IFAC Symposium on
Nonlinear control system design
Capri, June 1989

Nonlinear dynamics of adaptive linear systems: An elementary example

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We analyze the phase portrait and describe some non local non linear behavior of a closed loop system made of an adaptive proportional controller and a disturbed first order linear system. Our results are obtained applying perturbation methods — Poincaré method, structural stability of normally hyperbolic invariant sets. The conclusion is twofold:

• An adaptive linear system in the ideal case is very critical. In presence of perturbation, it can exhibit very rich and intricate dynamics.

• Looking at adaptive linear systems only from a linear point of view is very insufficient and in some cases misleading.

1 Introduction

Typically, adaptive linear controllers are designed from a linear point of view:

- First, a linear but parameterized linear controller is designed, applying linear system theory.
- Second, a parameter adaptation law is designed, applying linear estimation theory.
- Finally, an adaptive linear controller is obtained from the parameterized linear controller with those parameters given on line by the adaptation law.

Not only design but analysis is made from a linear point of view. Boundedness of the solutions is established applying the robust linear stability theory. Their properties are studied similarly with, some times, the help of averaging theory.

However, an adaptive linear controller is a dynamic nonlinear controller. For example, in this paper we will be concerned with the following proportional controller in its robust adaptive version:

$$\left.\begin{aligned}
s(t+1) &= \mu^2 s(t) + u(t)^2 + y(t)^2 \\
\theta_+(t) &= \theta(t) + \frac{y(t)\,(y(t+1) - u(t) - \theta(t)y(t))}{\gamma_1 \sup\{1, s(t+1)\} + y(t)^2} \\
\theta(t+1) &= \theta_0 + \min\left\{1, \frac{R}{|\theta_+(t) - \theta_0|}\right\}(\theta_+(t) - \theta_0) \\
u(t) &= -\theta(t)y(t) + r(t)
\end{aligned}\right\} \quad (1)$$

u, y and r are the system input, system output and set point signal respectively. (s, θ) is the controller state, θ having being given the interpretation of an adapted parameter. R, θ_0, μ and γ_1 are design parameters, interpreted as:

- the adapted parameter is looked for in the interval $[\theta_0 - R\,,\,\theta_0 + R]$.
- the larger the positive γ_1 is, the slower the adaptation, i.e. the θ-dynamics, is.
- the system to be controlled has no unmodelled almost cancellable pole-zero pair slower than μ, chosen strictly positive and smaller than 1.

This controller is known to give bounded solutions when placed in feedback with a system such that a sequence a exists to satisfy for each t, $\theta_0 - R \le a(t) \le \theta_0 + R$ and:

$$\frac{1}{q}\sum_{i=t+1}^{t+q}\left[\frac{y(i+1) - u(i) - a(i)y(i)}{\sup\{1, s(i+1)\}}\right]^2 \le \frac{C_1}{q} + \gamma_0^2 \quad (2)$$

$$\frac{1}{q}\sum_{i=t+1}^{t+q}|a(i) - a(i-1)| \le \frac{C_2}{q} + V \quad (3)$$

where:

$$\sqrt{\gamma_0^2 + 4R\gamma_1 V}\sqrt{\frac{1+\gamma_1}{\gamma_1}} < \frac{1-\mu^2}{4(R + |\theta_0|)} \quad (4)$$

