x and K_z C^1 functions in (z, u), such that the matrix $\begin{pmatrix} Q & R^\top \\ R & S \end{pmatrix}$ is symmetric positive definite, and for all $(z, u) \in \mathbb{R}^n \times \mathbb{R}^m$ and $(\widetilde{x}, \widetilde{z}) \in \mathbb{R}^l \times \mathbb{R}^n$, we have :

$$(\widetilde{x}^\top Q + \widetilde{z}^\top R)\left(A\widetilde{x} + K_x\widetilde{z}\right) \quad (6)$$
$$+ (\widetilde{x}^\top R^\top + \widetilde{z}^\top S)\left(C\widetilde{x} + K_z\widetilde{z}\right) \leq -\widetilde{x}^\top \Sigma_x \widetilde{x} - \widetilde{z}^\top \widetilde{z}$$

where Σ_x is a symmetric non negative matrix.

This is a strong assumption : observability is not allowed to depend on the input and the matrices Q, R and S are constant. However it is satisfied for example by systems in the form (2), or when $A(z, u)$ is skew symmetric (the case of adaptive nonlinear regulation). It is also implied by [2, assumption A2] and, locally, it is weaker than the exponential detectability assumption [5, H2-H3 sect. 7.2] or in [1, Lemma 3].

With assumptions S (3) and D (5) , we know, from [12, Theorem (2.29)], the existence of an output dynamic controller solving our regulation problem. Unfortunately, this controller is defined in a very abstract sense. So our task here is to find an explicit and implementable form. For this, one may first attempt to apply the separation "principle" and propose :

$$\begin{cases} \dot{\widehat{x}} = A(z, u)\widehat{x} + B(z, u) + K_x(z, u)\left(\widehat{z} - z\right) \\ \dot{\widehat{z}} = C(z, u)\widehat{x} + D(z, u) + K_z(z, u)\left(\widehat{z} - z\right) \\ u = u_{\mathrm{n}}(\widehat{x}, z) \end{cases} \quad (7)$$

In particular, if Σ_x, in (6), is positive definite uniformly in (z, u), the estimation errors $\widehat{x} - x$ and $\widehat{z} - z$ go exponentially to 0, whatever the control u is, but as long as the solution $(x(t), z(t))$ of (1) exists. Indeed, in this case, according to [1, Lemma 3] or to [17, Theorem 3.1] or its generalization in [16], we are guaranteed that our regulation problem is solved but only if the initial condition $(x(0), z(0))$ is sufficiently close to the desired rest values and the observer is initialized with $(\widehat{x}(0), \widehat{z}(0))$ sufficiently close to $(x(0), z(0))$. This means that the global regulation property may not hold. That such a property may fail follows from the well known fact that global asymptotic stability of the trivial solutions of the following two systems (see [15] or [11] and references therein) :

$$\dot{\zeta}_1 = f(\zeta_1) \quad , \quad \dot{\zeta}_2 = g(0, \zeta_2) , \quad (8)$$

is not sufficient to guarantee the global asymptotic stability of the trivial solution of the following composite system :

$$\begin{cases} \dot{\zeta}_1 = f(\zeta_1) \\ \dot{\zeta}_2 = g(\zeta_1, \zeta_2) \end{cases} \quad (9)$$

Extra conditions about g are needed. According to [14] (see also [7, section VI.3]), a sufficient condition for us is that the system :

$$\begin{cases} \dot{x} = A\left(z, u_{\mathrm{n}}(x + \varpi, z)\right) x + B\left(z, u_{\mathrm{n}}(x + \varpi, z)\right) \\ \dot{z} = C\left(z, u_{\mathrm{n}}(x + \varpi, z)\right) x + D\left(z, u_{\mathrm{n}}(x + \varpi, z)\right) \end{cases} \quad (10)$$

is input ϖ to state (x, z) stable. Such a property does not hold in general, but one may think of modifying the control law u_{n} according to [14, Theorem1]. Unfortunately the application of this Theorem is not straightforward since the full state is not measured. Nevertheless, for minimum phase linear systems with output nonlinearities such a modification of u_{n} has been implicitely obtained in [6] thanks to an iterative procedure. Here, we directly address the regulation problem (as in [2] or [6]) by applying a Lyapunov design starting from the feedback u_{n} and the "Lyapunov function" V given by assumption S (3) . The extra condition mentioned above we shall need here concerns the dependence of V on the unmeasured state component x. Precisely :

