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ADAPTIVE NONLINEAR REGULATION: EQUATION ERROR FROM THE LYAPUNOV EQUATION

J.-B. Pomet and L. Praly
CAI Automatique
École des Mines de Paris
35 Rue Saint Honoré
77305 Fontainebleau FRANCE

Abstract

This paper presents an adaptive controller for nonlinear linearly parametrized systems. The new features introduced in the design are:
- The estimation of the parameter performed on the scalar Lyapunov equation instead of the n-dimensionnal equation of the system itself. It allows us to tolerate non-Lipschitz uncertainties, especially when the stabilizing laws are not feedback linearisation+linear control.
- The double estimation: one estimate is used for the stabilizing control, the other for cancelling the perturbation terms introduced by the adaptation, if possible. We propose this as a solution to the implicit definition of the controller which arises when trying to do this cancellation.

1 Problem Statement

We consider the following family of systems, indexed by p:

$$\dot{x} = a^0(x) + b^0(x)u + \sum_{i=1}^{l} p_i \left(a^i(x) + b^i(x)u\right) \qquad (S_p)$$

where x lives in an n-dimensional C^∞ manifold M, u is in $\mathbf{R}^m$, the a^i's (resp. b^i's) are **known** C^2 vector (resp. matrix) fields and the parameter vector

$$p = (p_1 \dots p_l)^T \qquad (1)$$

belongs to $\mathbf{R}^l$. Since the S_p systems may not make sense for some p, we restrict p to lie in a **known** open set Π of $\mathbf{R}^l$.

Our problem is to design a controller to stabilize the zero solution of the particular S_p system obtained for $p = p^\star$, $p^\star$ being unknown in Π.

Several answers have already been proposed in the literature. In [10], [4] and [11] the problem is particularized to specific systems: robot arms and a continuous stirred tank reactor. More general purpose but feedback linearizable systems are considered in [13], [8], [3] and [2]. Finally Sastry and Isidori [9] study the case of exponentially minimum phase systems with globally Lipschitz nonlinearities. Here the S_p systems are specified by the following assumption:

Uniform Stabilizability (US) assumption: *There exist* **known** *u_n, later called the "nominal control field", and V, a C^1 and a C^2 function respectively, from $\Pi \times M$ to $\mathbf{R}^m$ and to $\mathbf{R}$ respectively, such that:*

1. *For all p in Π, $V(p,x)$ is positive for all x in M and zero if and only if x is zero.*
2. *For any real number K and any compact subset $\tilde{\Pi}$ of Π, the set:*

   $$\left\{ x \,\middle|\, V(p,x) \leq K \,,\, p \in \tilde{\Pi} \right\}$$

   is a compact subset strictly contained in M.
3. *For all (p,x) in $\Pi \times M$, we have:*

   $$L_{s(p,x)} V(p,x) \leq -c\, V(p,x) \qquad (2)$$

   where c is a strictly positive constant and s denotes the "nominal closed loop field":

   $$s = a^0 + b^0 u_n + \sum_{i=1}^{l} p_i \left(a^i + b^i u_n\right) \qquad (3)$$

Besides [9] where the function V is (implicitely) assumed to be only partially known, assumption US is required in all the references quoted above, V being a quadratic function of the linearising coordinates.

Clearly assumption US implies the stabilization problem would be solved for each S_p system if its parameter vector p were known. Therefore the actual problem concerns the possibility of making the nominal control u_n adaptive. Our solution is to design a dynamic controller:

$$\left.\begin{array}{l} \dot{\hat{p}} = \text{dynamic function of } (\hat{p}, x) \\ u = u_n(\hat{p}, x) + v \end{array}\right\} \qquad (4)$$

such that, for any initial condition $(\hat{p}(0), x(0))$ in $\Pi \times M$, the corresponding solution remains in a compact subset of $\Pi \times M$ and its x-component tends to zero as time t tends to infinity.

To illustrate our topic in this paper, we will work out the following example on $\mathbf{R} \times \mathbf{R}^3$:

$$\left.\begin{array}{l} \dot{x}_1 = x_2 + p x_1^2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = u \end{array}\right\} \qquad (5)$$

Following the Lyapunov design proposed in [7], assumption US is met with:

$$\begin{aligned} u_n(p,x) = &- a_3\xi_3 - (a_1 + a_2 + 2p\xi_1)\left(\xi_3 - a_2\xi_2 - \xi_1^{2k-1}\right) \\ &- \left[2p\left(\xi_2 + 2p\xi_1^2\right) + (2k-1)\xi_1^{2k-2}\right](\xi_2 - a_1\xi_1) \\ &- \xi_2 \left(\frac{\xi_2^2}{2} + \frac{\xi_1^{2k}}{2k}\right)^{j-1} \end{aligned} \qquad (6)$$

$$V(p,x) = U(\xi) = \frac{\xi_3^2}{2} + \frac{1}{j}\left(\frac{\xi_2^2}{2} + \frac{\xi_1^{2k}}{2k}\right)^j \qquad (7)$$

where k and j are strictly positive integers and $\xi = (\xi_1, \xi_2, \xi_3)$ is given by the following p-dependant diffeomorphism φ:

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$$\xi = \varphi(p,x) = \begin{pmatrix} x_1 \\ x_2 + a_1\, x_1 + p\, x_1^2 \\ x_3 + a_2\,(x_2 + a_1 x_1 + p x_1^2) \\ \quad + (a_1 + 2px_1)(x_2 + px_1^2) + x_1^{2k-1} \end{pmatrix} \quad (8)$$

Notice that if $k = j = 1$, u_n is a linearizing feedback.

We make the following additional assumption on the parameter set Π.

Imbedded Convex Sets (ICS) assumption: *There exists a* **known** *C^1 function $\mathcal{P}$ from Π to $\mathbf{R}$ such that:*

1. *the sets:*

   $$\{p \mid \mathcal{P}(p) \leq \lambda\}\ , \quad 0 \leq \lambda \leq 1$$

   are convex and contained in Π,

2. *the row vector $\frac{\partial \mathcal{P}}{\partial p}(p)$ is non zero for all p such that $\mathcal{P}(p)$ is in $[0\,,\,1]$,*

3. *the parameter vector $p^\star$ of the particular system to be actually controlled satisfies:*

   $$\mathcal{P}(p^\star) \leq 0 \qquad (9)$$

For our example (5), since $\Pi = \mathbf{R}$, this function $\mathcal{P}$ may be chosen indentically zero.

2 An Adaptive Controller

Let $\hat{p}$ be a C^1 time function to be precised later. Given a control law u and a solution x of the closed loop system u-$S_{p^\star}$, with assumption US, we may define the time function:

$$V(t) = V(\hat{p}(t), x(t)) \qquad (10)$$

Along the solutions of $S_{p^\star}$, we have:

$$\begin{aligned} \dot{V} = &\ L_{s(\hat{p},x)}V(\hat{p},x) + L_{g(p^\star,x)(u-u_n(\hat{p},x))}V(\hat{p},x) \qquad (11) \\ &+ Z_p(\hat{p},x)(p^\star - \hat{p}) + \frac{\partial V}{\partial p}(\hat{p},x)\dot{\hat{p}} \end{aligned}$$

where s is the nominal closed loop field (3) and the Z_p row vector is defined by:

$$Z_p = (L_{a^1+b^1u_n}V, \ldots, L_{a^l+b^lu_n}V) \qquad (12)$$

When compared to the nominal case as defined by assumption US, we see that $\hat{p}$ not being constant equal to $p^\star$, creates two disturbances: the $\frac{\partial V}{\partial p}\dot{\hat{p}}$ term and what is usually called the equation error: $Z_p(p^\star - \hat{p})$. The second term in the first line of (11) is not zero if, as originally proposed by Middleton and Goodwin [4], we augment the nominal control u_n:

$$u = u_n + v \qquad (13)$$

to try to counteract these disturbances. As it will be explained later, it is then appropriate to introduce a second C^1 time function $\hat{q}$ in Π and to rewrite (11) in:

$$\begin{aligned} \dot{V} = &\ L_{s(\hat{p},x)}V(\hat{p},x) + \Delta(\hat{p},\hat{q},\dot{\hat{p}},x,v) \qquad (14) \\ &+ Z_p(\hat{p},x))(p^\star - \hat{p}) + Z_q(\hat{p},x,v)(p^\star - \hat{q}) \end{aligned}$$

where we have defined the row vector Z_q by:

$$Z_q = (L_{b^1v}V, \ldots, L_{b^lv}V) \qquad (15)$$

and the scalar function Δ on $\Pi \times \Pi \times \mathbf{R}^l \times M \times \mathbf{R}^m$ by:

$$\Delta(p,q,\delta,x,v) = L_{g(q,x)v}V(p,x) + \frac{\partial V}{\partial p}(p,x)\delta\ . \qquad (16)$$

(14) may also be seen as an observation equation for the $(p^{\star T}\ p^{\star T})^T$ vector:

$$z(t) = (Z_p(t)\ Z_q(t))\begin{pmatrix} p^\star \\ p^\star \end{pmatrix} \qquad (17)$$