In particular, this result applies to the simplest of these systems, object of the forthcoming study:

$$y(t) = ay(t-1) + u(t-1) + \delta \quad (5)$$

with a belonging to $[\theta_0 - R\,,\,\theta_0 + R]$. Moreover, if δ is zero and r is constant, $(\theta(t), y(t))$ converges globally to (a, r), in an l_2-sense and exponentially if r is not zero (Goodwin and Sin, 1984). However, for r zero, any arbitrarily small disturbance δ leads to an intermittent phenomenum (Pomeau and Manville, 1980), whose characteristics on the y-component suggested the name bursting (Anderson, 1985; Jaïdane-Saïdane and Macchi, 1988). From a linear point of view, a constant exogenous signal leading to such a very high frequency output is very disappointing. In practice, several fixes are added — dead zone (Egardt, 1979), internal model principle (Elliott and Goodwin, 1984), filtering (Anderson et al., 1986), ... However, if those fixes are not appropriately chosen, a qualitatively similar behavior may be observed for these more intricate cases (Praly, 1988). This motivates our interest in the closed loop system (1), (5) and the objective of this paper is to analyze its phase portrait and to explain some non local nonlinear behavior of its solutions. This will implicitly demonstrate that non linear system theory is full of very appropriate results and the linear point of view can be advantageously compensated for.

To appply these results, it is more appropriate to write the system in the so called standard form. This is done as follows:
when restricted to the set:

$$\mathcal{W} = \left\{ (y, s, \theta) \,\middle|\, s \leq 1 , |\theta - \theta_0 + y \frac{\delta - (\theta - a)\, y}{\gamma_1 + y^2}| \leq R \right\} (6)$$

with r constant and letting:

$$x = y/\delta \; ; \quad \psi = \theta - a \; ; \quad \alpha = r/\delta \; ; \quad d = \frac{\delta^2}{\gamma_1} \quad (7)$$

the closed loop system is decribed by the map Σ from $\mathbf{R}^2$ into $\mathbf{R}^2$, given by:

$$\Sigma(\psi, x) = \begin{pmatrix} \psi + d \dfrac{x\,(1 - \psi\, x)}{1 + d\, x^2} \\ -\psi x + 1 + \alpha \end{pmatrix} \quad (8)$$

The equation for the s-component is omited since it has no influence on the remaining part of the system. The standard form (8) puts in relief the role played, in the qualitative behavior of the solutions, by the set point-to-disturbance ratio α and by the positive disturbance-to-adaptation speed ratio d. In particular this shows that d is the effective adaptation speed control. As most of the adaptive linear systems, this system belongs to the family of systems which can be described, with A, B, C smooth functions, by the map Σ_g from $\mathbf{R}^n$ into $\mathbf{R}^n$, given by:

$$\Sigma_g(\psi, x, k) = \begin{pmatrix} \psi + d\, C(\psi, x, d, k)) \\ A(\psi)\, x + B(\psi, k) \end{pmatrix} \quad (9)$$

For d small, this map appears locally as a perturbation of a map made of a family of linear maps. This remark motivates for applying perturbation methods — Poincaré method, structural stability of normally hyperbolic invariant sets, averaging. By doing so, and for the general system (9), can be obtained:

- existence of limit sets (Bodson et al., 1986; Praly and Pomet, 1987),
- existence of invariant sets (Praly, 1985, 1989; Riedle and Kokotovic, 1986). Their attractivity and / or repulsivity gives then information on the solution behavior.
- description of the motion along these invariant sets (Ljung, Söderström, 1984; Anderson et al., 1986; Riedle, 1986; Benveniste et al., 1987; Praly, 1989).

Here considering our disturbance-to-adaptation speed ratio d as a small parameter for the perturbation analysis, we apply these methods to our elementary example. From their generality, we expect that several of our conclusions extend to more general situations. Also, our analysis can be completed to obtain more precise local results by a bifurcation analysis (see (Golden and Ydstie, 1988) or (Mareels and Bitmead, 1986, 1988) for example).

To obtain the basis of our perturbation analysis, we start by studying the system obtained by taking d equal to zero and called the frozen system. Then, fixed points and periodic solutions of the map Σ are considered. In Section 4, we establish existence and properties of locally invariant sets. Critical elements and locally invariant sets are combined in Section 5 to obtain theoretical results on the system global dynamics. These results are interpreted in Section 6.

