Adaptive Control of Feedback Equivalent Systems

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Abstract *We address the problem of stabilizing a nonlinear system depending on some unknown parameters in such a way that all the systems obtained by varying these parameters are equivalent to one system, supposed to be state-feedback stabilizable. The description of the adaptation laws make use of passivity. We consider both the "general" case, and the case where some "matching assumptions" hold.*

1 Introduction

We consider a family of nonlinear affine-in-the-control systems, indexed by a parameter vector p:

$$p = (p_1 \dots p_l)^{\top} \in \mathbf{R}^l \ . \qquad (1)$$

The system $\mathcal{S}_p$ corresponding to a given value of p is described by :

$$\mathcal{S}_p \ : \ \dot{x} = f(p,x) + g(p,x)\,u \qquad (2)$$

$$\triangleq f(p,x) + \sum_{k=1}^{m} u^k g_k(p,x) \qquad (3)$$

where the state x lives in an n-dimensional C^∞ manifold M^n and is completely measured,

$$u = (u_1, \dots, u_m) \qquad (4)$$

is in $\mathbf{R}^m$, and f and the g_k's are **known** smooth vector fields smoothly depending on the parameter p; g is the matrix field defined by (3).

One particular $p^\star$ in $\mathbf{R}^l$ will be called the "true value of the parameter p", and our problem is to stabilize the system $\mathcal{S}_{p^\star}$, $p^\star$ being unknown. This will be done by means of a *dynamic* controller, i.e. of a system with a certain state, to be determined, input x, and output u :

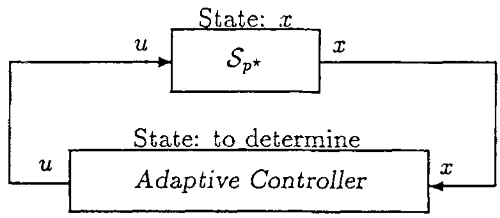

Fig. 1 : *Closed-loop system*

The state of the overall closed-loop system is composed of x and the dynamic variables (or the state) of the controller. By "stabilize $\mathcal{S}_{p^\star}$" we then mean that both x and the dynamic variables of the adaptive controller must be bounded, and x must tend to a certain point 0 of M^n, for all the solutions of the closed loop system (global properties), or only for some (local properties). By "$p^\star$ being unknown", we mean that the adaptive controller must not depend on, or use the value of, $p^\star$.

This general problem, as well as the distinction between global and local results, is extensively discussed in [10], [8], or [11], where a complete bibliography may be found. The present paper is specifically devoted to the case when all the systems

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$\mathcal{S}_p$ are equivalent to one another by state feedabck and diffeomorphism.

In section 2, we state precisely our assumptions about the systems $\mathcal{S}_p$. In section 3, we describe some general adaptation schemes. Section 4 presents some adaptive controllers designed for the general case where no more assumption is satisfied. Section 5 is devoted to the case where some "matching assumption" is satisfied and presents some modified controllers. Finally, in section 6 we give some geometric conditions allowing to construct a diffeomorphism meeting this "matching assumption".

2 Assumptions

Linear Parametrisation (LP) assumption: *The fields f and g in (2) depend linearly in the parameter p :*

$$f(p,x) = a^0(x) + \sum_{i=1}^{l} p_i\, a^i(x) \quad (5)$$

$$g_k(p,x) = b_k^0(x) + \sum_{i=1}^{l} p_i\, b_k^i(x) \quad , (6)$$

where the a^i's and the b_k^i's are smooth vector fields (independant of p).

Numerous practical examples, as position control of a DC-motor [6] or control of biochemical processes [4] satisfy this linear dependance on the parameters. All the existing schemes for adaptive nonlinear control require linear parametrization.

Feedback and Diffeomorphism Equivalence (FDE) assumption: *There exists three smooth maps, α from $\mathrm{R}^l \times M^n$ to R^m, β from $\mathrm{R}^l \times M^n$ to $M_{m\times m}(\mathrm{R})$, and φ from $\mathrm{R}^l \times M^n$ to M^n such that*

1. *For each p, $\varphi(p,.)$ is a diffeomorphism on M^n, and the "matrix field" $\beta(p,.)$ is such that*

$$\operatorname{rank} g(p,x)\beta(p,x) = \operatorname{rank} g(p,x) \quad (7)$$

2. *There is a system*

$$\dot{x} = \breve{f}(x) + \breve{g}(x)\, u \quad (8)$$

independant of p such that, for any p,

$$\xi = \varphi(p,x) \quad (9)$$

$$u = \alpha(p,x) + \beta(p,x)\, w \quad (10)$$

transforms (2) into

$$\dot{\xi} = \breve{f}(\xi) + \breve{g}(\xi)\, w \quad (11)$$

Local Stabilizability of the Transformed System (STS(Ω)) assumption : *There exist* **known** *functions v_{nom} and U, of class (at least) C^1 and C^2 respectively, from a neighborhood Ω of 0 in M^n to R^m and to R respectively, such that:*

1. *$U(\xi)$ is nonnegative, and zero if and only if ξ is zero, and for any $K > 0$,*

$$\{\, \xi \,/\, U(\xi) \leq K \,\} \quad (12)$$

is a bounded subset of M^n.