Assumption GC (Growth Condition) (11)
There exists a constant γ such that, for all (x, z) in $\mathbb{R}^l \times \mathbb{R}^n$:

$$\left\|\frac{\partial V}{\partial x}(x, z)\right\| \leq \gamma \quad , \quad \left\|\frac{\partial^2 V}{\partial x^2}(x, z)\right\| \leq \gamma^2 \quad . \quad (12)$$

A characteristic of assumption GC (11) is that the system nonlinearities are not explicitely involved. This differs from some more classical global Lipschitz condition or [16, Condition iii Theorem 3.1] for example.

Unfortunately, we do not know any characterization of systems in the form (1) such that GC (11) holds.

A straightforward consequence of the first order Taylor's expansion and assumption GC (11) is :

Lemma 1 *Under assumption GC (11) , for all* $(\widehat{x}, x, z)$ *in* $\mathbb{R}^l \times \mathbb{R}^l \times \mathbb{R}^n$, *we have :*

$$0 \leq V(x,z) \leq V(\widehat{x}, z) + \gamma \|\widehat{x} - x\| . \quad (13)$$

Up to now, we have introduced no assumption about the radial unboundedness of V. The motivation is : the less V will depend on x, the more easily we will be able to meet assumption GC (11) . Nevertheless this function V should give some information on the state vector (x, z). Here, we require only a boundedness observability, i.e. we assume that if the "output" trajectory $\{V(x(t), z(t))\}_{t\in[0,T)}$ is bounded then the state trajectory $\{(x(t), z(t))\}_{t\in[0,T)}$ is also bounded and this only for a particular class of inputs. Precisely :

Assumption BO (Boundedness observability) (14)
For all positive real number v, *all compact subset* $\mathcal{K}$ *of* $\mathbb{R}^l$, *and all vector* (x_0, z_0) *in* $\mathbb{R}^l \times \mathbb{R}^n$, *there exists a compact subset* Γ *of* $\mathbb{R}^l \times \mathbb{R}^n$ *such that for any* C^1 *function* $\widetilde{x} : \mathbb{R}_+ \to \mathcal{K}$ *and any solution* $(x(t), z(t))$ *of :*

$$\begin{cases} \dot{x} = A\left(z, u_{\mathrm{n}}(x + \widetilde{x}(t), z)\right) x + B\left(z, u_{\mathrm{n}}(x + \widetilde{x}(t), z)\right) \\ \dot{z} = C\left(z, u_{\mathrm{n}}(x + \widetilde{x}(t), z)\right) x + D\left(z, u_{\mathrm{n}}(x + \widetilde{x}(t), z)\right) \end{cases} \quad (15)$$

with $(x(0), z(0)) = (x_0, z_0)$ *and defined maximally on* $[0, T)$, *we have the following implication :*

$$V(x(t), z(t)) \leq v \quad \forall t \in [0, T) \quad (16)$$
$$\Longrightarrow \quad (x(t), z(t)) \in \Gamma \ \forall t \in [0, T) .$$

This assumption is trivially satisfied if the system (10) is input ϖ to state (x, z) stable. It is also met for minimum phase systems with an extra growth condition (see [11, 15]) when V is proper in the output and its time derivatives. Another interesting case implying that BO (14) holds has been considered in [2]. It is when :

There exists a C^1 *function* $\mathcal{V} : \mathbb{R}^l \times \mathbb{R}^n \mapsto \mathbb{R}_+$ *such that : 1.* $V + \mathcal{V}$ *is a proper function on* $\mathbb{R}^l \times \mathbb{R}^n$,
2. for all $(x, z, u) \in \mathbb{R}^l \times \mathbb{R}^n \times \mathbb{R}^m$, *we have :*

$$\frac{\partial \mathcal{V}}{\partial x}([Ax + B] + \frac{\partial \mathcal{V}}{\partial z}[Cx + D] \leq 0 . \quad (17)$$

With assumptions D (5) , S (3) , GC (11) and BO (14) we have our ingredients to find a dynamic controller guaranteeing solution boundedness. As far as the stronger property of regulation is concerned, as will be seen later, everything will depend on the singularity or nonsingularity of the matrix Σ_x in assumption D (5) and the properties of the "time derivative" W introduced in assumption S (3) .