This paper is a short version of a report written by Martin España and the author (España and Praly, 1988). Due to space limitations, no simulation results will be presented in this paper. They can be easily reproduced by the reader (take for example $d = 0.01$).

2 The frozen system

Our analysis will be done by considering Σ as a small perturbation of the so called frozen system:

$$\Sigma_f(\psi, x) = \begin{pmatrix} \psi \\ -\psi x + 1 + \alpha \end{pmatrix} \quad (10)$$

As mentioned in Introduction, it is a family of linear systems parameterized by ψ. The linear system theory allows us to completely describe its phase portrait:

Property 1 (Frozen system):
A frozen system solution satisfies one and only one of the following property:

- *if $|\psi|$ is strictly smaller than 1, it converges exponentially to (ψ, x), point of the graph:*

$$x = \frac{1 + \alpha}{1 + \psi} \quad (11)$$

- *if $|\psi|$ is strictly larger than 1, it diverges exponentially to infinity from the point (ψ, x) of the same graph.*
- *if ψ is equal to 1, it is a period-2 solution.*
- *if ψ is equal to -1, its x-component grows linearly towards $+\infty$.*

As will become clear later, the important facts in this result are:

- the set:

$$S_f = \left\{ (\psi, x) \,\middle|\, x = \frac{1 + \alpha}{1 + \psi} \; , \; \psi \neq -1 \right\} \quad (12)$$

which is a graph and is invariant under Σ_f,
- the frozen system stability boundary $|\psi| = 1$ which separates this invariant set into normally hyperbolic locally invariant components, namely:
- $S_f \cap \{(\psi, x) \,|\, |\psi| > 1\}$ is exponentially repellent,
- $S_f \cap \{(\psi, x) \,|\, |\psi| < 1\}$ is exponentially attractive,
- the period-2 solutions which are critical.

3 Equilibrium point and period-2 solutions

Going back to the actual system, let us see how the limit sets of the frozen system are disturbed. An elementary continuity argument shows (Praly and Pomet, 1987):

Lemma 1 (Existence of periodic solutions):
A necessary condition for a solution to be a period-T solution of Σ which remains bounded as d goes to 0, is that the accumulation point of its initial condition be one of the following 3 points:

$$\left. \begin{array}{llll} \psi_o = \alpha^{-1} & , & x_o = \alpha \\ \psi_{1,2} = 1 & , & x_{1,2} = \dfrac{1 + \alpha \pm \sqrt{1 - \alpha^2}}{2} \end{array} \right\} \quad (13)$$

Consequently, from the the frozen system equilibrium set S_f, only one point is of interest. Similarly from the set of critical period-2 solutions, only one persists by a continuous perturbation. To precise this necessary condition, we apply Poincaré method (Lefschetz, 1977) and get:

Property 2 (Critical elements):
i) The map Σ *has a unique fixed point for all* α *different from* 0 *or* -1. *It is exponentially stable for* α^{-1} *in* $(-1, -P_1)$ *and exponentially unstable for* α^{-1} *outside* $[-1, -P_1]$ *, with* P_1 *the unique solution of:*

$$2(P+1) = \frac{d}{P^2+d} \tag{14}$$

ii) For any α *strictly smaller than 1 in absolute value, one can find a strictly positive constant* d_o *, such that for all* d*, smaller than* d_o*, there exit two locally unique period-2 solutions which can be approximated by:*

$$\begin{aligned} \psi &= 1 - d\frac{\alpha}{2}\frac{1+\alpha \mp \sqrt{1-\alpha^2}}{2} + O(d^2) \\ x &= \frac{1+\alpha \pm \sqrt{1-\alpha^2}}{2} + O(d) \end{aligned} \tag{15}$$