2. *For all ξ in M^n, we have:*

$$\dot{U}_{\dot{\xi}=\breve{s}(\xi)} = L_{\breve{s}} U(\xi) \leq -c\, U(\xi) \quad (13)$$

where c is a strictly positive constant and $\breve{s}$ denotes the "nominal transformed closed loop field":

$$\breve{s}(\xi) = \breve{f}(\xi) + \breve{f}(\xi)\, v_{\mathrm{nom}}(\xi) \quad (14)$$

Global Stabilizability of the Transformed System (global STS) assumption : *The same as STS(Ω), with $\Omega = M^n$.*

For the sake of simplicity, we suppose that φ, α and β are defined globally in assumption FDE. If they were defined only for p in a certain domain, but for any x, we could modify our algorithms, adding a projection to maintain the estimates of the parameter in this domain (see [10] or [11]). If they were defined locally with respect to x too, only local results could be obtained.

The most touched-on situation, in the present litterarure is concerned with fully state-feedback linearisable systems. Assumption FDE includes it as a particular case and is clearly more general. For linearisable systems, STS is satisfied too, v_{nom} being a stabilizing linear feedback.

Notice that with assumptions FDE and STS, defining u_{nom} by :

$$u_{\text{nom}}(p,x) = \alpha(p,x)+\beta(p,x)v_{\text{nom}}(\varphi(p,x)), \tag{15}$$

for any p the feedback $u = u_{\text{nom}}(p,x)$ stabilises $\mathcal{S}_p$. In particular, $u = u_{\text{nom}}(p^\star,x)$ stabilises $\mathcal{S}_{p^\star}$. This is not however a solution to our problem, because $p^\star$ is unknown, and the controller must not depend on its value.

3 Adaptation laws

From the f and g in (2) and (3), we define a new family $(\mathcal{S}'_{p,q})$, in which, for any p and q, the system $\mathcal{S}'_{p,q}$ is :

$$\mathcal{S}'_{p,q} \ : \ \dot{x} = f(p,x) + g(p,x)u_1 + g(q,x)u_2 \tag{16}$$

Remark 1 : In general, $q = (q^1,\dots,q^l)$ will have the same dimension as p, but we may omit the q^i's such that the b_k^i's (see (6)) are identically zero, i.e. such that $g(q,x)$ does not depend on q^i. In particular, if g does not depend at all on the parameters, q may be dropped in the parametrization (16). This is also the case if for some reason w_2 is identically zero. ∎

Setting

$$u = u_1 + u_2 \ , \tag{17}$$

the family $(\mathcal{S}_p)$ is imbedded in the family $(\mathcal{S}'_{p,q})$ in an obvious way :

$$\mathcal{S}_p \equiv \mathcal{S}'_{p,p} \ . \tag{18}$$

We may consider that the system $\mathcal{S}_{p^\star}$ to be controlled is just $\mathcal{S}'_{p^\star,q^\star}$, with $q^\star = p^\star$. In the following, it will often be convenient to write $p^\star$ and $q^\star$, keeping in mind that $q^\star = p^\star$.

The controls u_1 and u_2 will be computed thanks to some "estimates" of the parameters : u_1 will depend on $\hat{p}$, and u_2 on $\hat{p}$ and $\hat{q}$, some "estimates" of $p^\star$ and $q^\star$.

The adaptation is the part of the controller which gives these estimates $\hat{p}$ and $\hat{q}$. The following remarks are some guides to design the adaptation.

Considering φ, α and β given by assumption FDE, if we define ξ and w_1 by

$$\xi = \varphi(\hat{p},x) \tag{19}$$
$$u_1 = \alpha(\hat{p},x) + \beta(\hat{p},x)\,w_1 \ , \tag{20}$$

we have, from assumptions FDE and LP,

$$\dot{\xi} = \breve{f}(\xi) + \breve{g}(\xi)w_1 + A(\hat{p},x,w_1)(p^\star - \hat{p}) + \frac{\partial\varphi}{\partial x}(\hat{p},x)g(q^\star,x)u_2 + \frac{\partial\varphi}{\partial p}(\hat{p},x)\dot{\hat{p}} \tag{21}$$

with

$$A(p,x,u) = \left(\frac{\partial\varphi}{\partial x}(\hat{p},x)\left[a^1(x) + \sum_{k=1}^{m} u^k\, b_k^1(x)\right],\dots \right. \left. \dots,\ \frac{\partial\varphi}{\partial x}(\hat{p},x)\left[a^l(x) + \sum_{k=1}^{m} u^k\, b_k^l(x)\right]\right), \tag{22}$$

where the u_1^k's are the functions of $\hat{p}$, x and w_1 given by (20).