3 Lyapunov design of a dynamic output feedback

To find a solution to our problem, we restrict our attention to a dynamic output feedback made of :

1. an observer $(\widehat{x}, \widehat{z})$ for the state vector (x, z),
2. an extra dynamical extension with state χ whose dimension is to be chosen,
3. the control itself chosen as the certainty equivalent control $u_{\mathrm{n}}(\widehat{x}, z)$.

Namely, we choose :

$$\begin{cases} \dot{\widehat{x}} = u_x(\widehat{x}, z, \widehat{z}, \chi) , & \dot{\chi} = u_\chi(\widehat{x}, z, \widehat{z}, \chi) \\ \dot{\widehat{z}} = u_z(\widehat{x}, z, \widehat{z}, \chi) , & u = u_{\mathrm{n}}(\widehat{x}, z) \end{cases} \quad (18)$$

To obtain explicit expressions for the new control u_x, u_z, and u_χ we have introduced, we apply a Lyapunov design. The control Lyapunov function we choose for this design is :

$$\begin{aligned} U(x, z, \widehat{x}, \widehat{z}, \chi) &= \alpha\left(1 + V(\widehat{x}, z)\right) \quad (19) \\ &- \chi^\top \widetilde{x} + \tfrac{1}{2}\widetilde{x}^\top Q \widetilde{x} + \widetilde{z}^\top R \widetilde{x} + \tfrac{1}{2}\widetilde{z}^\top S \widetilde{z} \\ &+ \tfrac{\beta}{2}\left(\chi^\top - \alpha \frac{\partial V}{\partial x}(\widehat{x}, z)\right) T \left(\chi - \alpha \frac{\partial V}{\partial x}(\widehat{x}, z)^\top\right) \end{aligned}$$

where to simplify the forthcoming notations we let :

$$\widetilde{x} = \widehat{x} - x \quad , \quad \widetilde{z} = \widehat{z} - z . \quad (20)$$

In (19), α and β are strictly positive real numbers and T is a symmetric positive definite matrix to be chosen later. With this choice for U, we have specified that the dynamical extension χ is in $\mathbb{R}^l$ like the unmeasured state component x. This expression for U follows from a straightforward extension of a control Lyapunov function we have built step by step in [9] to solve an adaptive nonlinear regulation problem. A first property of this function U is :

Lemma 2 *Under assumption GC (11) , if :*

$$\alpha < \frac{2\mu}{\gamma^2 \lambda_{\max}\{T\}} \quad , \quad \beta > \frac{2}{2\mu - \alpha\gamma^2 \lambda_{\max}\{T\}} , \quad (21)$$

with μ *a strictly positive real number such that :*

$$\mu T^{-1} < (Q - R^\top S^{-1} R) , \quad (22)$$

then there exists a strictly positive real number ε *such that, for all* $(x, z, \widehat{x}, \widehat{z}, \chi)$, *we have :*

$$\begin{aligned} &U(x, z, \widehat{x}, \widehat{z}, \chi) \quad (23) \\ &\geq \tfrac{\varepsilon}{2}\left[\alpha\left(1 + V(\widehat{x}, z)\right) + \widetilde{x}^\top Q \widetilde{x} + \widetilde{z}^\top S \widetilde{z} + \chi^\top T \chi\right] \\ &> 0 . \end{aligned}$$

Proof : It is sufficient to use assumption GC (11) and to show that for ε sufficiently small and σ defined by :

$$\frac{1 + \sigma}{(1 + \sigma)(\beta - \varepsilon) - \beta} = \mu - \varepsilon . \quad (24)$$

we have the following inequality :

$$\begin{aligned} &U(x, z, \widehat{x}, \widehat{z}, \chi) \quad (25) \\ &\geq \alpha\left(1 + V(\widehat{x}, z)\right) + \frac{\varepsilon}{2}\left(\widetilde{x}^\top Q \widetilde{x} + \widetilde{z}^\top S \widetilde{z} + \chi^\top T \chi\right) \\ &\quad - \tfrac{\sigma\beta\alpha^2}{2} \tfrac{\partial V}{\partial x}(\widehat{x}, z) T \tfrac{\partial V}{\partial x}(\widehat{x}, z)^\top . \quad \square \end{aligned}$$