These solutions are foci, exponentially stable for α *strictly positive, exponentially unstable for* α *strictly negative with a pseudo period:*

$$T = \frac{2\pi}{\sqrt{2d(1-\alpha^2)} + O(d)} \tag{16}$$

From this result, it follows that the control objective: *output = set point* cannot be met asymptotically if the set point-to-disturbance ratio is too small (Narendra and Annaswamy, 1986). Even more, with this small ratio, there **exists a period-2 solution lying close to the frozen system stability boundary. This solution being a focus, it** explains locally and at least transiently the intermittent behavior mentioned in Introduction. Also, being proportional to $\sqrt{d(1-\alpha^2)}$, the frequency of the corresponding bursts decreases as the disturbance-to-adaptation speed ratio goes to zero or the set point-to-disturbance ratio goes to 1. Though not established here, it is also noticeable that, for α equal to zero, the period-2 solution is:

$$\psi = 1 \ , \quad x = 0 \text{ or } 1 \tag{17}$$

It lies exactly on the frozen system stability boundary and is a critical attractor.

4 Locally invariant sets

In Section 2, we have noticed that the graph S_f is invariant for the frozen system. For the actual system, we get (when this makes sense):

$$\begin{aligned} \left(x(t+1) - \frac{1+\alpha}{1+\psi(t+1)}\right) &= -\psi(t)\left(x(t) - \frac{1+\alpha}{1+\psi(t)}\right) \\ &+ d\frac{(1+\alpha)x(t)(x(t)\psi(t)-1)}{(1+\psi(t))(1+\psi(t)+dx(t)(1+x(t)))} \end{aligned} \tag{18}$$

With d in the second term on the right hand, we see that S_f is close to be a locally invariant set of Σ. In fact, S_f being locally normally hyperbolic for the frozen system, for the actual system with d small enough, we can expect the existence of locally invariant sets, close to S_f, being repellent in the set $\{|\psi| > 1\}$ and attractive in the set $\{|\psi| < 1\}$ (Shub, 1978; Hirsch, Pugh and Shub, 1976). To prove this existence we apply the graph transform technique (Shub, 1978).

Property 3 (Repellent locally invariant set):
For any non-zero d*, let* ϵ *be the smallest positive root of:*

$$\Delta(\epsilon) = \left(\epsilon - \frac{\sqrt{d}}{1+\epsilon}\right) - 2\sqrt{\frac{|1+\alpha|\sqrt{d}(1+\epsilon+\sqrt{d})}{\epsilon}} \tag{19}$$

There exists a bounded Lipschitz continuous function H *defined on* $\{|\psi| \geq 1+\epsilon\}$ *and such that, with the function* ϕ_H *defined by:*

$$\phi_H(\psi) = \psi + d\,H(\psi)\frac{(1-\psi H(\psi))}{1+dH(\psi)^2} \tag{20}$$

i) If:

$$|\psi| \geq 1+\epsilon, \ |\phi_H(\psi)| \geq 1+\epsilon, \ sgn(\phi_H(\psi)) = sgn(\psi) \tag{21}$$

then

$$H(\phi_H(\psi)) = 1+\alpha - \psi H(\psi) \tag{22}$$

ii) There exists ρ *positive such that:* $(\phi, y) = \Sigma(\psi, x)$ *and* (ψ, x) *in* $\{|\psi| \geq 1+\epsilon\} \times \mathbf{R}$ *with* $\psi\phi$ *positive implies:*

$$|y - H(\phi)| \geq (1+\rho)\,|x - H(\psi)| \tag{23}$$

iii) Approximation of H*:* $\sup_{\{|\psi| \geq 1+\epsilon\}} \left\{ \frac{1}{d}\left| H(\psi) - \frac{1+\alpha}{1+\psi}\right| \right\}$ *is bounded.*

Hence, in fact for any d, there exists an exponentially repellent locally invariant set. It can be approximated by the frozen parameter invariant set S_f, for d sufficiently small.