The adaptation will be based on linear estimation, and must rely on an "observation equation", i.e. an equation, satisfied by the signals present in the system, which is linear with respect to $(p^\star, q^\star)$. If h is any function of ξ, we may choose as an observation equation the one describing the evolution of $h(\xi(t))$, namely :

$$\begin{aligned} \dot{h} \;=\; & \frac{\partial h}{\partial \xi}(\xi)\left[\breve{f}(\xi) + \breve{g}(\xi)\, w_1\right] \\ & + \frac{\partial h}{\partial \xi}(\xi) A(\hat{p}, x, w_1)\,(p^\star - \hat{p}) \\ & + \frac{\partial h}{\partial \xi}(\xi)\frac{\partial \varphi}{\partial x}(\hat{p}, x)\, g(q^\star, x)\, u_2 \\ & + \frac{\partial h}{\partial \xi}(\xi)\frac{\partial \varphi}{\partial p}(\hat{p}, x)\, \dot{\hat{p}}. \end{aligned} \tag{23}$$

To stress the fact that this equation is linear with respect to $p^\star$ and $q^\star$, we may rewrite it as :

$$z(\dot{h}, \dot{\hat{p}}, \hat{p}, \hat{q}, x, u) \;=\; \mathcal{Z}(\dot{\hat{p}}, \hat{p}, \hat{q}, x, u)\begin{pmatrix} p^\star \\ q^\star \end{pmatrix} \tag{24}$$

where

$$\begin{aligned} z \;=\; & \dot{h} - \frac{\partial h}{\partial \xi}(\xi)\left[\breve{f}(\xi) + \breve{g}(\xi) w_1\right] \\ & + \frac{\partial h}{\partial \xi}(\xi) A(\hat{p}, x, w_1)\,\hat{p} \\ & - \frac{\partial h}{\partial \xi}(\xi)\frac{\partial \varphi}{\partial x}(\hat{p}, x)\sum_{k=1}^{m} u_2^k\, b_o^k(x) \\ & - \frac{\partial h}{\partial \xi}(\xi)\frac{\partial \varphi}{\partial p}(\hat{p}, x)\, \dot{\hat{p}}, \end{aligned} \tag{25}$$

and the matrix field $\mathcal{Z}$ is naturally defined from (23), (24), and (25).

Then, a general, and implementable, gradient-type adaptation algorithm is

$$\begin{pmatrix} \dot{\hat{p}} \\ \dot{\hat{q}} \end{pmatrix} \;=\; -\mathcal{Z}^{\mathsf{T}} e \tag{26}$$

where the filtered equation error e is of the same dimension as h, and is given as the output of a "strictly passive" filter :

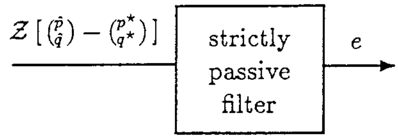

Fig. 2 : *The filtered equation error*

such that e may be computed from available signals (e is *defined* by fig. 2 but cannot be obtained this way, since $p^\star$ and $q^\star$ are not available).

The ideas of hyperstability and passsivity were introduced in adaptive control in [7], and one may find in [8] what we precisely mean here by "stricly passive".

A possible choice of the filter in fig. 2 is :

$$\dot{e} + r\,e \;=\; \mathcal{Z}\left[\begin{pmatrix} \hat{p} \\ \hat{q} \end{pmatrix} - \begin{pmatrix} p^\star \\ q^\star \end{pmatrix}\right] \tag{27}$$

where r is a positive function of the available signals. Using (25). e may then be obtained by :

$$\dot{\eta} \;=\; -r\,e + (\dot{h} - z) + \mathcal{Z}\begin{pmatrix} \hat{p} \\ \hat{q} \end{pmatrix} \tag{28}$$

$$e \;=\; \eta - h(\xi)\ . \tag{29}$$

(28)-(29) only makes use of available signals (from (25), $\dot{h} - z$ only depends on controls and state variables), and gives an e which, from (24) and (25), satisfies (27). This adaptation scheme is the one described in [11] (with $\mathcal{S}'_{p,q}$ here instead of $\mathcal{S}_p$ there). We then have the following property :