With this Lemma 2 and assumption BO (14) , we see that boundedness of the solutions will follow from the boundedness of U. Therefore, let us choose the new controls u_x, u_z and u_χ such that the time derivative of U along the solutions of (1)-(18) is negative. This time derivative is :

$$\dot{U} = \alpha \frac{\partial V}{\partial x}\dot{\widehat{x}} + \alpha \frac{\partial V}{\partial z}(Cx + D) - \widetilde{x}^\top \dot{\chi} - \chi^\top \left(\dot{\widehat{x}} - Ax - B\right) + \left(\widetilde{x}^\top Q + \widetilde{z}^\top R\right)\left(\dot{\widehat{x}} - Ax - B\right) + \left(\widetilde{x}^\top R^\top + \widetilde{z}^\top S\right)\left(\dot{\widehat{z}} - Cx - D\right) + \beta\left(\chi^\top - \alpha\frac{\partial V}{\partial x}\right) T \left(\dot{\chi} - \alpha \dot{\overline{\frac{\partial V}{\partial x}}}^\top\right) \quad (26)$$

On the other hand, using equations (1), we get :

$$\dot{\overline{\frac{\partial V}{\partial x}}}^\top = H(\widehat{x}, z)\,\dot{\widehat{x}} + F(\widehat{x}, z, u)\,x + G(\widehat{x}, z, u) \quad (27)$$

where :

$$\begin{cases} H(x,z) = \dfrac{\partial}{\partial x}\left(\dfrac{\partial V}{\partial x}^\top\right)(x,z)\,, \\ F(x,z,u) = \dfrac{\partial}{\partial z}\left(\dfrac{\partial V}{\partial x}^\top\right)(x,z)\,C(z,u)\,, \\ G(x,z,u) = \dfrac{\partial}{\partial z}\left(\dfrac{\partial V}{\partial x}^\top\right)(x,z)\,D(z,u)\,. \end{cases} \quad (28)$$

Then, it can be seen that, by choosing $u_x = \dot{\widehat{x}}$, $u_z = \dot{\widehat{z}}$ and $u_\chi = \dot{\chi}$ as follows :

$$\begin{pmatrix} Q & R^\top & -I \\ R & S & 0 \\ I + \alpha\beta T H & 0 & -\beta T \end{pmatrix} \begin{pmatrix} \dot{\widehat{x}} - A\widehat{x} - B - K_x\widetilde{z} \\ \dot{\widehat{z}} - C\widehat{x} - D - K_z\widetilde{z} \\ \dot{\chi} \end{pmatrix} \quad (29)$$

$$= \begin{pmatrix} \left(\alpha C^\top \frac{\partial V}{\partial z}^\top + A^\top \chi\right) - \beta\alpha F^\top T\left(\chi - \alpha \frac{\partial V}{\partial x}^\top\right) \\ 0 \\ \Lambda\left(\chi - \alpha\frac{\partial V}{\partial x}^\top\right) \\ \quad - \alpha\beta T\left[H\left(A\widehat{x} + B + K_x\widetilde{z}\right) + F\widehat{x} + G\right] \end{pmatrix}$$

when this makes sense and with Λ a symmetric positive definite matrix, we get :

$$\dot{U} \le -\alpha W(\widehat{x}, z) - \widetilde{x}^\top \Sigma_x \widetilde{x} - \widetilde{z}^\top \widetilde{z} - \left(\chi^\top - \alpha\frac{\partial V}{\partial x}\right)\Lambda\left(\chi - \alpha\frac{\partial V}{\partial x}^\top\right). \quad (30)$$

From assumption D (5) , the matrix $Q - R^\top S^{-1} R$ is nonsingular. Therefore, the linear system (29) can be solved if the following inequality between quadratic forms is satisfied for all $(x, z) \in \mathbb{R}^l \times \mathbb{R}^n$:

$$\alpha\,\beta\, H(x,z) \le \beta\left(Q - R^\top S^{-1} R\right) - T^{-1}. \quad (31)$$

But, with the definition (28) of H, assumption GC (11) and (22), this inequality holds if α and β are chosen such that :

$$\alpha\,\beta\,\gamma^2 < \frac{\mu\,\beta - 1}{\lambda_{\max}\{T\}}. \quad (32)$$

To summarize we have designed with (29) and $u = u_n(\widehat{x}, z)$ a dynamic output feedback. This feedkack incorporates four parameters α, β, T and Λ to be chosen such that (22), (21) and (32) are satisfied. Compared with the dynamic output feedback (7) given by a straightforward application of the separation "principle", we have obtained correction terms characterized by the right hand side of (29).

