Oscillatory Behaviour and Fixes in Adaptive Linear Control: A Worked Example

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Modern adaptive linear controllers guarentee solution boundedness under more realistic assumptions. But "bounded" does not mean "stable" or "satisfactory". Typically, in non ideal conditions, one may observe unacceptable oscillatory solutions. This paper is devoted to the study of a simple adaptive proprotional controller in feedback with a disturbed first order linear system. We explain the mechanism creating these oscillations and we analyze how fixes — dead zone, persistent excitation or internal model — deal with this problem. We establish the necessary knowledge required by each of these fixes to be efficient. We precise the type of convergence. We estimate asymptotic performance criteria — mean value, mean square value and sup value. This analysis is done by exhibiting critical elements — equilibrium points or periodic solutions - proving (or conjecturing) their attractiveness.

1 INTRODUCTION

Modern adaptive linear controllers guarentee solution boundedness under more realistic assumptions. In this paper we are interested in the simplest controller, namely the following adaptive proportionnal controller intended to regulate the plant output y around zero:

$$\left.\begin{array}{rcl} s(k)^2 & = & \mu^2 s(k-1)^2 + \frac{u(k)^2+y(k)^2}{\lambda} \\ \theta_1(k) & = & \theta(k) + \dfrac{y(k)\, y(k+1)}{max\{1, s(k)^2\} + y(k)^2} \\ \theta(k+1) & = & \theta_a + min\left\{1, \frac{R}{|\theta_1(k)-\theta_a|}\right\}(\theta_1(k) - \theta_a) \\ u(k) & = & -\theta(k) y(k) \end{array}\right\} \quad (1)$$

Applying Theorem 2 of (Praly [1986]), it provides solution boundedness when placed in feedback with any plant whose input-output signals satisfy for example:

$$A(q)y(k) = B(q)u(k-1) + d(k) \quad (2)$$

where d is any bounded sequence, A and B are proper rational fractions in the forward shift operator q, with A monic and:

$$\inf_{|\theta-\theta_a|\le R} \left\{ \sup_{|z|\le\mu} \left|1 - z^{-1}\theta - \frac{A(z)}{B(z)}\right|^2 \right\} < \frac{1}{1+\lambda} \quad (3)$$

We even know that the mean square value of $\frac{y}{s}$ is bounded by a constant times $\sup_k\{|d(k)|\}$. Unfortunately, neither boundedness nor this mean square performance give any information on stability or convergence. Indeed, if the plant satisfies:

$$y(k+1) = ay(k) + u(k) + d \quad (4)$$

with d a constant disturbance and a smaller than $\theta_a + R - 1$, the solutions exhibit more or less rapidly the oscillatory behavior depicted on Figure 1. We could call this phenomenum self-oscillations to emphasize that they are not induced by a forcing term but simply generated by the autonomous nonlinear system (1),(4).

These self-oscillations have been described by Egardt [1979] and Anderson [1985] and received an heuristic explanation by Macchi and Jaidanne [1986]. Praly and España [1987] have given a theoretical support to this explanation.

In this paper, we briefly review the explanation of this behavior (Section 2) and we study how fixes such as persistent excitation (Section 3), dead zone (Section 4) and internal model (Section 5) handle this problem. Being interested in regulation of y around zero, we evaluate performance with the following criteria:

$$I_a = \left|\lim_{T\to\infty} \frac{1}{T}\sum_{k=0}^{T} y(k)\right| \quad (5)$$

$$I_2 = \limsup_{T\to\infty} \frac{1}{T}\sum_{k=0}^{T} |y(k)|^2 \quad (6)$$

$$I_\infty = \limsup_{T\to\infty} |y(k)| \quad (7)$$

Would a be known, the best (while preserving stability) proportional linear controller would give:

$$I_a = \frac{|d|}{2}, \quad I_2 = \frac{d^2}{4}, \quad I_\infty = |d| \quad (8)$$

To allow theoretical arguments to be developped without too much complexity, we restrict ourselves with simpler systems. It is however important to mention that our conclusions confirm observations from real life experiments, at least in their qualitative view point.