Lemma 1 ([11]) *No matter what the control laws u_1 and u_2, or the smooth function h are, for any $x(t)$ solution of the controlled system (u_1, u_2)-$\mathcal{S}'_{p^\star,q^\star}$, and any solution of (26)-(27), defined on $[0, T)$ ($0 < T \leq +\infty$), we have, for all t in $[0, T)$,*

$$\begin{aligned} & \tfrac{1}{2}\left\|\left(\begin{smallmatrix}\hat{p}(t)\\ \hat{q}(t)\end{smallmatrix}\right) - \left(\begin{smallmatrix}p^\star\\ q^\star\end{smallmatrix}\right)\right\|^2 + \tfrac{1}{2}\|e(t)\|^2 + \int_0^t r\|e\|^2 \\ & \leq \tfrac{1}{2}\left\|\left(\begin{smallmatrix}\hat{p}(0)\\ \hat{q}(0)\end{smallmatrix}\right) - \left(\begin{smallmatrix}p^\star\\ q^\star\end{smallmatrix}\right)\right\|^2 + \tfrac{1}{2}\|e(0)\|^2\ . \end{aligned} \tag{30}$$

The various adaptation schemes we will use in the two following sections are obtained by precising the choice of both the function h and the filter in figure 2. This filter may be different from (27).

4 Adaptive Control in the General Case

Here, we make no more structural assumption besides LP, FDE and STS, and do not use the control u_2; u_1 is given by (32) :

$$u_2 \equiv 0 \ , \qquad (31)$$
$$u = u_1 = u_{\text{nom}}(\hat{p}, x) \ . \qquad (32)$$

To describe completely the adaptive controller, we have to define the adaptation, i.e. how $\hat{p}$ in (32) is obtained. The adaptation scheme (section 3) will contain no $\hat{q}$ (the term depending on q in (16) being now identically zero, we just omit q, see remark 1). We propose two choices for the function h, with corresponding choices of the filter of figure 2, both of the type (27).

1. A first possible choice is to take for h some coordinates on M^n and to define the filter (fig. 2) by (27), with no $\hat{q}$ and q^*, and r a constant real positive number. We only mention the results obtained when the systems are fully feedback linearizable, and we choose for h some coordinates in which (11) and $\breve{s}$ are linear ($\dot{\xi} = \breve{s}(\xi)$ implies $\dot{h} = Ah$ with A a Hurwitz matrix). Then, the adaptive controller (26)-(60)-(28)-(29) for feedback linearisable systems is similar to those described in [3], though the point of view is different there. We have the following results for these controllers (see [3] or [8]) :
If FDE and LP are satisfied, and STS is satisfied locally, i.e. STS(Ω) is satisfied for some Ω, then any solution of the closed-loop system (figure 1) such that $x(0)$ and $\eta(0)$ are closed enough to zero and $\hat{p}(0)$ is close enough to p^* is bounded and such that $x(t)$ goes to zero.
As a global result, we have (from [8]),
If FDE and LP are satisfied, and STS is satisfied globallyally, **and if in addition** *all the vector fields are globally Lipschitz, then any solution of the closed-loop system is bounded and such that $x(t)$ goes to zero.*

2. Another possible choice for h is the real positive function U given by assumption STS. The filter of figure 2 is again of the type (27), and we altogether obtain the following controller, which is the one we presented in [11] or [10], particularized to the case of feedback equivalent systems :

Adaptive Controller $\mathcal{AC}_1(U)$:

$$u = u_{\text{nom}}(\hat{p}, x) \qquad (33)$$
$$\dot{\hat{p}} = -\, e\, Z(\hat{p}, x, u_{\text{nom}}(\hat{p}, x))^{\top} \qquad (34)$$
$$\dot{\eta} = -\, r\, e + \frac{\partial U}{\partial x}(\varphi(\hat{p}, x))\, \breve{s}(\varphi(\hat{p}, x)) + W(\hat{p}, x).\dot{\hat{p}} \qquad (35)$$

where

$$e = \eta - U(\varphi(\hat{p}, x)) \qquad (36)$$
$$r = 1 + \|Z\|\,\|W\| \qquad (37)$$
$$W(p, x) = \frac{\partial U}{\partial x}(\varphi(p, x))\frac{\partial \varphi}{\partial p}(p, x) \qquad (38)$$
$$Z(p, x, u) = \left(z^1(p, x, u), \ldots \ldots, z^l(p, x, u)\right) \qquad (39)$$
$$z^i(p, x, u) = \frac{\partial U}{\partial x}(\varphi(p, x))\frac{\partial \varphi}{\partial x}(\hat{p}, x) \left[a^i(x) + \sum_{k=1}^{m} u^k b_k^i(x)\right] \qquad (40)$$

Notice that η is the state of the small system necessary to realize (27) using only available signals.