Property 4 (Attractive locally invariant set):
For any d *such that:*

$$0 < d < d^* = \frac{1}{2|1+\alpha|^2}\left(\sqrt{\frac{1+3|1+\alpha|}{1+2|1+\alpha|}} - 1\right) \tag{24}$$

let η *be the smallest positive root of:*

$$\Delta(\eta) = \left(\eta - \frac{dn_0^2}{1+dn_0^2}\right) - 2\sqrt{\frac{(1+2n_0)n_0 d}{1+dn_o^2}} \tag{25}$$

where n_0 *and* n_1 *are defined by:*

$$n_0 = \frac{|1+\alpha|}{\eta} \ , \quad n_1 = \sqrt{\frac{(1+dn_0^2)\,n_o}{(1+2n_0)\,d}} \tag{26}$$

There exists a bounded continuous Lipschitz function G *such that, with* $\psi_G(\phi)$ *a function implicitly defined by:*

$$\phi = \frac{\psi_G(\phi) + d\,G(\psi_G(\phi))}{1+d\,G(\psi_G(\phi))^2} \tag{27}$$

(i) If:

$$|\phi| \leq 1-\eta \ \text{ and } \ |\psi_G(\phi)| \leq 1-\eta \tag{28}$$

then:

$$\left.\begin{aligned} G(\phi) &= 1+\alpha - \psi G(\psi) \\ \phi &= \psi + d\,\frac{G(\psi)(1-\psi G(\psi))}{1+dG(\psi)^2} \end{aligned}\right\} \tag{29}$$

(ii) Let ξ *satisfy:*

$$n_o < \xi < n_o + \frac{n_o^2 + n_1(1+2n_o)}{n_1} \tag{30}$$

there exists σ*, depending on* ξ *and strictly smaller than* 1 *such that:* $(\phi, y) = \Sigma(\psi, x)$ *and* (ψ, x) *in* $\{|\psi| < 1-\eta\} \times \{|x| \leq \xi\}$ *implies:*

$$|y - G(\phi)| \;\leq\; \sigma\; |x - G(\psi))| \tag{31}$$

(iii) Approximation of G: $\sup_{\{|\psi|<(1-\eta)\}} \left\{ \frac{1}{d} \left| G(\psi) - \frac{1+\alpha}{1+\psi} \right| \right\}$ *is bounded.*

Consequently, for d sufficiently small, there exists an exponentially attractive locally invariant set which can be approximated by the frozen parameter invariant set S_f.

With these two properties, we see that for d small enough the linear analysis made for the frozen system gives a good approximation of the behavior of the actual system as long as we are far enough from the frozen system stability boundary $|\psi| = 1$ and its invariant set S_f.

5 Behavior of the solutions: technical results

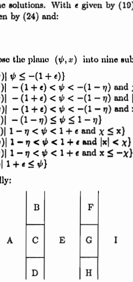
Knowing the existence of critical elements and locally invariant sets, we are now in position for studying the behavior of the solutions. With ϵ given by (19), η given by (25), d^* given by (24) and:

$$\chi \;>\; \frac{1}{1-\eta} \tag{32}$$

we decompose the plane (ψ, x) into nine subsets:

$A = \{(\psi,x)|\ \psi \leq -(1+\epsilon)\}$
$B = \{(\psi,x)|\ -(1+\epsilon) < \psi < -(1-\eta) \text{ and } \chi \leq x\}$
$C = \{(\psi,x)|\ -(1+\epsilon) < \psi < -(1-\eta) \text{ and } |x| < \chi\}$
$D = \{(\psi,x)|\ -(1+\epsilon) < \psi < -(1-\eta) \text{ and } x \leq -\chi\}$
$E = \{(\psi,x)|\ -(1-\eta) \leq \psi \leq 1-\eta\}$
$F = \{(\psi,x)|\ 1-\eta < \psi < 1+\epsilon \text{ and } \chi \leq x\}$
$G = \{(\psi,x)|\ 1-\eta < \psi < 1+\epsilon \text{ and } |x| < \chi\}$
$H = \{(\psi,x)|\ 1-\eta < \psi < 1+\epsilon \text{ and } x \leq -\chi\}$
$I = \{(\psi,x)|\ 1+\epsilon \leq \psi\}$

or graphically:



In the following we state several properties which are proved in (España and Praly, 1988). They are given without any comment, their interpretation being the topic of the next Section.

