Adaptive non linear stabilization and robust Lagrange stability

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Abstract: We are concerned with the problem of stabilizing the equilibrium point of a non linear system in presence of both parametric and dynamic uncertainties. For the parametric uncertainty, we propose a new adaptive controller based on a Lyapunov design and guaranteeing Lagrange stability if a growth condition is satisfied. For the dynamic uncertainty, we propose a new way of characterizing the unmodelled effects which encompasses at least some singular perturbations as illustrated by an example. Finally we show how, by modifying the above controller, Lagrange stability can be made robust to these unmodelled effects.

Keywords: Adaptive control, unmodelled effects, stabilization, robustness.

1 Introduction

Important progress has been done in adaptive control of non linear systems. Main difficulties are now well understood and some very sophisticated solutions are available. Most of the results are synthetized in [17] and the references therein with more recent developments in [9,10]. However, these studies concern the ideal case, the case where the system to be controlled is exactly modelled up to the knowledge of some constant parameters. We know from the linear case that robustness to unmodelled effects of the properties of adaptive systems is a very difficult issue. For the non linear case, results are already available for particular systems about the robustness of Lagrange stability to some unmodelled effects : Taylor et al. [19] and Kanellakopoulos et al. [8] have studied feedback linearizable systems in presence of singular perturbations, Campion and Bastin [1,2] and Reed and Ioannou [18] have considered manipulators under bounded disturbances and singular perturbations. The objective of this paper is to report some preliminary results for more general circumstances about the following two aspects :

1 – introduce a new way of characterizing unmodelled effects,
2 – study the robustness of Lagrange stability given by an adaptive stabilizer based on a Lyapunov design.

In proposing a characterization of the unmodelled effects in section 2, our goal is to study to what class of uncertainties Lagrange stability is robust. We look for a as general as possible description which could encompass as many types of effects as possible. However, to remain simple, we shall focus our attention on qualitative more than quantitative results. The idea to get this characterization is to generalize to non linear systems what was proposed in the linear case in [14], namely the so called normalizing signal technique. It has been shown to be a very powerful concept and its ability to describe all the possible linear unmodelled effects has been established in [6,15,16].

The adaptive stabilizer will be proposed in section 3. Based on a Lyapunov design, it will allow us to stabilize a larger class of ideal systems than the one considered in [12], namely those which are stabilizable by a state feedback such that some particular growth condition about the non linearities are satisfied. Moreover, for the purpose of robustness, specific modifications will be introduced – parameter update projection and signal normalization – .

To remain as simple as possible in this preliminary study, we have considered only the global Lagrange stability case. This will be the reason for some over restrictive assumptions.

Due to space limitations, the proofs will be omitted. They can be found in [7].

2 Unmodelled effects

Let the system to be controlled admit a finite state representation on $\mathbf{R}^N$ and its dynamics, may be augmented by input and output filters, be described globally by :

$$\dot{X} = F(X,t,u) , \quad x = H(X,t) \tag{1}$$

where the vector X is the state in $\mathbf{R}^N$ which is not measured and the dimension N is unknown, u is the input vector in $\mathbf{R}^m$, x is a measured output in $\mathbf{R}^n$ and, finally, F and H are C^1 unknown functions with $\frac{\partial H}{\partial x}(X,t)$, $\frac{\partial H}{\partial t}(X,t)$ and $F(X,t,u)$ bounded for all (X,u) in compact sets and $t \geq 0$. We assume also :

Assumption BO (Boundedness Observability) (2)
For all compact subsets $\mathcal{K}_x$ in $\mathbf{R}^n$ and $\mathcal{K}_u$ in $\mathbf{R}^m$ and for all initial condition $X(0)$ in $\mathbf{R}^N$, we can find a compact subset $\mathcal{K}_X$ in $\mathbf{R}^N$ such that, for the corresponding solution $X(t)$ of (1) defined on $[0,T)$,
$x(t) \in \mathcal{K}_x$ and $u(t) \in \mathcal{K}_u$ $\forall t \in [0,T)$ implies $X(\tau) \in \mathcal{K}_X$ $\forall \tau \in [0,T)$.

Namely to know that the trajectory $\{X(t)\}_{t\in[0,T)}$ is bounded, it is sufficient to observe that the trajectories $\{x(t)\}_{t\in[0,T)}$ and $\{u(t)\}_{t\in[0,T)}$ are bounded.