2 ADAPTIVE P CONTROLLER

To understand the self-oscillations observed for the solutions of (1),(4), we simplify this system by looking at it through a window. We see on Figure 1 that $\theta - a$ evolves around 1, y remains of the order of d and therefore s^2 remains of the order of $\dfrac{d^2(1+(1+a)^2)}{\lambda(1-\mu^2)}$. This motivates the choice of the following window:

$$\mathcal{W} = \left\{ (y, s, \theta) \;\middle|\; s \le 1,\ |\theta - \theta_a + \frac{y\,(d-(\theta-a)y)}{1+y^2}| \le R \right\} \quad (9)$$

Restricted to $\mathcal{W}$, the system (1),(4) is, with $\varphi = \theta - a$:

$$\left.\begin{aligned} y(k+1) &= -\varphi(k)y(k) + d \\ \varphi(k+1) &= \varphi(k) + \frac{y(k)\,(d - \varphi(k)y(k))}{1+y(k)^2} \end{aligned}\right\} \quad (10)$$

where we omit the equation in s since it has no influence on the remaining part of the system and its solutions are easily obtained from those of (10). Notice that there is no approximation but simply a clipping procedure and the "clipped" controller is nothing but the proportional adaptive controller described by Goodwin and Sin [1984].

We can assume d to be positive (if not transform d in $-d$ and $y(k)$ in $-y(k)$). All our simulation results will be presented for $d = 0.1$.

Now, let us look at Figure 2, the phase portrait of (10) when d is zero (i. e. the ideal case) and showing only one point out of two. Two important facts to be noticed:

• $\mathcal{S}_0 = \{(\varphi, y) \mid y = 0\}$ is a set of equilibrium points and therefore an invariant set. It is exponentially attractive for $|\varphi|$ smaller than and away from 1 and exponentially repulsive for $|\varphi|$ larger than and away from 1. In more rigorous terms, this set is locally normally hyperbolic (see Hirsch et al. [1976]).

• The solutions make "U-turn" around either $(1,0)$ or $(-1,0)$:

- We first observe an escape from the repulsive part of the invariant set $\mathcal{S}_0$ with an exponential growth of the y-component.

- Secondly, as the y-component becomes larger, the φ-component is more exponentially contracted, forcing it to enter the domain $\{\,|\varphi| < 1\,\}$.

- Finally, the solution being in the attraction domain of the attractive part of $\mathcal{S}_0$, it runs exponentially fast to it and freezes there, since no motion is allowed on this set.

To go to the case d non zero, we remark:

• A typical property of normally hyperbolic invariant sets is the permanence of their existence and their normal hyperbolicity property in presence of small perturbations (see Hirsch et al. [1976] for an example of a rigorous statement of such a property).

• For large $y(k)$ the map $(\varphi(k), y(k)) \to (\varphi(k+1), y(k+1))$ is nearly unaffected by d.

For d small enough, España and Praly [1988] (see also Praly, España [1987]) have established the existence of invariant sets $\mathcal{S}_{dr}$ and $\mathcal{S}_{da}$, defined at least for $|\varphi|$ away from 1 and on $\{|\varphi| > 1\}$ and $\{|\varphi| < 1\}$ respectively, close to $\mathcal{S}_0$. As $\mathcal{S}_0$, $\mathcal{S}_{dr}$ is repulsive and $\mathcal{S}_{da}$ is attractive . As a consequence, the solutions escape from $\mathcal{S}_{dr}$. But, since this escape is only possible by an increase of $|y(k)|$, the second remark above tells us that the solutions behave as for the unperturbed case and enter the domain $\{\,|\varphi| < 1\,\}$, where they are attracted to $\mathcal{S}_{da}$. Moreover, $\mathcal{S}_{dr}$ and $\mathcal{S}_{da}$ are graphs of functions M_{dr} and M_{da}, respectively:

$$\mathcal{S}_{dr} = \{(\varphi, y) \mid y = M_{dr}(\varphi)\,\}\ ,\ \mathcal{S}_{da} = \{(\varphi, y) \mid y = M_{da}(\varphi)\,\} \quad (11)$$

and, for some η depending on d,

$$\frac{1}{d^2}\left[M_{da}(\varphi) - \frac{d}{1+\varphi}\right] \text{ is bounded on } \{|\varphi| < 1 - \eta\}$$

It follows that the motion on $\mathcal{S}_{da}$ can be approximated, up to a term of order d^4, as follows:

$$\left.\begin{aligned} y(k+1) &= -\varphi(k)y(k) + d \\ \varphi(k+1) &= \varphi(k) + d^2\frac{1}{(1+\varphi(k))^2} + O(d^4) \end{aligned}\right\} \quad (12)$$

This implies that the solutions, when they remain in the neighborhoud of $\mathcal{S}_{da}$, have their φ-component strictly, though slowly, increasing. This explains why they have to leave the set $\{|\varphi| < 1\}$.