4 Properties of the closed loop system

Proposition 1 *Let assumptions D (5) , S (3) , GC (11) and BO (14) hold and α, β, T and Λ be chosen such that T and Λ are symmetric positive definite matrices and :*

$$\begin{cases} 0 < \alpha < \dfrac{\mu}{\gamma^2 \lambda_{\max}\{T\}}\,, \\ \beta > \dfrac{1}{\mu - \alpha\gamma^2\lambda_{\max}\{T\}}\,, \\ \mu T^{-1} < 2\left(Q - R^\top S^{-1} R\right). \end{cases} \quad (33)$$

Under these conditions, for any initial condition, there exists a unique solution of (1)-(18) with (29). This solution is bounded on $[0, +\infty)$ and converges to the set $\mathcal{I}$ of points $(x, z, \widehat{x}, \widehat{z}, \chi)$ satisfying :

$$\left.\begin{matrix} W(\widehat{x}, z) = 0\,,\ (\widehat{x} - x)^\top \Sigma_x (\widehat{x} - x) = 0\,, \\ \widehat{z} = z\,,\ \chi = \alpha\frac{\partial V}{\partial x}(\widehat{x}, z) \end{matrix}\right\}. \quad (34)$$

Proof : The closed loop system we consider has a C^1 right hand side in $\mathbb{R}^l \times \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^n \times \mathbb{R}^l$. It follows that, for any initial condition, there exists a unique solution $(x(t), z(t), \widehat{x}(t), \widehat{z}(t), \chi(t))$ defined on a right maximal interval $[0, T)$, with T may be infinite. From our design, we know that the time derivative of $U(x(t), z(t), \widehat{x}(t), \widehat{z}(t), \chi(t))$ satisfies (see (30)) for all t in $[0, T)$:

$$\dot{U}(t) \le -W(\widehat{x}(t), z(t)) - \left[\widehat{x}(t) - x(t)\right]^\top \Sigma_x \left[\widehat{x}(t) - x(t)\right] - \|\widehat{z}(t) - z(t)\|^2 - \left[\chi(t)^\top - \alpha\frac{\partial V}{\partial x}(\widehat{x}(t), z(t))\right]\Lambda\left[\chi(t) - \alpha\frac{\partial V}{\partial x}(\widehat{x}(t), z(t))^\top\right] \quad (35)$$

It follows that, for all t in $[0, T)$, U decreases with time. With Lemma 2, we know the existence of a strictly positive real number ε such that this inequality yields, for all t in $[0, T)$,

$$\alpha V(\widehat{x}(t), z(t)) + (\widehat{x}(t) - x(t))^\top Q\,(\widehat{x}(t) - x(t)) + (\widehat{z}(t) - z(t))^\top S\,(\widehat{z}(t) - z(t)) + \chi(t)^\top T\,\chi(t) \le \frac{2\,U(x(0), z(0), \widehat{x}(0), \widehat{z}(0), \chi(0))}{\varepsilon} - \alpha\,. \quad (36)$$

Now, we are in a position to use assumption BO (14) . For this, we define a positive real number v by :

$$v = \frac{2U(x(0), z(0), \widehat{x}(0), \widehat{z}(0), \chi(0))}{\alpha\varepsilon} - 1 + \gamma\sqrt{\frac{2U(x(0), z(0), \widehat{x}(0), \widehat{z}(0), \chi(0)) - \alpha\varepsilon}{\varepsilon\,\lambda_{\min}\{Q\}}} \quad (37)$$

and a compact subset $\mathcal{K}$ of $\mathbb{R}^l$ whose elements $\widetilde{x}$ satisfy :

$$\widetilde{x}^\top Q\,\widetilde{x} \le \frac{2\,U(x(0), z(0), \widehat{x}(0), \widehat{z}(0), \chi(0))}{\varepsilon} - \alpha\,. \quad (38)$$