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Our problem is : *design a controller such that the solutions $X(t)$ of (1) are bounded and $\lim_{t\to+\infty} H(X(t),t) = \mathcal{E}$, a desired set point for the measurement x.*

Since the system to be controlled is only partially known, we shall work from a reduced order model whose state is x. This is why it may be interesting to augment the dynamics of the system to be controlled by filters (see [17, Example (24)]). The dynamics of this model are chosen as being described by an equation involving an unknown constant parameter vector $p^\star$:

$$\dot{x} = a(x,u) + A(x,u)\,p^\star \tag{3}$$

where the functions a and A are known and continuously differentiable, and $p^\star$ is an unknown parameter vector in a known compact convex subset Π of $\mathbf{R}^l$. This model is said to be linearly parameterized in explicit form. This is more restrictive than the case of linear parameterization in implicit form :

$$\left(b(x) + B(x)\,p^\star\right)\dot{x} = a(x,u) + A(x,u)\,p^\star \tag{4}$$

as obtained with manipulators and considered in [1, 2,18]. Our model is supposed to be stabilizable for all p :

Assumption S (Stabilizability) (5)
There exist three known functions u_n, Υ and V such that : 1 - $u_n : \mathbf{R}^n \times \Pi \to \mathbf{R}^m$ *is of class* C^1,
2 - Υ *is positive, of class* C^1, $\Upsilon(v) = 0$ *iff* $v = 0$ *and* $\liminf_{v\to+\infty} \Upsilon(v) > 0$, 3 - V *is of class* C^2, *positive,* $V(x,p) = 0$ *iff* $x = \mathcal{E}$ *and, for any positive real number* K_v, *the set :* $\{x \mid V(x,p) \leq K_v\ ,\ p \in \Pi\}$ *is a compact subset of* $\mathbf{R}^n$, 4 - *for all* (x,p) *in* $\mathbf{R}^n \times \Pi$, *we have :*

$$\frac{\partial V}{\partial x}\left[a(\cdot,u_n) + A(\cdot,u_n)\,p\right] \leq -\Upsilon(V) \tag{6}$$

Namely, for any p in Π, $\mathcal{E}$ is a globally asymptotically stable equilibrium point of the system :

$$\dot{x} = a(x,u_n(x,p)) + A(x,u_n(x,p))\,p \tag{7}$$

and V is a corresponding Lyapunov function for this closed loop model with time derivative $-\Upsilon(V)$.

Knowing that the model can be stabilized whatever the (constant) value of the parameter vector is, we need now to characterize the discrepancy between our model (3) and the actual system (1). For this, we need to choose two strictly positive real numbers α and r_0 and a strictly increasing C^1 and convex function $\Psi : \mathbf{R}_+ \to \mathbf{R}_+$, with $\Psi(0) = 0$. Then, we assume :

Assumption UEC(α, r_0, Ψ) (Unmodelled Effects Characterization) (8)
There exist an open subset $\mathcal{X}$ of $\mathbf{R}^N$, with $H(\mathcal{X},0) = \mathbf{R}^n$, and positive real numbers μ_1, μ_2, D and Σ such that, for any C^1 time function $\widehat{p} : \mathbf{R}_+ \to \Pi$ and any solution $X(t)$ of :

$$\dot{X} = F\left(X\,,\,t\,,\,u_n(x,\widehat{p})\right)\ ,\ x = H(X,t)\ ,\ X(0) \in \mathcal{X} \tag{9}$$

defined on $[0,T)$, there exists a C^1 time function $p^\star : [0,T) \to \Pi$, with $\left\|\dot{p^\star}\right\| \leq \Sigma$, satisfying for all $t \in [0,T)$:

$$\begin{aligned}&\left|\tfrac{\partial V}{\partial x}(x,\widehat{p})\left[\dot{x} - a(x,u_n(x,\widehat{p})) - A(x,u_n(x,\widehat{p}))\,p^\star\right]\right| \\ &\qquad \leq \mu_1\,\Upsilon(V(x,\widehat{p})) + \mu_2\,\Psi^{-1}(r) + D\end{aligned} \tag{10}$$

where r, called the normalizing signal, is defined by :

$$\dot{r} = -\alpha\left(r - \Psi(\Upsilon(V(x,\widehat{p})))\right)\ ,\quad r(0) = r_0 \tag{11}$$

Note that $p^\star$ is allowed to be time dependent and to depend on $\widehat{p}$. Also, the condition $H(\mathcal{X},0) = \mathbf{R}^n$ means that we are looking for results which are global with respect to the model state initial condition $x(0)$.

This assumption is the unmodelled effects characterization we mentioned in Introduction. This characterization is in some sense a closed loop one. Instead of asking for inequality (10) to hold for all possible input function u which would be a very stringent requirement, we need only it holds for the particular class of input functions $u_n(\cdot,\widehat{p})$, among which will be the one actually used. However, one open loop aspect remains, since not knowing a priori what will be the time function $\widehat{p}$, we are led to ask for (10) to hold for all possible time function $\widehat{p}$. Also, the closed loop model Lyapunov function V is involved in inequality (10). One way to understand this is : the control law u_n should be designed in such a way that the corresponding V satisfies (10), i.e. the unmodelled effects should be taken into account in the control design.