Besides the torsion of the solutions around the point $(-1,0)$ where $\mathcal{S}_0$ was not hyperbolic, the main difference between the ideal case and this case "small non-zero d" is this slow drift which completes into "circles" the fast "U-turns" around $(1,0)$ of the unperturbed case (see Figure 3). A "U-turn" corresponds to a burst in Figure 1 while the slow drift corresponds to a quiet period.

To complete this analysis we have to notice the existence of two period 2 solutions:

$$y = 0 \text{ or } d\ ,\quad \varphi = 1 \quad (13)$$

For these limit solutions, we have the following values for the criteria mentionned in the Introduction:

$$I_a = \frac{d}{2}\ ,\ I_2 = \frac{d^2}{2}\ ,\ I_\infty = d \quad (14)$$

These solutions are not exponentially attractive but a Hopf bifurcation analysis leads us to think that they are attractive, at least for d small enough. On the other hand, Praly and España [1987] have established that the solutions, lying in the set $\mathcal{S}_{dr}$, go to infinity and therefore leave the window $\mathcal{W}$.

From this, we conjecture:

• The above values of the criteria hold for all but solutions entering $\mathcal{S}_{dr}$.

• The self-oscillations of Figure 1 are only transitory towards a period 2 oscillatory behavior. However, they do not disappear exponentially fast.

This analysis leads us to propose to classify the remedies to the self-oscillations into two classes:

• Those - dead-zone or persistent excitation - which prevent the drift stage by creating exponentially stable limit solutions .

• Those - internal model - trying to reduce the y-component of the limit solutions lying in a neighborhoud of the boundary of the y-stability domain $\{|\varphi| = 1\}$.

3 PERSISTENT EXCITATION

The structural instability of the system (10) for $d = 0$ can be conjectured from the existence of the continuous set of equilibrium points $\mathcal{S}_0$. As proposed by Anderson [1982], for example, we can try to change this situation by isolating an exponentially stable limit solution. In such a circumstance, as a particular case of hyperbolic set, this limit solution and its stability property are preserved under perturbation. In practice, this may be realized by introducing a dither r. For our study, we consider the case:

$$r(k) = \Re\{rz^k\}\ ,\quad z = \exp(\frac{2in\pi}{N}) \neq \pm 1 \quad (15)$$

with r a complex number and n, N integers. This dither is introduced in the control law by transforming the proportional into a dead-beat controller:

$$u(k) = -\theta(k)y(k) + r(k) \quad (16)$$

In this case, (10) becomes:

$$\left.\begin{aligned} y(k+1) &= -\varphi(k)y(k) + d + r(k) \\ \varphi(k+1) &= \varphi(k) + \frac{y(k)\,(d - \varphi(k)y(k))}{1+y(k)^2} \end{aligned}\right\} \quad (17)$$

For d small enough, the only periodic solutions which remain bounded as d goes to zero, with $\frac{|r|}{d}$ remaining fixed are (see Praly , Pomet [1987]):

- If $d^2 < \frac{|r|^2}{1+\Re\{z\}}$ (see Figure 4), a single period N solution, approximated by:

$$\varphi(k) = \varphi^* + O(d^2) \tag{18}$$

$$y(k) = \frac{d}{1+\varphi^*} + \Re\left\{\frac{r}{z+\varphi^*}z^k\right\} + O(d^2) \tag{19}$$

where φ^*, lying in $]0,1[$, is the unique solution of:

$$\frac{|r|^2}{2|z+\varphi^*|^2} = \frac{d^2}{\varphi^*(1+\varphi^*)^2} \tag{20}$$

It is exponentially stable, with the following characteristic multipliers:

$$\varphi^{*N} + o(1)\ ,\ 1 - d^2\frac{2N}{(1+\varphi^*)^3}\left(1+\frac{d^2(1-\varphi^*)^2}{\varphi^{*2}|r|^2(1+\varphi^*)^2}\right) + o(d^2)$$