Measuring x and computing $\hat{p}$, $\hat{q}$, u and v, Z_p and Z_q are available on-line. However, z defined by:

$$z = \dot{V} - L_{a^0+b^0u}V - \frac{\partial V}{\partial p}\dot{\hat{p}}\ , \qquad (18)$$

cannot be available, $\dot{V}$ being unmeasurable. This difficulty can be rounded by integration. This leads to the following dynamic controller (see Pomet's dissertation [6] for more details), where η is the additional dynamic variable introduced for this integration:

$$\dot{\hat{p}} = \mathrm{Proj}\left[\hat{p}\,,\, Z_p^T(\hat{p},x)\,(V(\hat{p},x) - \eta)\right] \qquad (19)$$

$$\dot{\hat{q}} = \mathrm{Proj}\left[\hat{q}\,,\, Z_q^T(\hat{p},x,v)\,(V(\hat{p},x) - \eta)\right] \qquad (20)$$

$$\dot{\eta} = r\,(V(\hat{p},x) - \eta) + L_{s(\hat{p},x)}V(\hat{p},x) + \Delta(\hat{p},\hat{q},\dot{\hat{p}},x,v) \qquad (21)$$

$$u = u_n(\hat{p},x) + v \qquad (22)$$

$$r = |V(\hat{p},x) - \eta|^{m_1}\,(1 + Z_pZ_p^T + Z_qZ_q^T)^{m_2} \qquad (23)$$

where the initial conditions are:

$$\mathcal{P}(\hat{p}(0)) \leq 0\ ,\ \hat{q}(0) = \hat{p}(0)\ ,\ \eta(0) = 0 \qquad (24)$$

m_1 and m_2 are two positive real numbers, Proj is the following locally Lipschitz continuous function:

$$\mathrm{Proj}(p,y) = \begin{cases} y \ \text{ if } \ \mathcal{P}(p) \leq 0 \\ y \ \text{ if } \ \mathcal{P}(p) \geq 0 \ \text{ and } \ \frac{\partial \mathcal{P}}{\partial p}(p)\,y \leq 0 \\ y - \frac{\mathcal{P}(p)\frac{\partial \mathcal{P}}{\partial p}(p)y}{\|\frac{\partial \mathcal{P}}{\partial p}(p)\|^2}\frac{\partial \mathcal{P}}{\partial p}(p) \ \text{ if not} \end{cases} \qquad (25)$$

and v is computed to make $\Delta(\hat{p},\hat{q},\dot{\hat{p}},x,v)$, defined in (16), non positive if possible (see (14)). Notice that the V function given by assumption US is explicitely used in the controller. A different choice of V would give a different controller.

If we had not introduced $\hat{q}$, the $\dot{\hat{p}}$ equation would have been:

$$\dot{\hat{p}} = \mathrm{Proj}\left[\hat{p}\,,\,(Z_p + Z_q)^T\,(V(\hat{p},x) - \eta)\right] \qquad (26)$$

If one of the b_i's, $i = 1,\ldots,l$ is not zero, Z_q depends on v. Since, in general, v depends on $\dot{\hat{p}}$, equation (26) defines $\dot{\hat{p}}$ only implicitly and an extra assumption may be needed for the controller to be well defined (assumption I in [3], assumption A4 in [2], no assumption thanks to filtering in [4]). In our case $\dot{\hat{p}}$, $\dot{\hat{q}}$ and $\dot{\eta}$ are defined explicitely. Note that $\hat{q}$ could be reduced to incorporate only those parameters corresponding to the non zero b_i's.

For our example (5), with no b_i term, the adaptive controller is:

$$\dot{\hat{p}} = Z_p\left(V(\hat{p},x)-\eta\right) \tag{27}$$

$$\dot{\eta} = |V(\hat{p},x)-\eta|^{m_1}\left(V(\hat{p},x)-\eta\right)\left(1+Z_p^2\right)^{m_2} \tag{28}$$

$$-a_3\,\xi_3^2 \;-\; \left(\frac{\xi_2^2}{2}+\frac{\xi_1^{2k}}{2k}\right)^{j-1}\left(a_2\,\xi_2^2+a_1\,\xi_1^{2k}\right)+\Delta$$

$$Z_p = \frac{\partial U}{\partial \xi}(\xi)\,\frac{\partial \xi}{\partial x_1}\,\xi_1^2 \tag{29}$$

$$\begin{aligned}\Delta \;=\; & \xi_3\,v \\ & + \{ \left(\frac{\xi_2^2}{2}+\frac{\xi_1^{2k}}{2k}\right)^{j-1} \xi_2\xi_1^2 \\ & \quad + \xi_3\left[a_2\,\xi_1^2+2\xi_1\left(\xi_2-a_1\xi_1\right)+\left(a_1+2\hat{p}\xi_1\right)\xi_1^2\right] \} \\ & \qquad \times\, Z_p\left(V(\hat{p},x)-\eta\right)\end{aligned} \tag{30}$$