To help the designer in choosing the constant α and the function Υ which are involved in assumption UEC(α, r_0, Ψ) (8) and will be explicitely used in the controller we shall propose in section 3.3, we have the following Lemma whose proof is straightforward from the arguments of [4, Chapter 3] :

Lemma 1 : *Let $v(t)$ be a C^1 function defined on $[0,T)$. We have the following properties :*
1 - If Ψ is a strictly increasing convex function with $\Psi(0) = 0$ then $\frac{\Psi(x)}{x}$ (resp. $\frac{\Psi^{-1}(x)}{x}$) is non decreasing (resp. increasing) and, for all positive x, y and $k \geq 1$,

$$\begin{gathered}\Psi^{-1}(kx) \leq k\,\Psi^{-1}(x) \\ \Psi^{-1}(x+y) \leq \Psi^{-1}(x) + \Psi^{-1}(y)\end{gathered} \tag{12}$$

2 - Let r_1 and r_2 be positive and such that :

$$\dot{r}_1 \leq -\alpha_1\left(r_1 - \Psi_1(v)\right)\ ,\ \dot{r}_2 = -\alpha_2\left(r_2 - \Psi_2(v)\right) \tag{13}$$

where $\alpha_1 \geq \alpha_2 > 0$ are constant and Ψ_1, Ψ_2 are strictly increasing functions with $\Psi_1(0) = \Psi_2(0) = 0$ and Ψ_2 and $\Psi_2\Psi_1^{-1}$ are convex. For all $t \in [0,T)$, we have :

$$\begin{aligned}&\Psi_1^{-1}(r_1(t)) \leq \tfrac{\alpha_1}{\alpha_2}\,\Psi_2^{-1}(r_2(t)) \\ &+ \Psi_2^{-1}\left(\mathrm{Max}\left\{0\,,\,\left(\Psi_2\Psi_1^{-1}(r_1(0)) - \tfrac{\alpha_1}{\alpha_2}r_2(0)\right)e^{-\alpha_1 t}\right\}\right)\end{aligned} \tag{14}$$

3 - Let r_3 and r_4 be positive and such that :

$$\dot{r}_3 \leq -\alpha r_3 + \beta\Psi(v) + \gamma\ ,\ \dot{r}_4 = -\alpha\left(r_4 - \Psi(v)\right) \tag{15}$$

where $\alpha > 0$, $\beta > 0$ and $\gamma \geq 0$ are constant and Ψ is a strictly increasing and convex function with $\Psi(0) = 0$. For all $t \in [0,T)$, we have :

$$\begin{aligned}&\Psi^{-1}(r_3(t)) \leq \mathit{Max}\left\{1, \tfrac{\beta}{\alpha}\right\}\Psi^{-1}(r_4(t)) \\ &\quad + \Psi^{-1}\left(\mathrm{Max}\left\{0\,,\,\tfrac{\gamma + (\alpha r_3(0) - \gamma - \beta r_4(0))e^{-\alpha t}}{\alpha}\right\}\right)\end{aligned} \tag{16}$$

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Inequality (10) in assumption UEC(α, r_0, Ψ) (8) captures – at least qualitatively – a wide variety of unmodelled effects. In order to help the reader to get some grip on this assumption, we propose the following example :

Example 1 (Singular perturbations) : Let the state $X = \binom{x}{z}$ of the system to be controlled satisfy the following equation :

$$\begin{cases} \dot{x} = f_{11}(x,u) + f_{12}(x,u)\, z \\ \varepsilon \dot{z} = f_{21}(x) + f_{22}(x)\, z \end{cases} \tag{17}$$

where ε is a small positive real number, $x \in \mathbf{R}^n$ is measured, $z \in \mathbf{R}^{N-n}$ is not and the functions f_{11}, f_{12}, f_{21} and f_{22} are partially known (see (21) below) and continuously differentiable on $\mathbf{R}^n \times \mathbf{R}^m$.

Assumption BO (2) holds if : *There exist a positive definite symmetric matrix P and a strictly positive real number λ such that, for all $x \in \mathbf{R}^n$:*

$$P f_{22}(x) + f_{22}(x)^\top P \leq -\lambda I_{N-n} \tag{18}$$

Indeed, the time derivative of $V_1 = z^\top P z$ along the solutions of (17), when they exist, satisfies :

$$\varepsilon \dot{V}_1 \leq -\frac{\lambda}{2\lambda_{\max}(P)} V_1 + \frac{2\lambda_{\max}(P)^2}{\lambda} \|f_{21}(x)\|^2 \tag{19}$$

The conclusion follows from continuity of f_{21}.

The reduced order model used for the control design is obtained by setting $\varepsilon = 0$, i.e. :

$$\dot{x} = a(x,u) + A(x,u)\, p^\star \tag{20}$$

where, precising the a priori knowledge on f_{ij}, the known functions a and A and the unknown vector $p^\star$ satisfy :

$$\begin{aligned} a(x,u) + A(x,u)\, p^\star &\stackrel{\text{def}}{=} \\ & f_{11}(x,u) - f_{12}(x,u) f_{22}(x)^{-1} f_{21}(x) \end{aligned} \tag{21}$$

For this model, we assume the existence of u_n, Υ and V such that assumption S (5) holds. For example, this model could be globally feedback linearizable as assumed – locally – by Taylor et al. in [19] and Kanellakopoulos et al. in [8].