For this solution our performance criteria are:

$$I_a = \frac{d}{1+\varphi^*} + O(d^2) \tag{21}$$

$$I_2 = \frac{d^2}{\varphi^*(1+\varphi^*)} + O(d^3) \tag{22}$$

$$I_\infty = \frac{d}{1+\varphi^*}\left(1+\sqrt{\frac{2}{\varphi^*}}\right) + O(d^2) + O(\frac{1}{N}) \tag{23}$$

- If $d^2 > \frac{|r|^2}{1+\Re\{z\}}$ (see Figure 5), a period N unstable solution satysfying (18)-(20), with φ^* lying in $]1,\infty[$, and a pair of even period (N or $2N$) solutions given by:

$$\varphi(k) = 1 + O(d^2) \tag{24}$$

$$y(k) = \frac{d}{2} + \Re\left\{\frac{rz^k}{z+1}\right\} \pm \frac{(-1)^k}{2}\sqrt{d^2 - \frac{|r|^2}{1+\Re\{z\}}} + O(d^2) \tag{25}$$

These solutions have of the same characteristics as the period two solutions of the unmodified case. In particular, it seems from simulations that they are foci and at least one of them is (weakly) attractive. They lie in a neighborhoud of the boundary of the y-stability domain $\{\varphi = 1\}$ and consequently we have an oscillatory transient.

We conclude that, to eliminate the self-oscillations by using a dither $\Re\{rz^k\}$, r and z should satisfy:

$$d^2 < \frac{|r|^2}{1+\Re\{z\}} \tag{26}$$

This shows that to be efficient an upperbound on d has to be known. On the other hand, as far as our performance criteria are concerned, φ^* should be as large as possible and therefore $\frac{|r|^2}{1+\Re\{z\}}$ should be closer to d^2. But in this case the attractiveness is weaker and the transient longer. In any case, when compared with the unmodified controller, persistent excitation replaces self-oscillations by forced oscillations. This gives a faster transient towards worse performance criteria.

4 DEAD-ZONE

In Section 2, we mentionned that, when compared with the case $d = 0$, the self-oscillations result from a slow drift of the φ-component, observed when the solution is in the set $\{|\varphi| < 1\}$. Hence, freezing the parameter when it starts drifting should solve our problem.

How can we get the information that the slow drift has started?

We noticed that the drift takes place when the solution lies in a neighborhoud of the set $\mathcal{S}_{da}$. Moreover, this set can be approximated, for d small enough, by the graph:

$$y = \frac{d}{1+\varphi} \tag{27}$$

Consequently, a necessary condition for the drift to take place is that the y-component be of the order of the disturbance d, i. e. small. Therefore we can think of using a dead-zone technique for realizing our freezing process. Several types of dead-zone algorithms have been proposed in the litterature (Egardt [1979], Praly, Redjah [1982]). Here, we choose the simplest to analyse. Restricted to the window $\mathcal{W}$ of Section 2, this gives:

$$\varphi(k+1) = \begin{cases} \varphi(k) + \frac{y(k)y(k+1)}{1+y(k)^2} & \text{if } |y(k+1)| > \delta \\ \varphi(k) & \text{if } |y(k+1)| \le \delta \end{cases} \tag{28}$$

with δ a positive real number to be determined. This dead-zone makes the closed-loop system to switch between two systems:

- the system (10) studied in Section 2,
- the following, made of a family of linear systems indexed in φ:

$$\left.\begin{aligned} y(k+1) &= -\varphi y(k) + d \\ \varphi(k+1) &= \varphi \end{aligned}\right\} \tag{29}$$

Its only limit sets are:

$$\mathcal{S}_{za} = \{(\varphi,y) \mid y = \frac{d}{1+\varphi} \text{ and } |\varphi| < 1\} \tag{30}$$

$$\mathcal{S}_{zc} = \{(\varphi,y) \mid \varphi = 1\} \tag{31}$$

$$\mathcal{S}_{zr} = \{(\varphi,y) \mid y = \frac{d}{1+\varphi} \text{ and } |\varphi| > 1\} \tag{32}$$

$\mathcal{S}_{za}$ and $\mathcal{S}_{zr}$ are sets of equilibrium points, attractive and repulsive respectively. $\mathcal{S}_{zc}$ is a set of period 2 solutions. For this system, the performance criteria depend on φ and make sense only for $|\varphi| < 1$:

$$I_a = \frac{d}{1+\varphi} \quad I_2 = \frac{d^2}{(1+\varphi)^2}\ ,\ I_\infty = \frac{d}{1+\varphi} \tag{33}$$