Since $(x(t), z(t))$ is also a solution of (15) with, for all t in $[0, T)$, $\tilde{x}(t) = \hat{x}(t) - x(t)$ in $\mathcal{K}$ and $V(x(t), z(t)) \leq v$ (see Lemma 1), assumption BO (14) implies the existence of a compact set Γ, depending on $(x(0), z(0), \hat{x}(0), \hat{z}(0), \chi(0))$, such that, for all t in $[0, T)$, we have :

$$(x(t), z(t)) \in \Gamma . \qquad (39)$$

Since Q, S and Λ are positive definite matrices, this membership property and inequality (36) proves that the solution $(x(t), z(t), \hat{x}(t), \hat{z}(t), \chi(t))$ is bounded on $[0, T)$. Therefore, by contradiction, $T = +\infty$ and the first part of our Proposition is established. Our last statement is a straigthforward application of [4, Theorem X.3.2.a]. □

With this Proposition, we know that the output dynamic feedback we have designed in the previous section guarantees that all the solutions of the closed loop system are bounded and converge to the set $\mathcal{I}$ defined in (34). Then the fact that we have solved or not the regulation problem stated in section 1 depends only on the properties of the bounded solutions of :

$$\begin{cases} \dot{x} = A\left(z\,,\,u_{\mathrm{n}}(\hat{x}(t)\,,\,z)\right)\,x \,+\, B\left(z\,,\,u_{\mathrm{n}}(\hat{x}(t)\,,\,z)\right) \\ \dot{z} = C\left(z\,,\,u_{\mathrm{n}}(\hat{x}(t)\,,\,z)\right)\,x \,+\, D\left(z\,,\,u_{\mathrm{n}}(\hat{x}(t)\,,\,z)\right) \\ \dot{\hat{x}} = A\left(z\,,\,u_{\mathrm{n}}(\hat{x}(t)\,,\,z)\right)\,\hat{x} \,+\, B\left(z\,,\,u_{\mathrm{n}}(\hat{x}(t)\,,\,z)\right) \\ \qquad\qquad -\,\alpha\,\beta\,T\,F\left(\hat{x}, z, u_{\mathrm{n}}(\hat{x}, z)\right)\left(\hat{x} - x\right) \end{cases} \qquad (40)$$

which satisfy, for all $t \in \mathbb{R}_+$,

$$W(\hat{x}(t), z(t)) = \left(\hat{x}(t) - x(t)\right)^{\top} \Sigma_x \left(\hat{x}(t) - x(t)\right) = 0 \qquad (41)$$

To obtain the last equation of (40), we have simply used (27), $\tilde{z} \equiv 0$ and $\chi \equiv \alpha\frac{\partial V}{\partial x}$ in the last component of the vector equation (29). A straightforward consequence is that, if we can find a symmetric positive definite matrix T such that $A - \alpha\beta T F$ is stable in the appropriate sense or if Σ_x is positive definite, i.e. the unmeasured component x is strongly detectable (see (6)), the same regulation properties hold as for the closed loop system with x measured and $u = u_{\mathrm{n}}(x, z)$.

5 Examples

Going back to the linear system with output nonlinearities (2), we have established in [10] that Proposition 1 applies at least in the case where $\mathcal{B}$ is Hurwitz and $\mathrm{degree}(\mathcal{B}) \geq \mathrm{degree}(\mathcal{A}) - 3$. But extension to cases where $\mathcal{B}$ is not Hurwitz is also possible (see [10, Example 2]).

Let us now consider the following system which cannot be represented in the form (2) :

$$\begin{cases} \dot{z} \;= x_1 \,+\, z\,x_2 \\ \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 + u \end{cases} \qquad (42)$$

Assumption D (5) is satisfied with :

$$\begin{cases} K_x = \begin{pmatrix} -1 \\ -z \end{pmatrix} \,,\; K_z = -1\,, \\ Q = \begin{pmatrix} 1\,0 \\ 0\,1 \end{pmatrix} \,,\; R = 0\,,\; S = 1\,,\; \Sigma = 0\,. \end{cases} \qquad (43)$$

Assumptions S (3) , BO (14) and GC (11) are met if we choose :

$$V(x_1, x_2, z) \;=\; \log\left(1 + \overline{V}(x_1, x_2, z)\right)\;, \qquad (44)$$

with :

$$\overline{V}(x_1, x_2, z) \;= \qquad (45)$$
$$\tfrac{1}{2}\left[x_1^2 + \left(1 + (z-1)\exp(-x_1)\right)^2 + (x_1 + x_2)^2\right]\;,$$

and :

$$u_{\mathrm{n}}(x_1, x_2, z) \;=\; -x_1 \;-\; 2\,x_2 \qquad (46)$$
$$\qquad -\;\left(1 + (z-1)\exp(-x_1)\right)\exp(-x_1)\,.$$