Let us show now that assumption UEC(α, r_0, Ψ) (8) is satisfied if : *there exist positive real numbers k_1, D_1 and $\beta \in (0,2]$ such that, for all $(x, \widehat{p}, \eta) \in \mathbf{R}^n \times \Pi \times \mathbf{R}^{N-n}$,*

$$\begin{aligned} \left\| \tfrac{\partial h}{\partial x} \left[a(\cdot,u_n) + A(\cdot,u_n)p^\star\right] \right\|^2 &\leq k_1 \Upsilon(V)^\beta + D_1 \\ \left\| \tfrac{\partial h}{\partial x} f_{12}(\cdot,u_n)\eta \right\|^2 &\leq k_1 \left[\Upsilon(V)^\beta + \|\eta\|^2\right] + D_1 \\ \left\| \tfrac{\partial V}{\partial x} f_{12}(\cdot,u_n) \right\| &\leq k_1 \Upsilon(V)^{1-\beta/2} + D_1 \end{aligned} \tag{22}$$

where all the functions are evaluated at $(x, \widehat{p})$ and :

$$h(x) \stackrel{\text{def}}{=} f_{22}(x)^{-1} f_{21}(x) \tag{23}$$

Indeed, as in [19], we define a new variable η :

$$\sqrt{\varepsilon}\, \eta = z + h(x) \tag{24}$$

In the coordinates (x, η), the system is:

$$\begin{aligned} \dot{x} &= a(x,u) + A(x,u)\, p^\star + \sqrt{\varepsilon}\, f_{12}(x,u)\, \eta \\ \varepsilon \dot{\eta} &= f_{22}(x)\, \eta + \sqrt{\varepsilon}\, h' \\ h' &\stackrel{\text{def}}{=} \tfrac{\partial h}{\partial x}(x) \left[a(x,u) + A(x,u)p^\star + \sqrt{\varepsilon} f_{12}(x,u)\eta\right] \end{aligned} \tag{25}$$

Then let $r_1 \stackrel{\text{def}}{=} (\eta^\top P \eta)^\gamma / \gamma$ with $\gamma \geq \frac{1}{\beta}$. With our assumption and Young's inequality, the time derivative of r_1, along the solutions of this system with $u = u_n(x, \widehat{p})$, satisfies :

$$\dot{r}_1 \leq -\frac{c_1}{\varepsilon} r_1 + \frac{c_2}{\sqrt{\varepsilon}} \Upsilon^{\beta\gamma} + D_2 \tag{26}$$

where $c_1 > 0$, $c_2 > 0$ and $D_2 \geq 0$ are real numbers. With point 2 of Lemma 1 and ε sufficiently small, this leads us to choose α and Ψ as :

$$\frac{c_2}{\sqrt{\varepsilon}} \leq \alpha \leq \frac{c_1}{\varepsilon} \quad \text{and} \quad \Psi(\Upsilon) = \Upsilon^{\beta\gamma} \tag{27}$$

Then, defining r by :

$$\dot{r} = -\alpha r + \alpha \Upsilon^{\beta\gamma}, \quad r(0) = r_0 > 0 \tag{28}$$

we get with point 3 of Lemma 1 :

$$\begin{aligned} r_1^{\frac{1}{\beta\gamma}} &\leq r^{\frac{1}{\beta\gamma}} \\ &+ \mathrm{Max}\left\{0, \tfrac{D_2}{\alpha} + \left(r_1(0) - \tfrac{D_2}{\alpha} - r_0\right) e^{-\alpha t}\right\}^{\frac{1}{\beta\gamma}} \end{aligned} \tag{29}$$

This together with (22) and Young's inequality, implies the existence of a positive real number c_3 such that :

$$\begin{aligned} &\left| \tfrac{\partial V}{\partial x}(x,\widehat{p}) \sqrt{\varepsilon} f_{12}(x, u_n(x,\widehat{p}))\eta \right| \\ &\leq c_3\sqrt{\varepsilon}\left[\Upsilon + r^{\frac{1}{\beta\gamma}} + \left(D_1 + \left(\tfrac{D_2\sqrt{\varepsilon}}{c_2}\right)^{\frac{1}{\beta\gamma}}\right)\right] \\ &+ c_3\sqrt{\varepsilon}\, \mathrm{Max}\left\{0, \left(\tfrac{\lambda_{\max}(P)^\gamma \|\eta(0)\|^{2\gamma}}{\gamma} - r_0\right) e^{-\alpha t}\right\}^{\frac{1}{\beta\gamma}} \end{aligned} \tag{30}$$

This is (10) in assumption UEC(α, r_0, Ψ) (8) with the last line defining the set $\mathcal{X}$. Precisely, in this line, we notice the presence of the unmodelled dynamics initial condition $\eta(0)$ but multiplied by $\sqrt{\varepsilon}$ times an exponentially decaying term. In the linear case, if the controller has a guaranteed bounded gain, there cannot be any finite escape time. In such a case it would be sufficient to take $\mathcal{X} = \mathbf{R}^N$. But in the non linear case, finite escape time is possible. So we let :

$$\mathcal{X} = \left\{ (x,\eta) \;\middle|\; \|\eta\|^{2\gamma} < \frac{\gamma}{\lambda_{\max}(P)^\gamma} r_0 \right\} \tag{31}$$

However, the consequent restriction on the initial condition $X(0)$ in UEC(α, r_0, Ψ) (8) can be omitted when the initial condition r_0 of the normalizing signal is arbitrary as it is the case when this signal is not used in the controller – it is then completely artificial – .