In the set $\{|\varphi| < 1\}$, both (10) and (29) are making the solutions to converge close to $\mathcal{S}_{za}$. And, since for $(\varphi(k), y(k))$ in $\mathcal{S}_{za}$, (10) makes $\varphi(k)$ strictly increasing, we guess that for the dead-zone to be of any interest, the intersection: $\mathcal{S}_{za} \bigcap \{(\varphi,y) \mid |y| \le \delta\}$ must be non empty (see Figure 6). Praly [1988] has established that for d small enough and $\delta < \frac{d}{2}$, there exists η, $0 < \eta < 1$, for which no solution of the closed-loop system with dead-zone satisfies for all k:

$$|\varphi(k)| \le 1 - \eta \tag{34}$$

Also, to prevent the solutions of (10) from (when looked at one point out of two) "circling" around $(1, d)$, which lies in $\mathcal{S}_{zc}$, this point should belong to the dead-zone $\{(\varphi,y) \mid |y| \le \delta\}$ (notice that the other point $(1,0)$ is always in this dead-zone). Indeed Praly [1988] has proved that there is no solution leaving the set $\{|\varphi| \le 1\}$ if and only if $\delta \ge d$.

We conclude that δ should be chosen larger than d for the dead-zone modification to eliminate the self-oscillations.

In this case, from our knowlege on the systems(10) and (29), we can guess that the dead-zone has made the points of $\mathcal{S}_{za} \bigcap \{(\varphi, y) \mid |y| \leq \delta\}$ exponentially stable equilibria (see Figure 7).

This leads us to conjecture an exponential convergence to one of the following values for the performance criteria:

$$\frac{d}{2} \leq I_a \leq \delta \ , \ \frac{d^2}{4} \leq I_2 \leq \delta^2 \ , \ \frac{d}{2} \leq I_\infty \leq \delta \tag{35}$$

except for solutions with initial value satisfying:

$$\varphi(0) = 1 \quad , \quad d - \delta \leq y(0) \leq \delta \tag{36}$$

Consequently, compared with the unmodified controller, we have a faster convergence to, depending on the initial conditions, possibly better, possibly worse performance criteria.

5 INTERNAL MODEL

From linear time invariant systems theory, the internal model principle tells us that the disturbance model should be introduced in the controller. In Section 2 we remarked that, for d small enough and after a finite time, the solutions in $\{|\varphi| < 1\}$ have increments for their φ-component which can be approximated by $d^2 \dfrac{1}{(1+\varphi(k))^2}$. It follows that the y-subsytem looks like a slowly time varying linear system as soon as $\varphi(k)$ is far away from -1. Therefore, we may hope the internal model principle to be applicable in our case, at least locally. As proposed by Elliott and Goodwin [1984]), we modify our adaptive controller in:

$$\left.\begin{array}{rcl} s(k)^2 &=& \mu^2 s(k-1)^2 + \frac{u_f(k)^2 + y(k)^2}{\lambda} \\ \theta_1(k) &=& \theta(k) + \dfrac{y_f(k)\, y(k+1)}{max\{1, s(k)^2\} + y_f(k)^2} \\ \theta(k+1) &=& \theta_a + min\left\{1, \frac{R}{|\theta_1(k) - \theta_a|}\right\}(\theta_1(k) - \theta_a) \\ u_f(k) &=& -\theta(k) y_f(k) + (y_f(k+1) - y(k+1)) \end{array}\right\} \tag{37}$$

with the subscript f for a sequence v standing for:

$$v_f(k) = F(q^{-1}) v(k) \tag{38}$$

where F, a polynomial in the unit delay operator, is supposed to be the annihilator of the disturbance. Since the closed-loop system order is increased by the degree of the polynomial F, simplicity imposes the choice:

$$F(q^{-1}) = 1 - f q^{-1} \tag{39}$$

Practically, this makes sense only for $f = \pm 1$. Other values of f will give us more insight on the effect of introducing this polynomial.