Property 5 (Solution boundedness, sets : A to I):
(i) If α lies in $(-1,0]$, Σ has unbounded solutions.
(ii) If α is not in $[-1,0]$, all the solutions of Σ are bounded.

Property 6 (Solutions in the repellent locally invariant set, sets: A , I): *For any non zero d:*
(i) The fixed point of Σ, belongs to the repellent locally invariant set if and only if $|1/\alpha| \leq 1+\epsilon$.

(ii) Let $(\psi(t), x(t))$ be a solution such that:

$$|\psi(0)| \geq 1+\epsilon \;\;,\quad x(0) \;=\; H(\psi(0)) \tag{33}$$

and let T be the largest integer such that:

$$\psi(t)\psi(t-1) > 0 \text{ and } |\psi(t)| \geq 1+\epsilon\,,\ \forall\, 0 \leq t < T \tag{34}$$

Case $\alpha < -1$: *For all t in $(0,T)$, $(\psi(t)-\psi(t-1))\psi(t-1)$ is negative. With (i), this implies that $|\psi(t)|$ goes monotonically to and crosses $1+\epsilon$. Consequently T is finite.*

Case $\alpha = -1$: *$H(\psi) \equiv 0$ and any point in the repellent locally invariant set is a fixed point of Σ. T is infinite.*

Case $-1 < \alpha \leq 0$: *If $\psi(0)$ is positive, $\psi(t)$ is strictly increasing and therefore converges to $+\infty$ while $x(t)$ goes to zero. Consequently, T is infinite.*
If $\psi(0) < \min\{1/\alpha, -(1+\epsilon)\}$, $\psi(t)$ is strictly decreasing to $-\infty$ while $x(t)$ goes to zero. Consequently, T is infinite.
If $1/\alpha < \psi(0) < -(1+\epsilon)$, $\psi(t)$ is strictly increasing and crosses $-(1+\epsilon)$. Consequently, T is finite.

Case $0 < \alpha$: *$\psi(t)$ is monotonically going towards $1/\alpha$. Consequently, T is finite if and only if $\psi(0)$ is negative and/or $1/\alpha < 1+\epsilon$.*

Property 7 (Solutions in the "strict instability" set outside the repellent locally invariant set, sets: A , I):
For any non zero d:
(i) If for some time t_0, a solution satisfies:

$$|\psi(t_0)| \;\geq\; 1+\epsilon \;\;,\quad x(t_0) \;\neq\; H(\psi(t_0)) \tag{35}$$

then there exists a (finite) time t_1 such that:

$$|\psi(t_1)| \;<\; 1+\epsilon \tag{36}$$

Hence, there is no solution satisfying for ever $x \neq H(\psi)$ and $|\psi| \geq 1+\epsilon$.

(ii) Moreover, while the solution remains in $\{\psi \geq 1+\epsilon\}$ (resp. $\{\psi \leq -(1+\epsilon)\}$), it exponentially diverges from the graph $\{(\psi, H(\psi))\}$ and crosses the repellent locally invariant set at each time t (resp. it remains on the same side).

Property 8 (Solutions in the attractive locally invariant set, set: E):
For any d, $0 < d < d^*$:
(i) The unique equilibrium point of Σ is in the locally invariant set if and only if $|\alpha|$ is strictly larger than $1/(1-\eta)$. And any solution starting on it, monotonically approaches the equilibrium point.