In particular, we get :

$$W(x_1, x_2, z) \;=\; x_1^2 \;+\; (x_1 + x_2)^2\,. \qquad (47)$$

Then, with the expression (47) of W and by applying Proposition 1, we are guaranteed that the closed loop solutions are bounded with the following property for their components :

$$\lim_{t \mapsto +\infty} (\hat{x}_1(t), \hat{x}_2(t)) \;=\; 0 \qquad (48)$$

With LaSalle's Theorem and (40), it follows that these solutions satisfy also the following equation asymptotically :

$$B\left(z\,,\,u_{\mathrm{n}}(0\,,\,z)\right) + \alpha\,\beta\,T\,F\left(0, z, u_{\mathrm{n}}(0, z)\right)\,x = 0. \qquad (49)$$

Choosing say $T = I$, with (46), (28), (44) and the definition of F, this yields :

$$\begin{pmatrix} 0 \\ \\ -z \end{pmatrix} + \alpha\,\beta \begin{pmatrix} \dfrac{(1 - 2z - \frac{1}{2}z^2)(x_1 + z x_2)}{(1 + \frac{1}{2}z^2)^2} \\ 0 \end{pmatrix} = 0\;. \qquad (50)$$

Therefore, we have also :

$$\lim_{t \mapsto +\infty} (x_1(t), z(t)) \;=\; 0\,. \qquad (51)$$

Then, again with LaSalle's Theorem and the system equation, we get finally :

$$\lim_{t \mapsto +\infty} x_2(t) \;=\; 0\,. \qquad (52)$$

6 Concluding remarks

We have proposed a dynamic output feedback for systems :

A1 : which are linear in their unmeasured state components,

A2 : such that the unmeasured state components are weakly detectable but uniformly with respect to the measured state components and the input,

A3 : and for which we know a state feedback and a corresponding Lyapunov function

A4 : whose partial derivatives with respect to the unmeasured state components are bounded.

In this feedback, the control is equal to the state feedback of A3 evaluated at an observed state vector. The update law for this observed state vector is designed by applying the control Lyapunov function technique (see [13]). This leads to a dynamic output feedback with an observer-controller structure. Compared with the feedback that the standard separation "principle" would give, we get correction terms involving mainly the partial derivatives of the Lyapunov function of A3.

Though, as seen from examples, our dynamic output feedback can be applied to systems for which no such feedback were known before – as far as we know –, more work needs to be done to simplify if not relax our assumptions A1 to A4.

It would be interesting to have a complete characterization of the sytems which can be written in the form (1) with the linearity assumption A1 satisfied. Such a form implies in particular that if the measured state components are given by an output function, this function cannot depend on the input. This constraint is also required in [3]. Nevertheless partial results about this characterization are available in [8] and references therein for example. This linearity assumption mentioned in A1 allows us to choose a control Lyapunov function quadratic in the state observation error. It is sometimes possible to meet this assumption by "immersing" the system into a higher dimensional one (see [2, Example 2] for an illustration).

The detectability assumption A2 is somewhat necessary in our approach. Its restrictiveness follows from the fact that we want the stability involved in the detectability notion to be described by a constant positive definite matrix. Allowing a non constant matrix would be very helpful as shown in [2, Example 2].

Assumption A3 characterizes one way of approaching the problem of designing dynamic output feedbacks. Namely this assumption means that a solution is already known if the complete state vector is measured. And it allows us to split the design problem into two parts : the complete information part and the incomplete one. The fact that a Lyapunov function is required leads us to prefer the control Lyapunov function approach to solve the first part. Restrictiveness appears really with assumption A4. We do not know any characterization of systems for which this assumption holds. In practice, the idea to meet A4 is, in the first part mentioned above, to find not only one state feedback or more precisely one control Lyapunov function but a family of it among which we may expect to find one satisfying A4. This procedure has been used in our example. Note that the Lyapunov function mentioned in A3 and A4 needs not be proper but satisfies the less stringent "boundedness observability" condition BO (14) .

In conclusion, what is proposed here is not a stabilizability result but a design tool among many others. And what should be mainly extracted from this paper is the control Lyapunov function (19) and the way we get the output feedback from this expression.

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