and, with φ given in (8), we compute:

$$\left(\xi_1\,,\,\xi_2\,,\,\xi_3\right)^T \;=\; \varphi(\hat{p},x) \tag{31}$$

The possibility of making Δ non positive is related to the sign of $\frac{\partial V}{\partial p}\dot{\hat{p}}$ when $L_{gv}V$ is zero. In general, we cannot expect any relation between these two quantities. However the following theorem established in [6] gives conditions implying a relation:
Let the usual f (and similarly g) be defined by:

$$f(p,x) \;=\; a^0(x) \;+\; \sum_{i=1}^{l} p_i\,a^i(x) \tag{32}$$

we have:

Theorem 1 (Pomet[6]) *Assume $g(p,x)$ has rank m on $\Pi\times M$ and for each fixed p, Range$\{g(p,x)\}$ is an involutive distribution on M. Under this condition, the following two propositions are equivalent:*

1- Range$\{g(p,x)\}$ does not depend on p and, for all i in $\{1,\ldots,l\}$, we have on $\Pi\times M$:

$$\frac{\partial f}{\partial p_i} \;\in\; Span\{g\,,\,[f,g]\} \tag{33}$$

2- For all (p_0,x_0) in $\Pi\times M$, there exist a neighborhood $\mathcal{N}(p_0,x_0)$ and C^1 functions α, β and φ, respectively, from $\mathcal{N}(p_0,x_0)$ to $\mathbf{R}^m$, $GL(\mathbf{R}^m)$ and M, respectively, such that:
- *for each p, φ is a diffeomorphism,*
- $$\left.\begin{array}{l} f(p_0,\varphi(p,x)) = L_{f(p,x)+g(p,x)\alpha(p,x)}\varphi(p,x) \\ g(p_0,\varphi(p,x)) = L_{g(p,x)\beta(p,x)}\varphi(p,x)\end{array}\right\} \tag{34}$$
- *For each i in $\{1,\ldots,l\}$, we have on $\mathcal{N}(p_0,x_0)$:*

$$\frac{\partial \varphi}{\partial p_i}(p,x) \;\in\; Range\left\{L_{g(p,x)\beta(p,x)}\varphi(p,x)\right\} \tag{35}$$

What is meant by (34) is that, by p-dependant diffeomorphism (φ) and regular feedback transformation (α,β), each S_p system can be transformed into one particular of them, S_{p_0} here. A straightforward consequence of this strong property is that u_n can be modified so that the V function of assumption US can be chosen to satisfy:

$$V(p,x) \;=\; U\left(\varphi(p,x)\right) \qquad \forall\,(p,x) \;\in\; \mathcal{N}(p_0,x_0) \tag{36}$$

where U is nothing but:

$$U(\xi) \;=\; V(p_0,\xi) \tag{37}$$

and the modified u_n^m is:

$$u_n^m(p,x) \;=\; \alpha(p,x) \;+\; \beta(p,x)\,u_n(p_0,\varphi(p,x)) \tag{38}$$

In this circumstance, Δ in (16) can be written:

$$\begin{aligned}\Delta(p,q,\delta,x,v) \;=\; & \mathrm{d}U\left(\varphi(p,x)\right) \\ & \times\left(L_{g(q,x)v}\varphi(p,x)+\frac{\partial\varphi}{\partial p}(p,x)\,\delta\right)\end{aligned} \tag{39}$$

But, the distribution Range$\{g(p,x)\}$ having constant rank and being independant of p (as assumed in Theorem 1), with (35), there exists a C^1 function v such that, for all (p,x) in $\mathcal{N}(p_0,x_0)$, all q in Π and δ in $\mathbf{R}^l$, (see[6])

$$L_{g(q,x)v(p,q,\delta,x)}\varphi(p,x)+\frac{\partial\varphi}{\partial p}(p,x)\,\delta \;=\; 0 \tag{40}$$

To summarize, we have:

Property 1 (Pomet[6]) *If assumption US holds and there exists a a neighborhood of $(p^\star,0)$ in $\Pi\times M$ such that, on this neighborhood:*
1- $g(p,x)$ has rank m and Range$\{g(p,x)\}$ is an involutive distribution on M for each p and does not depend on p.

$$2\text{-}\;\frac{\partial f}{\partial p_i} \;\in\; Span\{g\,,\,[f,g]\} \qquad \forall\, i \;\in\; \{1,\ldots,l\} \tag{41}$$

Then there exist a neighborhood of $(p^\star,p^\star,0,0)$ and a C^1 function $v(p,q,\delta,x)$, defined on this neighborhood, such that, may be by modifying u_n and V, $\Delta(p,q,\delta,x,v)$, defined in (16), is zero.