We remark also :

1 – α should be chosen depending on ε. Precisely, according to (27), α must be large when ε is small. This implies some knowledge on ε. In the case of linear systems, this strong requirement can be overcome by input filtering as shown by Ioannou and Tsakalis in

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[6].
2 – $\dot{z}$ in (17) does not depend on u. If u were present, we would have difficulties since a term $\dot{u}$ would appear in the $\dot{\eta}$ equation in (25). With $u = u_n(x, \widehat{p})$, this would imply the presence of a term $\dot{\widehat{p}}$ which is not bounded for all $((x,z),\widehat{p}) \in \mathbf{R}^N \times \Pi$. This differs from the local stability analysis of [2,8,19]. In any case, one way to make sure that the input will not appear in the $\dot{z}$ equation is, as in point 1 above, to add integrators on the input (see [5,6]). We know from [20, Theorem 3.c] (see also [3]) that if a system is smoothly enough stabilizable, it is still stabilizable if we add integrators on the input. However, in the context of adaptive control, adding integrators may cause problems, the parameter dependence of the closed loop system being reinforced. In particular for the case of manipulators as considered by Reed and Ioannou [18] and Campion and Bastin [2], we do not see how integrators could be added in the adaptive case due to the fact that the model is linearly parameterized only in the implicit form (4).
3 – Assumptions (18) and (22) are restrictive. As remarked in Introduction, this is a consequence of our will of establishing global Lagrange stability results – compare with the local analysis of [2,8,19]. ∎

3 Lagrange stability

Our objective is now to study if assumptions BO (2), S (5) and UEC(α, r_0, Ψ) (8) are sufficient to guarantee the existence of a controller solving our problem. We proceed by increasing order of difficulties.

3.1 $p^\star$ given and constant

When the vector $p^\star$ in (10) is given and constant, the value of $p^\star$ is available for computation. Therefore, we may propose the following control law :

$$u = u_n(x, p^\star) . \qquad (32)$$

We get :

Proposition 1 : *Let assumptions BO (2) and S (5) hold. Under these conditions if, for some α, r_0, Ψ, assumption UEC(α, r_0, Ψ) (8) holds with $p^\star$ known and constant, i.e. $\Sigma = 0$, and :*

$$1 - \mu_1 - 2\mu_2 - \limsup_{v \to +\infty} \frac{D}{\Upsilon(v)} > 0 \qquad (33)$$

then all the solutions of (1)-(32) are well defined on $[0, +\infty)$, unique and bounded. Moreover, if $D = 0$ in (10), then :

$$\lim_{t \to +\infty} x(t) = \mathcal{E} \qquad (34)$$

The proof of this Proposition is based on the following Lyapunov function :

$$\mathcal{V}(x, r) = I(V(x, p^\star)) + \frac{\varepsilon}{2} r^2 \qquad (35)$$

with $\varepsilon > 0$ and $I(V)$ the function defined by :

$$I(V) = \int_0^V \frac{\Psi(\Upsilon(v))^2}{\Upsilon(v)}\, dv \qquad (36)$$

Since Ψ, Υ, V and α are known, this function can be evaluated on line and therefore used in a Lyapunov design.

3.2 $p^\star$ unknown and time varying, V does not depend on p

When the vector $p^\star$ is unknown and time varying, the control law (32) cannot be implemented. Instead we use a dynamic controller with state $\widehat{p}$:

$$\dot{\widehat{p}} = \mathcal{F}(x, \widehat{p}) , \quad u = u_n(x, \widehat{p}) \qquad (37)$$

which is obtained by applying Parks' Lyapunov design [11]. We just mentioned that $\mathcal{V}$ defined in (35) is an appropriate Lyapunov function for the case where $p^\star$ is known. Since the model equation is affine in the parameter vector, we may try the following as a control Lyapunov function :

$$W(x, r, \widehat{p}) = I(V(x, \widehat{p})) + \frac{\varepsilon}{2} r^2 + \frac{\gamma}{2} \|\widehat{p} - p^\star\|^2 \quad (38)$$

Namely, let us design the function $\mathcal{F}$ in (37) so that the time derivative of W along the solutions of the closed loop model (3)-(11)-(37) – which is not (1)-(37) – be negative. Assuming that such solutions exist, we get with (11) in assumption S (5) :

$$\begin{aligned} \dot{W} \leq & -\Psi^2 - \varepsilon \alpha r^2 + \varepsilon \alpha r \Psi \\ & + \left(\gamma \dot{\widehat{p}}^{\mathsf{T}} - \frac{\Psi^2}{\Upsilon} \frac{\partial V}{\partial x} A\right)(\widehat{p} - p^\star) + \frac{\Psi^2}{\Upsilon} \frac{\partial V}{\partial p} \dot{\widehat{p}} \end{aligned} \qquad (39)$$