(ii) if $|\alpha|$ is strictly smaller than 1, all the solutions in the locally invariant set have their ψ-component strictly increasing while it remains in $\{|\psi| \leq 1-\eta\}$. Consequently, with (i), the solutions in the attractive locally invariant set leave the "strict stability" set in finite time, through the boundary $\psi = 1-\eta$.

Property 9 (Solutions in the "strict stability" set outside the attractive locally invariant set, set: E):
For any d, $0 < d < d^*$:
(i) Any solution in $\{|\psi| \leq 1-\eta\} \times \mathbf{R}$ exponentially approaches the graph $\{(\psi, G(\psi))\}$. Moreover a solution starting in $\{|\psi| \leq 1-\eta\} \times \mathbf{R}$ remains in this set as long as it remains in the set $\{|x| \geq \frac{1}{1-\eta}\}$.

(ii) if $|\alpha| < 1$ and for some time t_0, a solution satisfies:

$$|\psi(t_0)| \;\leq\; 1-\eta \tag{37}$$

then there exists a (finite) time t_1 such that:

$$|\psi(t_1)| \;>\; 1-\eta \tag{38}$$

Hence, for $|\alpha| < 1$, there is no solution satisfying for ever $|\psi| \leq 1 - \eta$.

(iii) Moreover, while a solution remains in the set $\{\frac{d(1+\xi+n_0)n_1}{1+dn_0^2} \leq \psi \leq 1-\eta\} \times \{|x| \leq \xi\}$ (resp. in the set $\{-(1-\eta) \leq \psi \leq -\frac{d(1+\xi+n_0)n_1}{1+dn_0^2}\} \times \{|x| \leq \xi\}$), it crosses the attractive locally invariant set at each time t (resp. it remains on the same side).

Property 10 (Solutions in the "critical stability" sets B , D , F , H): *As long as a solution remains in the set $\{(\psi, x) |\ 1-\eta < |\psi| < 1+\epsilon$ and $|x| \geq \chi\}$, its ψ component is exponentially decaying. Hence there is no solution satisfying for ever $1-\eta < |\psi| < 1+\epsilon$ and $|x| \geq \chi$.*

Property 11 (Solutions in the "critical stability" set G): *If $|1/\alpha|$ is larger than $1+\epsilon$ and, for d small enough, period-2 solutions exist and are in the set:*

$$G = \{(\psi, x) |\ 1-\eta < \psi < 1+\epsilon \text{ and } |x| < \chi\} \quad (39)$$

6 Behavior of the solutions: interpretation

The behavior of the solutions of Σ can be roughly explained with the following remarks:

According to Property 7, a solution in the set A or I, but not in the repellent locally invariant set, diverges exponentially from a set which is the graph of a uniformly bounded function of ψ. This explains an exponential growth of the x-component which becomes and remains large. Moreover, for a solution in the set I, at each time t, the x-component changes side with respect to this graph. This explains a burst with a very high frequency content of this component. Conversely, for a solution in the set A, the x-component remains on the same side of the graph. It corresponds a burst without oscillations. This behavior appears for any value of the disturbance and the set point.

According to Property 10, a solution in the set A or I with a large x-component or in the set B, D, F or H has its ψ-component exponentially decaying. This behavior exists for any value of the disturbance and the set point.

According to Property 9, as soon as a solution enters the set E, it is exponentially attracted towards a set which is once again the graph of a uniformly bounded function of ψ. This explains the exponential decrease of the x-component and the fast decay of its high frequency content if it were present. This behavior happens for any value of the set point-to-disturbance ratio and at least for small values of the disturbance-to-adaptation speed ratio.

While a solution, in the set E, goes to the attractive locally invariant set, its evolution is more and more similar to this of the solutions in this set. According to Properties 4 and 7, this explains a speed of the ψ-component of the order of d. Moreover, according to Properties 2 and 8, if the set point-to-disturbance ratio $|\alpha|$ is strictly larger than 1, the solutions converge to the fixed point which corresponds to the desired working conditions. But, if this ratio is strictly smaller than 1, according to Properties 8 and 9, the solutions leave the set E and, very likely, enter the set G. Among the conditions for existence of this behavior are smallness of the disturbance-to-adaptation speed ratio.