For our example, assumption 1 of this Property is satisfied but assumption 2 is not. Also it turns out that Δ in (30) cannot be guarenteed not positive since there is no reason for the expession

$$\left(\frac{\xi_2^2}{2}+\frac{\xi_1^{2k}}{2k}\right)^{j-1}\xi_2\xi_1^2 \;\times\; Z_p\left(V(\hat{p},x)-\eta\right)$$

to be negative when ξ_3 is zero. Nevertheless, assumption 2 is not necessary in general. In [1], we have shown that, for some planar systems, Δ can be made zero though this assumption fails.

In the case where the S_p systems are feedback linearizable, the assumptions in Property 1 (more precisely in proposition 1 of Theorem 1) have been introduced by Kanellakopoulos et al. [3] and called extended matching condition. These authors have established these assumptions are sufficient for solving in v the equation (see also assumption A4 in [2]):

$$L_{g(p,x)v}\varphi(p,x)+\frac{\partial\varphi}{\partial p}(p,x)\,\delta \;=\; 0 \tag{42}$$

where φ is a p-dependant diffeomorphism associated with the feedback linearization. We know with [6] these assumptions are also necessary for the existence of a p-dependant diffeomorphism such that, locally, (42) can be solved and (34) holds.

3 The Stabilization Property

Applying our adaptive controller to the $S_{p^\star}$ system leads to an autonomous non linear locally Lipschitz continuous system living in $M \times \Pi \times \Pi \times \mathbf{R}$ whose solutions $(x, \hat{p}, \hat{q}, \eta)$ are locally well defined and unique. We have:

Theorem 2 *Assume assuptions US and ICS are satisfied.*

1- If there exists a globally defined locally C^1 function $v(p, q, \delta, x)$ such that Δ defined in (16) is not positive (see Property 1), then, choosing $m_2 = 0$ in the controller, all the solutions are defined on $[0, \infty)$, remain in a compact set and their x-component tends to zero as t tends to infinity.

2- If we cannot choose v as specified in point 1 above, we take it identically zero (hence no $\hat{q}$). If there exist a C^0 function d on Π and positive constants σ and τ such that:

- *for all (p, x) in $\Pi \times M$, with Z_p defined in (12):*

$$\|Z_p(p,x)\| \leq d(p)\, Sup\{1, V(p,x)^\tau\} \tag{43}$$

$$\left\|\frac{\partial V}{\partial p}(p,x)\right\| \leq d(p)\, Sup\{1, V(p,x)^\sigma\} \tag{44}$$

- $\sigma \leq 1\,, \quad \sigma + \tau \leq 2$ (45)

Then, choosing m_1 and m_2 to satisfy:

$$\left.\begin{array}{l} m_1 \geq 0\,, \quad 1 \geq \frac{2m_2}{m_1+2}\,\tau \\ m_1 + 2 \geq 2m_2 \geq 0\,,\ 1 \geq \sigma + \left(1 - \frac{2m_2}{m_1+2}\right)\tau \end{array}\right\} \tag{46}$$

all the solutions are defined on $[0, \infty)$, remain in a compact set and their x-component tends to zero as t tends to infinity.

3- If the assumptions of points 1 and 2 above are not satisfied, but v is chosen to be zero or to be any locally Lipschitz continuous function of $(\hat{p}, \hat{q}, \eta, x)$ such that, with (19),

$$\Delta(\hat{p}, \hat{q}, \dot{\hat{p}}, x, v) \leq \frac{\partial V}{\partial p}(\hat{p}, x)\,\dot{\hat{p}} \quad , \tag{47}$$

then there exists an open neighborhood of $(0, p^\star)$ such that, for any initial condition $(x(0), \hat{p}(0))$ in this neigborhood, the corresponding $(x, \hat{p}, \hat{q}, \eta)$ solution exists on $[0, \infty)$, remains in a compact set and its x-component tends to zero as t tends to infinity.

For our example, we have already mentioned that point 1 of this Theorem does not apply. But we may look for k and j to meet point 2 assumptions. With Z_p given in (29) and V given in (7), we obtain:

$k = j = 1$ (feedback linearization) $\Longrightarrow \tau = \frac{5}{2}, \sigma = 2$ No
$k = 3, j = 2$ $\Longrightarrow \tau = \frac{13}{12}, \sigma = \frac{11}{12}$ Yes

Hence point 2 of Theorem 2 applies if the nominal control law is appropriately chosen. It turns out that feedback linearization does not give a robust enough global stabilization for this purpose.