Theorem 1 ([11])
If assumptions LP and FDE hold, and assumption STS holds locally, there exists an open neighborhood of $(p^*, 0, 0)$ such that

any solution $(\hat{p}, x, \eta)$ with initial condition $(\hat{p}(0), x(0), \eta(0))$ in this neighborhood, exists on $[0, \infty)$, remains in a compact set and its (x, η)-component tends to zero. In addition, the point $(p^*, 0, 0)$ is a (non asymptotically) stable equilibrium point.

Theorem 2 ([11]) *If assumptions LP and FDE hold, and assumption STS holds globally, and if in addition there exists a C^0 function d on Π such that for all (p, x) in $\Pi \times M$, with Z defined in (39),*

$$\begin{aligned} &\|Z(p,x)\| \left\|\tfrac{\partial V}{\partial p}(p,x)\right\| \\ &\quad \leq d(p)\,(1 + V(p,x)^2)\ , \end{aligned} \tag{41}$$

then all the solutions are defined on $[0, \infty)$, remain in a compact set and their (x, η)-component goes to zero.

This algorithm therefore either gives only local results or requires the bound (41) on the growth of the different vector fields. This was also the case for the first algorithm (h = some coordinates), but, as stressed in [11] or [10], if the systems are such that $\breve{s}$ is not Lipschitz in the coordinates h, we cannot guarantee global stability in the previous algorithm whereas we can here.

5 Adaptive control with matching assumptions

We call matching assumptions the fact that FDE holds with the following restriction on the dependance on p of φ :

Assumption FDEM2: *Assumption FDE is satisfied, and φ has the property that there exists a smooth map v_2 :*

$$(p, q, \dot{p}, x) \longmapsto v_2(p, q, \dot{p}, x) \in \mathbb{R}^m \tag{42}$$

meeting, for any $(p, q, \dot{p}, x)$,

$$\frac{\partial \varphi}{\partial x}(p,x) g(q,x) v_2(p,q,\dot{p},x) + \frac{\partial \varphi}{\partial p}(p,x)\dot{p} = 0 \tag{43}$$

We will use here $\hat{p}$ to compute u_1 with (see (15)) :

$$u_1 = u_{\text{nom}}(\hat{p}, x) \ , \tag{44}$$

and the estimate $\hat{q}$ of q^* to compute u_2 by :

$$u_2 = v_2(\hat{p}, \hat{q}, \dot{\hat{p}}, x) \ . \tag{45}$$

We then have

$$\begin{aligned} \dot{\xi} &= \breve{s}(\xi) + A(\hat{p}, x, u_{\text{nom}}(\hat{p}, x))\,(p^* - \hat{p}) \\ &\quad + A_2(\hat{p}, x, v_2(\hat{p}, \hat{q}, \dot{\hat{p}}, x))\,(p^* - \hat{q}) \end{aligned} \tag{46}$$

with A_2 the row vector defined by

$$\begin{aligned} &\frac{\partial \varphi}{\partial x}(p,x)\,[g(q_1,x) - g(q_2,x)]\,u_2 \\ &\quad = A_2(p,x,u_2)\,(q_1 - q_2) \end{aligned} \tag{47}$$

We will again use the adaptation schemes proposed in section 3 to obtain $(\hat{p}, \hat{q})$. We propose three different choices :

1. First, as in section 4, we may chose for h some coordinates on M^n, using the filter (27) with r some positive constant. Again, we only mention the results obtained if the systems are fully feedback linearizable and the coordinates h are these in which $\breve{s}$ is linear. Thee adaptive controller is given by (17)-(44)-(45)-(26)-(28)-(29). The same controllers are described in [3], but without the reparametrisation (and consequently with $\hat{p}$ in place of $\hat{q}$), and under an assumption different from FDEM2, see remark 2 below. We have the following result :

If FDEM2 and LP are satisfied, and STS is satisfied globally (resp. STS(Ω) is satisfied for some Ω), then any solution of the closed-loop system (resp. any such that

$x(0)$ and $\eta(0)$ are closed enough to zero and $\hat{p}(0)$ is close enough to $p^\star$) is bounded and such that $x(t)$ goes to zero.

Notice that, unlike in the situation of the preceeding section, we need no additionnal assumption like global Lipschitzness to get global stability. This would not be the case if the vector field h was not linear in the coordinates h (see [8]). This is one of the reasons to prefer the following choice of h.