Restricted to the window:

$$\left\{(y, y_f, s, \theta) \,\middle|\, s \leq 1\,, \, |\theta - \theta_a + y_f \frac{d(1-f) - (\theta - a) y_f}{1 + y_f^2}| \leq R\right\}$$

the closed-loop system is, omitting the s-equation:

$$\left.\begin{array}{rcl} y(k+1) &=& -\varphi(k) y_f(k) + d(1-f) \\ y_f(k+1) &=& -\varphi(k) y_f(k) - f y(k) + d(1-f) \\ \varphi(k+1) &=& \dfrac{\varphi(k) + d(1-f) y_f(k)}{1 + y_f(k)^2} \end{array}\right\} \tag{40}$$

This system is very similar to (10), replacing d by $d(1-f)$, y by y_f and adding the component y. We (briefly) follow the same steps as in Section 2:

• For $f = 1$ (see Figure 8), $\{(\varphi, y_f, y) \mid y_f = y = 0\}$ is a set of equilibrium points which is exponentially attractive for φ in $]-1, \frac{1}{2}[$ and away from the boundary values and exponentially repulsive for φ outside this set, away from its boundary values. Elliott and Goodwin [1984] have established that performance criteria are zero.

• For $f \neq 1$ (see Figure 9), as for (10), we predict a phase portrait very similar to the case $f = 1$, except for a slow drift of the solutions when their φ-component lies in the stability set of the linear (y_f, y)-subsystem, i. e. $f\varphi$ in $]-1, 1-|\varphi|[$. Applying the results of Praly [1987], for $d^2(1-f)^2$ small enough, there exist η and a locally attractive locally invariant set defined as

$$\{(\varphi, y_f, y) | \, y_f = L_f(\varphi), \, y = L(\varphi), \, f\varphi \in]1 - \eta, -1 + \eta - |\varphi|\,[\}$$

where the functions L_f and L are such that

$$\left|L_f(\varphi) - \frac{d(1-f)^2}{1 + \varphi(1-f)}\right| + \left|L(\varphi) - \frac{d(1-f)}{1 + \varphi(1-f)}\right|$$

divided by $d^2(1-f)^2$, is bounded on $\{\varphi \,|\, f\varphi \in]-1+\eta, 1-\eta-|\varphi|\,[\,\}$. The same type of results for $f\varphi$ outside $]-1-\eta, 1+\eta-|\varphi|\,[$ should hold with locally repulsive locally invariant sets, depending on how many poles of the linear (y_f, y)-subsystem are outside the unit circle. As for the case $f = 1$, we may expect the solutions to be rejected by these latter invariant sets. This implies a stronger contraction of their φ-component which makes them enter the set $\{\varphi | f\varphi \in]-1+\eta, 1-\eta-|\varphi|[\}$. They are later attracted to the former invariant set where their φ-component start moving as:

$$\varphi(k+1) = \varphi(k) + \frac{d^2(1-f)^3}{(1 + \varphi(1-f))^2} + O(d^4(1-f)^4) \tag{41}$$

The increments being strictly non zero, this forces the solutions to reenter the domain where they are again rejected by the repulsive invariant sets. Consequently, at least a self-oscillatory transient of the same type as for the unmodified controller can be predicted.

For d small enough, the only periodic solutions which remain bounded as $d(1-f)$ goes to zero with f fixed, are obtained as follows:

• For $f < -\frac{1}{2}$ or $1 < f$, they have period N, with $N \neq 2$, if and only if the following complex number:

$$z = \frac{1 + i\sqrt{4f^2 - 1}}{2f} \tag{42}$$

is an Nth root of unity. In this case, they can be approximated by

$$\varphi(k) = -\frac{1}{f} + O(d^2(1-f)^2) \tag{43}$$

$$y_f(k) = \frac{df(1-f)^2}{2f-1} + \Re\left\{bz^k\right\} + O(d^2(1-f)^2) \tag{44}$$

$$y(k) = \frac{df(1-f)}{2f-1} + \Re\left\{\frac{bz}{z-f} z^k\right\} + O(d^2(1-f)^2) \tag{45}$$

where b is a complex number whose modulus satisfies:

$$|b|^2 = \frac{2d^2 f^3 (f-1)^3}{(2f-1)^2} \tag{46}$$

For these solutions, the performance criteria are:

$$I_a = \frac{df(f-1)}{|2f-1|} + O(d^2(1-f)^2) \tag{47}$$

$$I_2 = \frac{d^2 f(f-1)^2}{(2f-1)} + O(d^3(1-f)^3) \tag{48}$$

$$I_\infty = \frac{d(f-1)}{2f-1}\left(f + \sqrt{2f(f-1)}\right) + O(d^2) + O(\frac{1}{N}) \tag{49}$$