After entering the set G, a solution may either leave it, going to the set I, F or H or remains in G. According to Properties 2 and 11, for a set point-to-disturbance ratio strictly smaller than 1 and for a disturbance-to-adaptation speed ratio sufficiently small, there exist two period-2 solutions in G . They are attractive if set point and disturbance have same signs and repellent in the opposite case. The former case explains why the solutions may remain in the set G and why we can expect the bursting phenomenum to disappear asymptotically. Moreover, also from Property 2, as the set point-to-disturbance ratio is closer and closer to (though smaller than) 1, the rotation of the solutions around these period-2 solutions is slower and slower and, therefore, the bursts are less and less frequent.

From simulations, it seems that the attractive locally invariant set and the repellent locally invariant set are smoothly connected through the set G. From a theoretical point of view, we know that if the fixed point lies in the set I, the repellent locally invariant set is the stable manifold of this critical element. Its intersection with the boundary $\psi = 1+\epsilon$ being transverse, we expect that it extends in the set E, giving a candidate for an attractive locally invariant set. Using this conjecture as a working hypothesis, the more a solution approaches the invariant set while it is in the set E, the more its evolution will be similar to the solutions in this invariant set even in the set I. But, according to Property 6, for a set point-to-disturbance ratio strictly smaller than 1 in absolute value and negative (resp. positive), the solutions in the set I and in the repellent locally invariant set are unbounded (resp. bounded). On the other hand, the bigger its ψ-component is, the more the x-component of a solution in the set I but not in the repellent locally invariant set, is pushed away (exponentially) from this invariant set. This reasoning explains the possibility of a very high sensitivity to initial conditions of solutions starting in the set E, close to the attractive locally invariant set or to remain simple and according to Property 4, close to the graph of the function:

$$x = \frac{1+\alpha}{1+\psi}. \quad (40)$$

According to Property 5, for α in $(-1, 0]$, Σ has unbounded solutions lying on the repellent locally invariant set. Hence, even though the set point is sufficiently exciting to estimate a single parameter, if its level does not exceed the disturbance, unbounded solutions are possible (Narendra and Annaswamy, 1986) and this for any value of the disturbance-to-adaptation speed ratio. Also, Properties 5 and 6 establish that all the solutions are bounded if these in the repellent locally invariant set are also bounded.

According to Property 2, a set point-to-disturbance ratio strictly larger than 1 corresponds to a exponentially stable fixed point. According to Property 5 each solution remains in a compact set. According to Lemma 1 and for a sufficiently small disturbance-to-adaptation speed ratio, there is no periodic solution other than the fixed point. This suggests that the fixed point is a global attractor. In this case, the bursting phenomenum should not take place. Qualitatively speaking, this case most resembles to the ideal case.

Summarizing for the case of a small disturbance-to-adaptation speed ratio, according to the value of α, the set point-to-disturbance ratio, three essentially different behaviors of the solutions of Σ can be predicted:

- $|\alpha| \gg 1$ (high level excitation): bounded solutions, no bursting, no periodic solution, a globally attractive fixed point is conjectured, behavior similar to the ideal case.
- $0 < \alpha \ll 1$ (low level excitation): bounded solutions, periodic solutions exist and are conjectured to be global attractors, the fixed point is a saddle, bursting is present but is conjectured to disappear asymptotically.
- $-1 \ll \alpha \leq 0$ (low level excitation): unbounded solutions exist, periodic solutions exist, bursting is present, the fixed point is an unstable node.

Since the set point-to-disturbance ratio α, is a relative quantity, drastic qualitative changes of the systems behavior may be expected when both r and δ are close to 0 which is the natural working condition for an adaptive linear controller.

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