Proof of Theorem 2

Let $(x, \hat{p}, \hat{q}, \eta)$ be a solution whose maximal interval of definition upperbound is T. First we notice that, thanks to the Proj function and the choice of $\hat{p}(0)$ and $\hat{q}(0)$ in (24), we have:

$$\mathcal{P}(\hat{p}(t)) \leq 1\,, \quad \mathcal{P}(\hat{q}(t)) \leq 1 \qquad \forall t \in [0, T). \tag{48}$$

Hence $\hat{p}$ and $\hat{q}$ remain in Π and even in a closed subset of Π.

Step 1: $\hat{p}$, $\hat{q}$ and $V - \eta$ are bounded:
Let the scalar e be defined by:

$$e = V(\hat{p}, x) - \eta \tag{49}$$

e is a C^1 time function defined on $[0, T)$. From (14) and (21), it satisfies:

$$\dot{e} + r e = (Z_p\ Z_q)\begin{pmatrix} p^\star - \hat{p} \\ p^\star - \hat{q}\end{pmatrix} \tag{50}$$

Notice also that (19) and (20) can be written:

$$\left.\begin{array}{l} \dot{\hat{p}} = \mathrm{Proj}\,(\hat{p}, Z_p^T e) \\ \dot{\hat{q}} = \mathrm{Proj}\,(\hat{q}, Z_q^T e) \end{array}\right\} \tag{51}$$

Now, we consider the comparison function:

$$W(e, \hat{p}, \hat{q}) = \frac{1}{2}\left(e^2 + \|p^\star - \hat{p}\|^2 + \|p^\star - \hat{q}\|^2\right) \tag{52}$$

Along the solutions of (50)-(51) for any t in $[0, T)$, we have:

$$\begin{aligned} \dot{W} &= -re^2 \\ &\quad + \left((p^\star - \hat{p})^T\,(p^\star - \hat{q})^T\right)\begin{pmatrix} Z_p^T e - \mathrm{Proj}\,(\hat{p}, Z_p^T e) \\ Z_q^T e - \mathrm{Proj}\,(\hat{q}, Z_q^T e)\end{pmatrix} \end{aligned} \tag{53}$$

From definition (25) of Proj and assumption ICS, we have:

$$(p^\star - \hat{p})^T \mathrm{Proj}(p, y) \geq (p^\star - \hat{p})^T y \tag{54}$$

Hence:

$$\dot{W} \leq -r\,e^2 \qquad \forall t \in [0, T) \tag{55}$$

With the choice of $\eta(0)$ and $\hat{q}(0)$ in (24), we have established:

$$\|p^\star - \hat{p}(t)\|^2 + \|p^\star - \hat{q}(t)\|^2 + e^2 \leq F(\hat{p}(0), x(0))^2 \tag{56}$$

$$\int_0^T \varepsilon(t)^{m_1+2}\,\mathrm{d}t \leq F(\hat{p}(0), x(0))^2 \tag{57}$$

with:

$$\varepsilon = \left(|e|^{m_1+2}\,(1 + Z_p Z_p^T + Z_q Z_q^T)^{m_2}\right)^{\frac{1}{m_1+2}} \tag{58}$$

and F, to play a key role in the following, is defined by:

$$F(\hat{p}(0), x(0))^2 = 2\,\|p^\star - \hat{p}(0)\|^2 + V(\hat{p}(0), x(0))^2 \tag{59}$$

In particular, this proves that $\hat{p}$ and $\hat{q}$ remain in a compact subset of Π. And, V being positive, η is also lower bounded on $[0, T)$:

$$\eta(t) \geq -F(\hat{p}(0), x(0)) \tag{60}$$

Step 2: To conclude the proof, we only have to show that V is bounded and x tends to zero. We will use the following straightforward consequence of Hölder and Bellman - Gronwall inequalities:

Lemma 1 *Let U be a C^1 time function defined on $[0\,,T)$ and satisfying:*

$$\dot U \;\leq\; -cU \;+\; (1+U(t))\sum_i f_i(t)\;,\; U(0)\;=\;0 \qquad (61)$$

where c is a strictly positive constant and (f_i) is a finite family of positive time functions such that:

$$\int_0^T f_i(t)^{k_i}\,dt \;=\; S_i \;<\; +\infty\;,\; k_i\;\geq\;1 \qquad (62)$$

Under this assumption, $U(t)$ satisfies with G a continuous function and $G(0,k)=0$:

$$U(t)\;\leq\;G(S_i,k_i) \qquad \forall t\;\in\;[0\,,T) \qquad (63)$$

Moreover, if T is infinite then:

$$\limsup_{t\longrightarrow\infty} U(t)\;\leq\;0 \qquad (64)$$

Now with (21), (58) and assumption US, we have:

$$\dot\eta\;\leq\;-c\,\eta+\Delta(\hat p,\hat q,\dot{\hat p},x,v)+(c+r)\,\varepsilon \qquad (65)$$

with r given by (23)