2. Another choice for h is $h = U$, the filter being given by (27) with r equal to 1. The expression of $\mathcal{Z}$ is then

$$\mathcal{Z}(\hat{p}, \hat{q}, \dot{\hat{p}}, x) = (Z \mid Y) , \qquad (48)$$

with, A_2 being defined by (47),

$$\begin{aligned} & Y(\hat{p}, x, v_2(\hat{p}, \dot{\hat{p}}, \hat{q}, x)) \qquad (49) \\ & = \frac{\partial \varphi}{\partial x}(\hat{p}, x) A_2(\hat{p}, x, v_2(\hat{p}, \dot{\hat{p}}, \hat{q}, x)). \end{aligned}$$

This yields the following controller :

Adaptive Controller $\mathcal{AC}'_1(U)$:

$$\begin{aligned} u &= u_{\text{nom}}(\hat{p}, x) + v_2(\hat{p}, \dot{\hat{p}}, \hat{q}, x) && (50) \\ \dot{\hat{p}} &= -\, e\, Z(\hat{p}, x, u_{\text{nom}}(\hat{p}, x)) && (51) \\ \dot{\hat{q}} &= -\, e\, Y(\hat{p}, x, v_2(\hat{p}, \dot{\hat{p}}, \hat{q}, x))^\top && (52) \\ \dot{\eta} &= -\, e + \frac{\partial U}{\partial \xi}(\xi).\check{s}(\xi) && (53) \end{aligned}$$

where

$$e = \eta - U(\xi) , \qquad (54)$$

and Z and Y are given by (39)-(40) and (49)-(47).

and the following result :

Theorem 3 *If FDEM2, LP and STS (resp. STS(Ω) for a certain neighborhood Ω of $(0,0)$ in $\mathbb{R}^l \times M^n$) hold, then all the solutions $(\hat{p}(t), \hat{q}(t), \eta(t), x(t))$ of the closed-loop system $\mathcal{S}_{p^\star}$-$\mathcal{AC}'_1(U)$ (resp. the solutions with initial conditions in a certain neighborhood of $(p^\star, p^\star, 0, 0)$) are defined on $[0, \infty)$, remain in a compact set and their (x,η)-component goes to zero. In addition, $(p^\star, p^\star, 0, 0)$ is a (non assymptotically) stable equilibrium point of this dynamical system.*

Here, we get a global result as soon as assumption STS is global, without a restriction like (41) (but under assumption FDEM2, which is stronger than FDE). Yet, STS being globally satisfied is very restrictive. If it is satisfied only locally , $\mathcal{AC}'_1(U)$ gives a good beheviour for only some solutions : ξ has to start close to 0, which is natural, but $(\hat{p}, \hat{q})$ also has to start close to $(p^\star, q^\star)$, which is more unconvenient. We will modify $\mathcal{AC}'_1(U)$ to get a better local result. Suppose that STS(Ω) is satisfied for a certain Ω, and U_o is a positive number such that

$$U(\xi) < U_o \Rightarrow \xi \in \Omega . \qquad (55)$$

We then define the following modified algorithm which is defined only for $U(\xi)$ smaller than U_o :

Adaptive Controller $\mathcal{AC}''_1(U, U_o)$:
The same as $\mathcal{AC}'_1(U)$, but U is replaced by

$$\frac{U_o\, U}{U_o - U} \qquad (56)$$

in (53) and in the definition of Z and Y.

We have the following result :

Theorem 4 *If FDEM2, LP and STS(Ω) hold, and U_o is chosen according to (55), the solutions $(\hat{p}(t), \hat{q}(t), \eta(t), x(t))$ of the closed-loop system $\mathcal{S}_{p^\star}$-$\mathcal{AC}''_1(U, U_o)$ such that $U(\xi(0)) < U_o$ are defined on $[0, \infty)$, remain in a compact set and their (x,η)-component goes to zero, $U(\xi)$ remaining smaller than U_o. In addition, $(p^\star, p^\star, 0, 0)$ is a (non assymptotically) stable equilibrium point of this dynamical system.*

Sketch of proof : We only state the proof of theorem 3; the proof of theorem 4 goes the same way, replacing U by (56) and considering that (56) is infinite when U is U_o.

Consider a solution of the closed-loop system. Lemma 1 (30) implies that $\hat{p}$ and $e = \eta - U$ are bounded on $[0,T)$, the right maximal interval of definition of the solution, and that the time-fuction $e(t)$ is in $L^2([0,T))$. In addition, from (53), (13) and (54), we have :

$$\dot{\eta} \leq -c\eta + (1+c)e \ , \qquad (57)$$

which, together with e being L^2, implies that η is bounded too. This implies that the solution itself is bounded on $[0,T)$ and therefore that $T = +\infty$. Then (57) and (30) applied for $T = +\infty$ give e and η, and therefore U, going to zero. □

3. A third choice for h in the algorithm of section 3 is

$$h(\xi) = \xi \ . \qquad (58)$$

In this case, we chose as a passive filter (fig. 2) the following, with state χ on M^n :

$$\begin{array}{rcl} \dot{\chi} &=& \breve{s}(\chi) - \mathcal{Z}\left[\binom{\hat{p}}{\hat{q}} - \binom{p^\star}{q^\star}\right] \\ e &=& -\frac{\partial U}{\partial \xi}(\chi)^\top . \end{array} \qquad (59)$$