Hence, in the case where V does not depend on p, this leads us to choose :

$$\mathcal{F}(x, \widehat{p}) = \frac{\Psi(\Upsilon(V(x)))^2}{\gamma\, \Upsilon(V(x))} A(x, u_n(x, \widehat{p}))^{\mathsf{T}} \frac{\partial V}{\partial x}(x)^{\mathsf{T}} \quad (40)$$

Note that, as a consequence, the values of ε and r are not needed to implement the controller. Moreover, we know that $p^\star$ is in the known convex compact subset Π. We use this a priori knowledge by projecting $\mathcal{F}$ onto the boundary of Π whenever $\widehat{p}$ is on this boundary and $\mathcal{F}$ is pointing outside Π. The following controller follows :

$$\begin{aligned} & \dot{\widehat{p}} = \mathrm{Proj}\left(\widehat{p}, \frac{\Psi(\Upsilon(V(x)))^2}{\gamma \Upsilon(V(x))} A(x, u_n(x, \widehat{p}))^{\mathsf{T}} \frac{\partial V}{\partial x}(x)^{\mathsf{T}}\right) \\ & u = u_n(x, \widehat{p}) \end{aligned} \qquad (41)$$

with $\widehat{p}(0) \in \overset{\circ}{\Pi}$ and, with some extra but weak restrictions on the set Π, the function Proj can be made locally Lipschitz continuous and have the following property (see [17] or [12]) :

$$\begin{aligned} (p - p^\star)^{\mathsf{T}}\, \mathrm{Proj}\,(p, y) &\leq (p - p^\star)^{\mathsf{T}}\, y \\ \|\mathrm{Proj}\,(p, y)\| &\leq \|y\| \end{aligned} \qquad (42)$$

for all $(p, p^\star, y)$ in $\mathbf{R}^l \times \overset{\circ}{\Pi} \times \mathbf{R}^l$. It remains to study the properties this controller provides to the actual closed loop system (1)-(41). We have :

Proposition 2 : *Let assumptions BO (2) and S (5) hold with V independent of p. Under these conditions if, for some α, r_0, Ψ, assumption UEC(α, r_0, Ψ) (8) is satisfied with :*

$$1 - \mu_1 - 2\mu_2 - \limsup_{v \to +\infty} \frac{D}{\Upsilon(v)} > 0 \qquad (43)$$

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and if γ is chosen sufficiently small so that :

$$\gamma \Sigma \sup_{(p_1,p_2)\in\Pi^2} \|p_1 - p_2\| < \left(1 - \mu_1 - 2\mu_2 - \limsup_{v\to+\infty} \frac{D}{\Upsilon(v)}\right) \liminf_{v\to+\infty} \Psi(\Upsilon(v))^2 \quad (44)$$

then all the solutions of (1)-(41) are well defined on $[0, +\infty)$, *unique and bounded. Moreover, if* $D = 0$ *and* $\Sigma = 0$, *then :*

$$\lim_{t\to+\infty} x(t) = \mathcal{E} \quad (45)$$

We remark :

1 – Inequality (44) implies that the larger the speed Σ of the unknown vector $p^\star$ or the larger the parametric uncertainty $\sup_{(p_1,p_2)\in\Pi^2} \|p_1 - p_2\|$ the faster the adaptation should be.

2 – Proposition 2 confirmes one of the conclusion which can be drawn from the work of Reed and Ioannou [18] and Campion and Bastin [2] for manipulators. In the case where we can choose V independant of the updated parameter vector $\widehat{p}$, the only modification which is needed compared with the known parameter vector case is a mechanism guaranteeing boundedness of the updated parameter vector $\widehat{p}$. Instead of the projection used here and in [2], Reed and Ioannou proposed the so called σ-modification.

3 – The normalizing signal r is not explicitely used in the controller (41). This implies that for the systems studied in Example 1, we have a global Lagrange stability.

4 – In the case where singular perturbations are present, only a local result is obtained in [2] and [18]. This follows from the fact that, in these two cases, the control appears in the fast subsystem (see Remark 2 of Example 1).

5 – Robustness of Lagrange stability has also been established locally by Taylor et al. [19] for feedback linearizable models with a parameter independent linearizing diffeomorphism. This independance implies our parameter independent V assumption. There is however a possibility to extend those results to the case where V depends on the updated parameter vector if a so called matching condition is satisfied (see [8,17]). Indeed, when $\frac{\partial V}{\partial p} \not\equiv 0$ but this condition holds, it is possible by augmenting the control in :

$$u = u_n(x, \widehat{p}) + v(x, \widehat{p}, \dot{\widehat{p}}) \quad (46)$$

to annihilate by v the term $\frac{\Psi^2}{\Upsilon}\frac{\partial V}{\partial p}\dot{\widehat{p}}$ in (39). Unfortunately in this case assumption UEC(α, r_0, Ψ) (8) is not sufficient since u is no more in the class $u_n(\cdot, \widehat{p})$. We need to make UEC(α, r_0, Ψ) (8) more restrictive by replacing :