Point 1: Δ is non positive and m_2 is zero. It follows that:

$$\dot\eta\;\leq\;-c\,\eta+c\,\varepsilon+\varepsilon^{m_1+1} \qquad (66)$$

Lemma 1 applies and therefore η is upperbounded on $[0\,,T)$. With the bounds obtained in Step 1, we have established for t in $[0\,,T)$:

$$V(\hat p(t),x(t))\;\leq\;F(\hat p(0),x(0))+G\left(F(\hat p(0),x(0)),m_1\right) \qquad (67)$$

But $\hat p(t)$ remaining in a compact subset of Π, the V properties given in assumption US imply that $x(t)$ belongs to a compact set strictly contained in M for t in $[0\,,T)$. To summarize, we know now that the solution under consideration remains in a compact subset of $M\times\Pi\times\Pi\times\mathbf{R}$. This implies:

$$T\;=\;+\infty \qquad (68)$$

Therefore we know also from Lemma 1:

$$\limsup_{t\longrightarrow+\infty}\eta(t)\;\leq\;0 \qquad (69)$$

Moreover, from Step 1, e is bounded and belongs to $L^{m_1+2}(0\,,+\infty))$. From (50) and boundedness of Z_p and Z_q, $\dot e$ is bounded. This implies that e goes to zero as time goes to infinity. Consequently V also goes to zero. The properties of V imply finally that x goes to zero.

Point 2: v is zero, there is no $\hat q$. (65) becomes:

$$\dot\eta\;\leq\;-c\,\eta+\left\|\frac{\partial V}{\partial p}\right\|\left\|\mathrm{Proj}(\hat p,Z_p^T e)\right\| \qquad (70)$$
$$+c\,\varepsilon+(1+Z_pZ_p^T)^{\frac{m_2}{m_1+2}}\,\varepsilon^{m_1+1}$$

But, by definition of Proj, we have with (58):

$$\left\|\mathrm{Proj}(\hat p,Z_p^Te)\right\|\leq\|Z_p\|\,|e| \qquad (71)$$
$$\leq\|Z_p\|^{1-\frac{2m_2}{m_1+2}}\,\varepsilon \qquad (72)$$

Also $p^\star$ being bounded from Step 1 and the d function in assumptions (43), (44) being continuous, there exists a constant K depending only on $F(\hat p(0),x(0))$ such that:

$$\dot\eta\leq-c\,\eta+K\mathrm{Sup}\{1,V^\sigma\}K^{1-\frac{2m_2}{m_1+2}}\mathrm{Sup}\{1,V^{\tau\left(1-\frac{2m_2}{m_1+2}\right)}\}\varepsilon$$
$$+c\,\varepsilon+K^{\frac{1}{m_1+2}}\mathrm{Sup}\{1,V^{\tau\frac{2m_2}{m_1+2}}\}\,\varepsilon^{m_1+1} \qquad (73)$$

With our choice of m_1, m_2 in (46), Lemma 1 applies and the proof is completed as in Point 1 above.

Point 3: Δ is upperbounded by $\frac{\partial V}{\partial p}\dot{\hat p}$. (65) becomes:

$$\dot\eta\;\leq\;-c\,\eta+\left(\left\|\frac{\partial V}{\partial p}\right\|\|Z_p)\|+c\right)\varepsilon \qquad (74)$$
$$+(1+Z_pZ_p^T+Z_qZ_q^T)^{\frac{m_2}{m_1+2}}\,\varepsilon^{m_1+1}$$

Then let μ be a strictly positive real number and $\mathcal{C}$ be the compact neighborhood of $(0,p^\star,p^\star,0)$ in $M\times\Pi\times\Pi\times\mathbf{R}$ defined by:

$$V(\hat p,x)\leq\mu\,,\;\|p^\star-\hat p\|\leq\mu\,,\;\|p^\star-\hat q\|\leq\mu\,,\;\|\eta\|\leq\mu \qquad (75)$$

All the function in the closed loop system being continuous, let $K(\mu)$ be a constant such that for all $(x,\hat p,\hat q,\eta)$ in $\mathcal{C}$, we have:

$$K\geq\left\|\frac{\partial V}{\partial p}\right\|\|Z_p\|+c \qquad (76)$$

$$K\geq\left(1+Z_pZ-p^T+Z-qZ-q^T\right)^{\frac{m_2}{m_1+2}} \qquad (77)$$

We get:

$$\dot\eta\;\leq\;-c\,\eta\;+\;K\left(\varepsilon+\varepsilon^{m_1+1}\right) \qquad (78)$$