• For $-\frac{1}{2} < f < 1$, they have period 2 and can be approximated by:

$$\varphi(k) = \frac{1}{1+f} + O(d^2(1-f)^2) \tag{50}$$

$$\frac{y_f(k)}{d(1-f)} = \frac{1+f}{2}\left(1-f \pm (-1)^k\sqrt{1-f^2}\right) + O(d(1-f)) \tag{51}$$

$$\frac{y(k)}{d(1-f)} = \frac{1}{2}\left(1+f \pm (-1)^k\sqrt{1-f^2}\right) + O(d(1-f)) \tag{52}$$

For these solutions, the performance criteria are:

$$I_a = \frac{d(1-f^2)}{2} + O(d^2(1-f)^2) \tag{53}$$

$$I_2 = \frac{d^2(1-f^2)(1-f)}{2} + O(d^3(1-f)^3) \tag{54}$$

$$I_\infty = \frac{d(1-f)}{2}\left(1+f+\sqrt{1-f^2}\right) + O(d^2(1-f)^2) \tag{55}$$

These solutions have of the same characteristics as the period two solutions of the unmodified case. They lie in a neighborhoud of the boundary of the linear (y_f, y)-subsystem stability domain.

We conclude that to eliminate the self-oscillations a perfect model of the disturbance must be known. If such a condition is not met, we conjecture the existence of self-oscillations similar to those of the unmodified case. And for the performance criteria, Figure 10 gives a comparison of the value of I_2 given by (48) for $f < -\frac{1}{2}$ or $1 < f$ and by (54) for $-\frac{1}{2} < f < 1$, on the one hand, and what we have obtained by simulations, on the other hand. We see that, for a range of values of f around the optimal $f = 1$, performance is improved compared with the unmodified case ($f = 0$).

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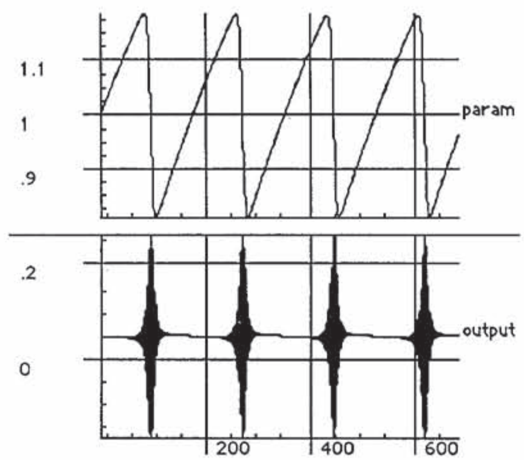

Figure 1: $\theta(k) - a$ and $y(k)$ in (1),(4)