for any C^1 time function $\widehat{p} : \mathbb{R}_+ \to \Pi$ and any solution $X(t)$ of :

$$\dot{X} = F(X, t, u_n(H(X,t), \widehat{p})) \ , \quad X(0) \in \mathcal{X} \quad (47)$$

by : *for any C^1 time function $\widehat{p} : \mathbb{R}_+ \to \Pi$ and $u : \mathbb{R}_+ \to \mathbb{R}^m$ and any solution $X(t)$ of :*

$$\dot{X} = F(X, t, u(t)) \ , \quad X(0) \in \mathcal{X} \quad (48)$$

3.3 $p^\star$ unknown but constant, V depends on p

As mentioned above, when V depends on p we have the extra term $\frac{\Psi^2}{\Upsilon}\frac{\partial V}{\partial p}\dot{\widehat{p}}$ in (39). If such a term cannot be annihilated via the control, we have to consider it as a disturbance and to design a controller which will guarantee robustness of Lagrange stability with respect to it. For this design, we propose to replace the control Lyapunov function W in (38) by :

$$W(x, r, \widehat{p}) = L\left[I(V(x, \widehat{p})) + \tfrac{\varepsilon}{2} r^2\right] + \frac{\gamma}{2}\|\widehat{p} - p^\star\|^2 \quad (49)$$

where the function L is to be designed. For this new function W, the same Lyapunov design as in section 3.2 – without projection – leads to the following inequality, replacing (39) :

$$\dot{W} \le -\left[\Psi^2 + \varepsilon\alpha r^2 - \varepsilon\alpha r\Psi - \left(\frac{\Psi^2}{\Upsilon}\right)^2 \frac{\partial V}{\partial p} A^\top \frac{\partial V}{\partial x}^\top L'\right] L' \quad (50)$$

where L' is the derivative of L. We conclude that this derivative should be positive but as small as possible while guaranteeing radial unboundedness and positive definiteness of L. This leads us to choose $L(x) = \log(1 + x)$ and to propose the following controller :

$$\begin{aligned} \dot{r} &= -\alpha\left(r - \Psi(\Upsilon(V(x, \widehat{p})))\right) \ , \quad r(0) = r_0 \\ \dot{\widehat{p}} &= \mathrm{Proj}\left(\widehat{p}, \frac{\frac{\Psi(\Upsilon(V(x,\widehat{p})))^2}{\Upsilon(V(x,\widehat{p}))} A(x, u_n(x,\widehat{p}))^\top \frac{\partial V}{\partial x}(x,\widehat{p})^\top}{\left(1 + I(V(x,\widehat{p})) + \frac{\varepsilon}{2} r^2\right)\gamma}\right) \quad (51) \\ u &= u_n(x, \widehat{p}) \end{aligned}$$

with $\widehat{p}(0) \in \overset{\circ}{\Pi}$ and I defined in (36). We have :

Proposition 3 : *Let assumptions BO (2) and S (5) hold with, for all $(x, p) \in \mathbb{R}^n \times \Pi$:*

$$\left\|\frac{\partial V}{\partial p}\right\| \left\|\frac{\partial V}{\partial x} A(\cdot, u_n)\right\| \le d\left(1 + \int_0^V \Upsilon(v)dv\right) \quad (52)$$

where d is a positive real number. We choose α, r_0, Ψ, ε and γ such that :

1 – $\frac{\Psi(\Upsilon(v))}{\Upsilon(v)(\int_0^v \Upsilon(t)dt)^k}$ is non increasing for $v \ge 0$, with k some positive real number,

$$2 - \qquad \alpha\varepsilon < 2(1 - \xi) \quad (53)$$

with v_0 defined by $\int_0^{v_0} \Upsilon(v)dv = 1$ and :

$$\xi \overset{\mathrm{def}}{=} \frac{2d}{\gamma} Max\left\{(1 + 2k), \frac{\Psi(\Upsilon(v_0))^2}{\Upsilon(v_0)^2}\right\} \quad (54)$$

Under these conditions if UEC(α, r_0, Ψ) (8) holds with the above given α, r_0, Ψ and moreover μ_1, μ_2, D, Σ satisfy $\Sigma = 0$ and :

$$(\alpha\varepsilon + \mu_2)^2 < 2\alpha\varepsilon\left(1 - \mu_1 - \mu_2 - \xi - \limsup_{v\to+\infty} \frac{D}{\Upsilon(v)}\right) \quad (55)$$

then all the solutions of (1)-(51) are well defined on $[0, +\infty)$, unique and bounded. Moreover, if $D = 0$, then :

$$\lim_{t\to+\infty} x(t) = \mathcal{E} \quad (56)$$

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We remark :

1 – In this case where V depends on the updated parameter vector $\widehat{p}$, together with the parameter update projection another modification is used : a normalization. Namely, compared with (41), we have introduced, in (51), the denominator $(1+I(V)+\frac{\varepsilon}{2}r^2)$. But consequently, the normalizing signal r appears explicitely in the controller. This implies in particular that the initial condition r_0 is no more the free parameter we can use to prove globality of the Lagrange stability. This is opposite to the case of Proposition 2 (see remark 3 following Proposition 2).