From assumption STS, this filter is (see [8]) "strictly passive with respect to U". Now, from (46), a particular solution (for χ) of the first equation in (59) is precisely ξ (notice that this would be faulse without assumption FDEM2, (42) and (45)). This means that we may compute e by

$$e = -\frac{\partial U}{\partial \xi}(\xi)^\top . \qquad (60)$$

Doing so, we obtain the following

Adaptive Controller $\mathcal{AC}'_2(U)$:

$$\begin{array}{rcll} u &=& u_{\mathrm{nom}}(\hat{p},x) + v_2(\hat{p},\dot{\hat{p}},\hat{q},x) & (61) \\ \dot{\hat{p}} &=& -Z(\hat{p},x,u_{\mathrm{nom}}(\hat{p},x))^\top & (62) \\ \dot{\hat{q}} &=& -Y(\hat{p},x,v_2(\hat{p},\dot{\hat{p}},\hat{q},x))^\top & (63) \end{array}$$

This controller is the same as these described in [5], except that the double parametrization is not used there (replace $\hat{q}$ by $\hat{p}$ and (62)-(63) by $\dot{\hat{p}} = -Z^\top - Y^\top$. The approach to the synthesi presented here is however different : the authors design the adaptation, and the control u_2, to make the positive function

$$U(\xi) + \tfrac{1}{2}\|p^\star - \hat{p}\| + \tfrac{1}{2}\|q^\star - \hat{q}\| \qquad (64)$$

decrease along the solutions (in fact, $U(\xi)$ is a quadratic function of ξ and the third term is absent there). The authors encounter the problem about implicit definition of $\dot{\hat{p}}$ which we mention in remark 2. We can, as for $\mathcal{AC}'_1$, modify it $\mathcal{AC}'_2$ into

Adaptive Controller $\mathcal{AC}''_2(U,U_o)$:
The same as $\mathcal{AC}'_2(U)$, but U is replaced by

$$\frac{U_o U}{U_o - U} \qquad (65)$$

in the definition of Z and Y.

Theorem 5 *The adaptive controllers $\mathcal{AC}'_2(U)$ and $\mathcal{AC}''_2(U,U_o)$ have the same properties as $\mathcal{AC}'_1(U)$ and $\mathcal{AC}''_1(U,U_o)$ given by theorems 3 and 4, replacing (x,η) by x.*

See [8] for a proof using hyperstability. A direct Lyapunov argument, using the function (64) is also described there. It is similar to a proof given in [5].

Remark 2 : Our algorithms are exlicitely defined, i.e. give u, $\dot{\hat{p}}$, $\dot{\hat{q}}$ (and $\dot{\eta}$) as explicit functions of the state variables x, $\hat{p}$, $\hat{q}$ (and η). Let us check this for $\mathcal{AC}'_1(U)$ (or $\mathcal{AC}''_1(U)$) : since ξ stands for $\varphi(\hat{p},x)$, and considering (54), $\dot{\hat{p}}$ and $\dot{\eta}$ are given by (51) and (53) as functions of $\hat{p}$, η, and x; then (52) gives $\dot{\hat{q}}$ as a function of x, $\hat{p}$, $\hat{q}$ and η. For $\mathcal{AC}'_2(U)$ (or $\mathcal{AC}''_2(U)$), the same holds, without η.

This would not have been the case if we had not used the two estimates $\hat{p}$ and $\hat{q}$ for

$p^\star$. In fact, using only $\hat{p}$, we would have defined u_2, instead of (45), by

$$u_2 = v_2(\hat{p}, \hat{p}, \dot{\hat{p}}, x) . \quad (66)$$

Using then the same adaptation law as above, but without overparametrization would give, for $\mathcal{A}C_2'(U)$, instead of (61)-(52),

$$u = u_{\text{nom}}(\hat{p}, x) + v_2(\hat{p}, \dot{\hat{p}}, \hat{p}, x) \quad (67)$$
$$\dot{\hat{p}} = -Z(\hat{p}, x, u)^{\mathsf{T}} \quad (68)$$

which no longer gives explicitly u and $\dot{\hat{p}}$, because $\dot{\hat{p}}$ depends on Z which depends on u which depends on $\dot{\hat{p}}$. Indeed, (67)-(68) give an algebraic equation in (for example) $\dot{\hat{p}}$, which may fail to have a solution. This problem of implicit definition had been first mentionned, in the context of feedback linearisation, in [5]. The overparametrisation (16) is the only way we presently know to solve this problem.