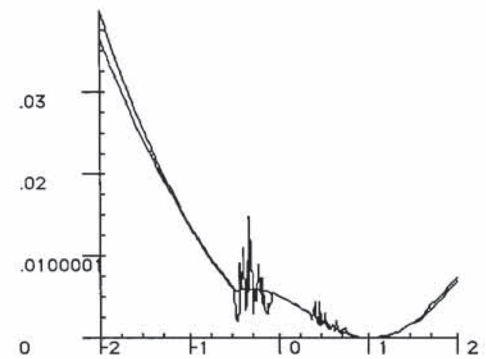

Figure 10: Theoretic and simulated I_2 criteria versus f

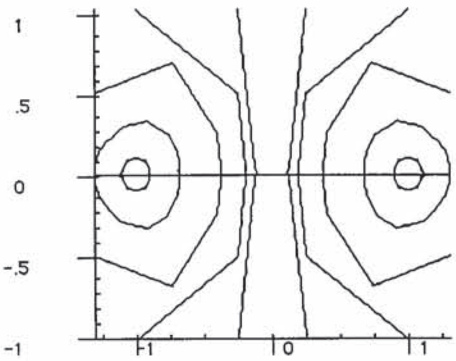

Figure 2: y versus φ in (10) for $d = 0$

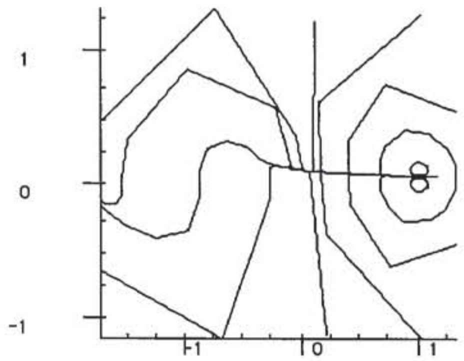

Figure 3: y versus φ in (10) for $d \neq 0$

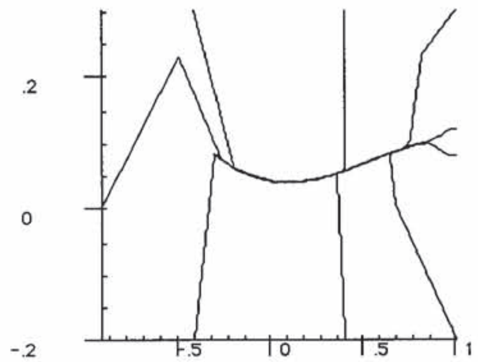

Figure 4: y versus φ in (17) (1 pt /6) for $\varphi^* = 0.5$, $z = \exp(\frac{2i\pi}{3})$

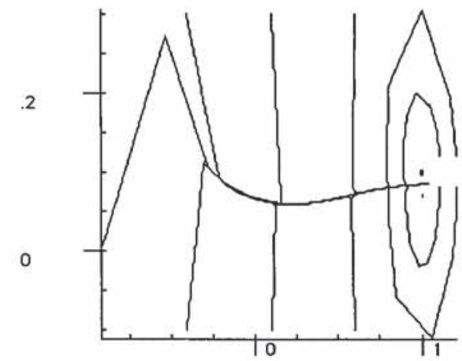

Figure 5: y versus φ in (17) (1pt/6) for $\varphi^* = 1.1$, $z = \exp(\frac{2i\pi}{3})$

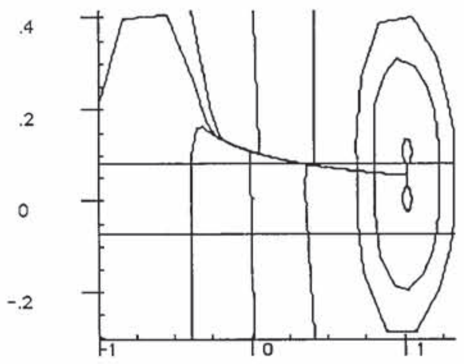

Figure 6: y versus φ in (28) for $\delta = \frac{3d}{4}$

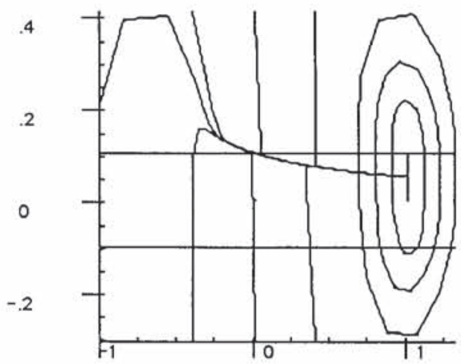

Figure 7: y versus φ in (28) for $\delta = d$

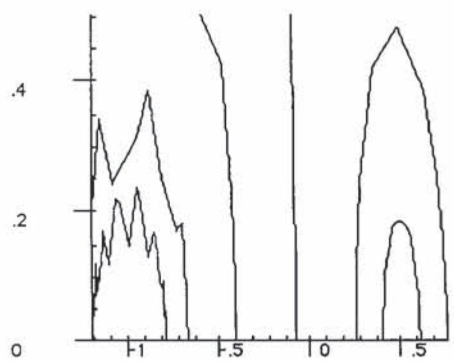

Figure 8: $\| (y_f, y) \|$ versus φ in (40) for $f = 1$

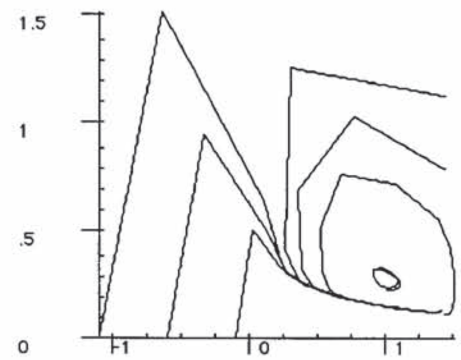

Figure 9: $\| (y_f, y) \|$ versus φ in (40) (1pt/3) for $f = -1$