With (57), Hölder and Bellman-Gronwall inequalities yield for all t in $[0\,,T)$, with F given in (59):

$$\eta(t)\;\leq\;K\left[\frac{1}{c^{\frac{1}{m_1+2}}}F^{2\frac{m_1+1}{m_1+2}}+\frac{1}{c^{\frac{m_1+1}{m_1+2}}}F^{\frac{2}{m_1+2}}\right] \qquad (79)$$

Hence, with (49) and (56), the solution under consideration remains in $\mathcal{C}$ and T is infinite if its initial condition satisfies:

$$F+K(\mu)\left[\frac{1}{c^{\frac{1}{m_1+2}}}F^{2\frac{m_1+1}{m_1+2}}+\frac{1}{c^{\frac{m_1+1}{m_1+2}}}F^{\frac{2}{m_1+2}}\right]\leq\mu \qquad (80)$$

which is always possible since F is zero at $(0,p^\star,p^\star,0)$. The proof is continued as in Point 1 above. □

Comments

1- In this proof, we see that, in any case, $V(p^\star,x)$ can be bounded by a function of $2\|p^\star-\hat p(0)\|^2+V(\hat p(0),x(0))^2$. Hence if assumption US were satisfied only locally in x, say on the compact subset of M:

$$\mathcal{P}(p)\;\leq\;0\;,\quad V(p,x)\;\leq\;V_0\;\neq\;0 \qquad (81)$$

Theorem 2 would still hold provided $2\|p^\star - \hat{p}(0)\|^2 + V(\hat{p}(0), x(0))^2$ is small enough. Another possibility to deal with a local asumption US is to replace, in the controller, $V(p, x)$ by a function of $V(p, x)$ which is infinite for $V = V_0$, say:

$$h(V(p,x)) \;=\; \frac{V(p,x)}{V_0 - V(p,x)} \tag{82}$$

Our controller guarentees the boundedness of the function it actually incorporates provided this function meets the properties inovoked in assumption US. Therefore, any solution with:

$$V(\hat{p}(0), x(0)) \;<\; V_0 \tag{83}$$

remains in the above compact set. Similarly, would u_n be smooth only on $\Pi \times M - \{0\}$ (see [12]), we would replace V by, say, $\mathrm{Sup}\{V - \varepsilon\,,\, 0\}^2$ with ε some strictly positive constant (see [1]).

2- Assumptions (43) and (44) describes the behavior of the norm of the regressor vector Z_p and the p-sensitivity $\frac{\partial V}{\partial p}$ as V goes to infinity. A key point of our controller stands in incorporating this information: m_1 and m_2 given by (46) are used in $\dot{\hat{p}}$. For our example, we have seen that, to get global stabilization, u_n has to be chosen with $k = 3$ and $j = 2$. But accordingly $\dot{\hat{p}}$ has to be computed with m_1 and m_2 satisfying:

$$\frac{m_2}{m_1 + 2} \;=\; \frac{6}{13} \tag{84}$$

4 Discussion

To conclude this paper, we compare our algorithm to those previously proposed in the litterature. Our criterion is: global stabilization. The objective being to evaluate if globalness, holding in the known parameter case, is preserved or lost when adaptation is introduced.

The first point to be mentioned is that our algorithm is of an equation error type. Nam and Arapostathis [8] and Bastin and Campion [2] have proposed algorithms of the same type. But, their equation error is directly obtained from the S_p equation or its form tranformed by a p-dependant diffeomorphism associated with feedback linearization. Our equation error is obtained from the Lyapunov equation. Though algorithms in [8] and [2] are presented only for feedback linearization, they can be extended to the assumption US case (see [6]). But applying this controller to our example, there is actually no proof of global stabilization whatever k and j are chosen if Δ in (30) cannot be guarenteed not positive. This follows from the fact that robustifying the controller by increasing k, j leads to non globally Lipschitz nominal closed loop system (see [6]).

Opposed to the equation error design is the Lyapunov design as introduced by Parks [5]. It has been used by Talor et al. [13] and extended by Kanellakopoulos et al. [3] in the case of feedback linearizable systems. Again, extension to the assumption US case can be done (neglecting the $\frac{\partial V}{\partial p}$ term). The difference is that we can always guarentee boundedness of the parameter vector $\hat{p}$ in our algorithm whereas we do not know how to do so in the Lyapunov design if Δ cannot be made non positive.

A last design which can be compared to ours is proposed by Sastry and Isidori [9] in the case of no zero dynamics. The algorithm is based on an equation error from the $S_{p^\star}$ equation transformed by the diffeomorphism $\varphi(p^\star, x)$ associated with the output linearization. There is no Δ term in this case but global stabilization is established only for a globally Lipschitz regressor vector. This assumption is not met in our example.

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