Notice however that it may just not occur, for instance if g does not depend on p at all, because then Z does not depend on u (this was pointed out in [5]), but we saw (remark 1) that in this case, we may drop all q in our reparametrization, which is therefore ineffective; indeed, v_2 in (43) does not depend on q in this case, and $\hat{q}$ is not needed in our algorithms (if we keep it, we just get $\dot{\hat{q}} = 0$). Also, in [3], an assumption ("strong feedback linearisation") is made that allows to avoid this problem. In these cases, since (43) is only used for $p = q$, FDEM2 can be replaced by the apparently weaker assumption FDEM1 (see section 6). ∎

6 Geometric Conditions

We give here some geometric conditions for assumption FDEM2. allowing to build the diffeomorphism φ *locally* around a certain $(\bar{p}, \bar{x})$.

Let, for any p, $\mathcal{G}_p$ be the distribution spanned by the control vector fields of $\mathcal{S}_p$:

$$\mathcal{G}_p(x) = \text{range}\, g(p, x) , \quad (69)$$

$\mathcal{K}_p$ be the distribution made of the vector fields in $\mathcal{G}_p$ whose Lie bracket with any vector field in $\mathcal{G}_p$ is in $\mathcal{G}_p$, and $\mathcal{L}_p$ be the distribution made of the vector fields in $\mathcal{K}_p$ whose Lie bracket with the drift vector field $f(p, .)$ of the system $\mathcal{S}_p$ are in $\mathcal{G}_p$.

$$\mathcal{K}_p = \{X \in \mathcal{G}_p / [X, \mathcal{G}_p] \subset \mathcal{G}_p\} \quad (70)$$
$$\mathcal{L}_p = \{X \in \mathcal{K}_p / [X, f(p, .)] \in \mathcal{G}_p\} \quad (71)$$

Proposition 1 *Suppose that the dimension of $\mathcal{G}_p(x)$ and $\mathcal{L}_p(x)$ are constant (around $(\bar{p}, \bar{x})$) and that $\mathcal{G}_p(x)$ is involutive.*

Then, there exists locally, around $(\bar{p}, \bar{x})$, some φ, α and β satisfying FDEM2 if and only if

1. *$\mathcal{G}_p(x)$ does not depend on p,*
2. *for $i = 1, \ldots, l$, we have, locally around $(\bar{p}, \bar{x})$,*

$$a^i \in \mathcal{G}_p + [f(p, .), \mathcal{G}_p] \quad (72)$$

Proposition 2 *Suppose that the dimension of $\mathcal{G}_p(x)$ and $\mathcal{L}_p(x)$ are constant (around $(\bar{p}, \bar{x})$) and that for any p, the distribution $\mathcal{G}_p(x)$ is involutive.*

Then, there exists locally, around $(\bar{p}, \bar{x})$, some φ, α and β satisfying FDEM2 if and only if, for $i = 1, \ldots, l$, we have, locally around $(\bar{p}, \bar{x})$,

$$a^i \in \mathcal{G}_p + [f(p, .), \mathcal{K}_p] \quad (73)$$

The proof of proposition 1 may be found in [8], and the proof of proposition 2 in the forthcoming [9]. Some more general properties, including global results, as well as a more extensive discussion may also be found in these references. Proposition 1 and 2 are only concerned with *local* existence of the diffeomorphism φ, whereas our assumption FDE needs a global definition. In fact, if the conditions of our propositions are satisfied everywhere, building φ locally may (or may not !) result in a global definition of φ. See also the comment after assumptions FDE and STS for the case where φ (and α and β) are defined only for p in a certain domain, but for all x.

In remark 2, we saw that using a double adaptation prevents the problem of implicit definition of the controller from occuring, but requires assumption FDEM2 instead of the weaker FDEM1 :

Assumption FDEM1: *FDE is satisfied, and φ has the property that there exists a smooth map $\bar{v}_2$ meeting, for any $(p, \dot{p}, x)$,*

$$\frac{\partial \varphi}{\partial x}(p,x)g(p,x)\bar{v}_2(p,\dot{p},x) + \frac{\partial \varphi}{\partial p}(p,x)\dot{p} = 0 \tag{74}$$

This is the assumption required in [5] ("extended matching assumption") for linearisable systems (though written a bit differently).

Clearly, if FDEM1 is satisfied and the control distribution $\mathcal{G}_p$ does not depend on p, FDEM2 is satisfied too. But (see [8]), if $\mathcal{G}_p$ is involutive, FDEM1 implies that $\mathcal{G}_p$ does not depend on p. Therefore, we could substitute FDEM1 to FDEM2 in our proposition 1. Indeed, the conditions given in [5] for the "extended matching condition" (equivalent to FDEM1) are exactly conditions 1 and 2 of proposition 1.

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