2 – Inequality (52) generalizes the growth condition introduced in [12] for the case $\Upsilon(v) = v$.

3 – Monotonicity of $\frac{\Psi(\Upsilon(v))}{\Upsilon(v)(\int_0^v \Upsilon(t)dt)^k}$ is a weak (technical) growth condition on the functions Ψ and Υ. For instance, it is satisfied when these functions are polynomials.

4 – In contrast with remark 1 following Proposition 2, γ should be large enough for (53) to hold. This is the well known robustness versus fast adaptation tradeoff.

5 – All our assumptions are satisfied if the system to be controlled is linear, V is quadratic in x, u_n is linear in x, $\Psi(\Upsilon) = \Upsilon$ and $\Upsilon(v) = c\,v$. In this case (51) is a new – as far as we now – robust adaptive linear controller which does not require any augmented error.

References

[1] Campion G., Bastin G. : *Lyapunov design of an adaptive external linearization feedback control for manipulators.* in : A. Bensoussan and J. Lions, Ed., *Analysis and optimization of systems.* Springer Verlag, 1986, pp. 172-187.

[2] Campion G., Bastin G. : *Analysis of an adaptive controller for manipulators : Robustness versus flexibility.* Systems & Control Letters 12 (1989) 251-258.

[3] Coron J.-M., Praly L. : *Adding an integrator for the stabilization problem.* Submitted for publication in Systems & Control Letters. October 1990.

[4] Hardy G., Littlewood J.E., Pólya G. : *Inequalities.* Second Edition. Cambridge Mathematical Library, 1989.

[5] Ioannou P.A., Kokotovic P.V. : *Adaptive systems with reduced order models.* Lecture Notes in Control and Information Sciences 47. Springer Verlag, 1983.

[6] Ioannou P., Tsakalis K. : *The class of unmodelled dynamics in robust adaptive control.* Proc. 1988 Amer. Contr. Conf., 1988, pp. 337-342.

[7] Jiang Z.-P. and Praly L. : *Addendum to : Adaptive non linear stabilization and robust Lagrange stability.* Internal Report. Centre Automatique et Systèmes, 35 rue Saint Honoré, 77305 Fontainebleau Cedex, France. January 1991.

[8] Kanellakopoulos I., Kokotovic P.V., Marino R. : *Robustness of Adaptive Nonlinear Control under an Extended Matching Condition.* IFAC symposium on Nonlinear Syst. Design, Capri, Italy, 1989.

[9] Kanellakopoulos I., Kokotovic P.V., Morse A.S. : *Adaptive output-feedback control of systems with output non linearities.* To appear in *Foundations of Adaptive Control,* P. V. Kokotovic, Ed., Springer-Verlag, Berlin, 1991.

[10] Kanellakopoulos I., Kokotovic P.V., Morse A.S. : *Systematic design of adaptive controllers for feedback linearizable systems* Technical Report UILU-ENG-90-2249, DC-124, University of Illinois at Urbana-Champaign, October 1990. Submitted for publication in IEEE Transactions on Automatic Control.

[11] Parks P. C. : *Lyapunov Redesign of Model Reference Adaptive Control Systems* I.E.E.E. Transactions on Automatic Control 11 No 3, pp 362-367, 1966.

[12] Pomet J.-B., Praly L. : *Adaptive non-linear stabilization: Estimation from the Lyapunov equation.* To appear in IEEE Transactions in Automatic Control. See also Proc. of 28th IEEE Conference on Decision and Control, Tampa, 1989.

[13] Popov V.M.: *Hyperstability of control systems.* Springer Verlag. 1973.

[14] Praly L. : *Commande adaptative indirecte multivariable: stabilité et robustesse.* Colloque national du CNRS. Belle-Ile. Septembre 1982.

[15] Praly L. : *Commande linéaire adaptative : Solutions bornées et leurs propriétés.* Docteur en Mathématiques et Automatique. Université de Paris IX Dauphine. UER Mathématiques de la Décision. Octobre 1988.

[16] Praly L. : *Almost exact modelling assumption in adaptive linear control.* Int. J. Control, 1990, VOL. 51, NO. 3, 643-668

[17] Praly L., Bastin G., Pomet J.-B., Jiang Z.P. : *Adaptive stabilization of non linear systems.* to appear in *Foundations of Adaptive Control,* Kokotovic P.V., Ed., Springer-Verlag, Berlin, 1991.

[18] Reed J.S., Ioannou P.A. : *Instability analysis and robust adaptive control of robotic manipulators.* I.E.E.E. Transactions on Robotics and Automation 5 No 3, June 1989.

[19] Taylor D.G., Kokotovic P.V., Marino R., Kanellakopoulos I. : *Adaptive Regulation of Nonlinear Systems with Unmodeled Dynamics* . I.E.E.E. Transactions on Automatic Control 34 No 4, April 1989.

[20] Tsinias J. : *Sufficient Lyapunov-like conditions for stabilization.* Math. Control Signals Systems 2 (1989) 343-357.

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