Proceedings of the 26th Conference
on Decision and Control
Los Angeles, CA • December 1987

A ROBUST ADAPTIVE MINIMUM VARIANCE REGULATOR

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ABSTRACT

In this paper we investigate the twin issues of *robustness* as well as *performance* in the adaptive control of linear, stochastic systems. Towards this end we present an adaptive controller which provides *optimal performance* under *ideal conditions*, while providing *stable behavior* when the idealness assumptions are violated. Specifically, our adaptive controller has the following properties:

(i) When the system under control is of the order assumed, with delay also as assumed, is of minimum phase, and the disturbance satisfies a positive real condition on its spectrum, then the mean-square tracking error is *optimal.*

(ii) When the positive realness condition on the disturbance is violated, then the system remains mean square stable.

(iii) When the system violates the "ideal" conditions and is merely in a graph-topological neighborhood of systems satisfying (i), then the adaptive controller still provides mean square stability.

(iv) When the adaptation gain is made non-vanishing, then stability is still preserved.

I. INTRODUCTION

In recent years much progress has been made on the study of adaptive controllers for linear stochastic systems.

In 1981, Goodwin, Ramadge and Caines [1] were able to exhibit a family of stochastic gradient based adaptive controllers which were *self-optimizing* in the sense that the mean square tracking error was optimal, i.e., equal to the minimum mean square error attainable if the true system were known to start with. In 1985, Becker, Kumar and Wei [2] were established to establish the *self-tuning* property for the *regulation* problem, i.e., the parameter estimates were shown to so converge that the resulting asymptotic regulator is optimal. Recently, in 1987, Kumar and Praly [3] exhibited adaptive controllers for the *tracking problem* which also possess the self-tuning property. They also showed how to construct such self-tuning trackers for importance classes of problems, such as the case of maintaining a given *set-point*, when the reference trajectory is insufficiently exciting.

Inevitably, these *exact* results are dependent on the true system under control satisfying some assumptions. Typically these assumptions are that:

(i) the *order* of the system is as assumed,

(ii) the *delay* of the system is known,

(iii) the disturbance entering into the system is a stochastic process satisfying a certain *positive-real* condition,

(iv) the system satisfies a *minimum-phase* condition.

The first two assumptions allow us to choose a structure for the controller, so that the parameters within this structure can then be tuned to asymptotic optimality. The third assumption above is essentially a *pseudo-gradient* assumption which guarantees that the adaptation mechanism tunes the parameters in the right direction. Such a positive-real condition arises in recursive identification; see Solo [4], Ljung and Soderstrom [5] and Kumar and Varaiya [6]. The last assumption above arises in minimum-variance control using a stationary (i.e. time-invariant) control law, and in such a case it is needed to guarantee that while the output variance of the system is minimized, the input sequence does not blow up. (However, as we show in the more detailed version of this paper [7], for some non-minimum phase systems one can achieve minimum variance control with finite power input when the controller is adaptive, and thus time varying.)

When adaptive controllers are used in practice, it is however inevitable that the so-called "idealized" assumptions will not be satisfied by the true system under control. For example, the true system may even be infinite-dimensional. Beginning with Ioannou and Kokotovic [8] and Rohrs, Valavani, Athans and Stein [9], the role of these assumptions has therefore come under increasing scrutiny, and the topic of robust adaptive control has attracted much research attention.

In this paper we pay simultaneous attention to both *performance* as well as *robustness*. Specifically, one would not want to achieve robustness by sacrificing performance.

We formulate these twin goals as follows. First we would like our adaptive controller to achieve optimal performance when the "idealized" assumptions are satisfied, thus ensuring that nominal behavior is good. Second, we would like the adaptive controller to be robust, i.e. stable, when applied to as large a class of systems as possible.

In attempting to make precise what is meant by a "large class of systems" we consider the *graph topology* on the space of linear systems; see Vidyasagar [10]. This is the *weakest* topology under which a constant linear feedback controller preserves stability when small perturbations from the nominal system, with respect to this topology, are allowed, and the frequency response changes only slightly (in "sup" norm). Viewing our adaptive controllers as conducting a real time search over the space of linear feedback controllers, we require that our adaptive controller preserves stability when the true system lies in a graph topological neighborhood of the set of "ideal" systems.

In the next section we describe an adaptive controller which has the performance and robustness properties described in the Abstract.

II. THE ADAPTIVE CONTROLLER

We choose three integers n_R, n_s and n_c which describe the dimensions of our adaptive controller, and an integer $d \geqslant 1$ to describe the "nominal" delay. In addition, we choose a "nominal" parameter vector θ^c, two positive numbers $0 < \lambda_0 < \lambda_1$ which will serve as bounds on the eigenvalues of the "covariance matrix," and three further positive integers P, σ_0 and K. All these are *a priori* constants chosen in the "design" phase.

Let u and y be the input and output of the system to be controlled, and let y^m be a bounded reference trajectory which we want the output y of the system to follow as closely as possible.

Our adaptive controller is then defined recursively by the

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following equations. Let

$$\phi(t) := (u(t),\ldots,u(t-n_s),y(t),\ldots,y(t-n_R),$$

$$y^m(t+d-1),\ldots,y^m(t+d-n_c))^T$$

be the "regression vector," and $\rho(t)$ defined by,

$$\rho(t) := \rho(t-1) + \max(\rho,\|\phi(t-d)\|^2)\ ,\ t \geqslant 1$$

be a "normalizing sequence." Set,

$$\bar{\phi}(t-d) := \frac{\phi(t-d)}{\rho^{1/2}(t)}\ ,$$

$$g(t) := \frac{1}{1+\bar{\phi}^{T}(t-d)F(t-d)\bar{\phi}(t-d)}$$

$$e(t) := y(t)-\theta^T(t-d)\phi(t-d)$$

and

$$\bar{e}(t) := \frac{e(t)}{\rho^{1/2}(t)}$$

where the "covariance matrix" $F(t)$ and the "parameter estimate" $\theta(t)$ are defined below,

$$F^1(t-d) := F(t-d)-g(t)\,F(t-d)\bar{\phi}(t-d)\bar{\phi}^T(t-d)F(t-d)\ ,$$

$$F(t) := (1-\frac{\lambda_0}{\lambda_1})F^1(t) + \lambda_0 I\ (\text{where } \lambda_0 I \leqslant F(0) \leqslant \lambda_1 I)\ ,$$

$$\theta^1(t) := \theta(t-d) + g(t)F(t-d)\bar{\phi}(t-d)\bar{e}(t)\ ,$$

$$\theta^2(t) := \theta^1(t) + \max(0,\sigma_0-s_0^1(t))\frac{F_1(t)}{F_{11}(t)}\ .$$

Above, $s_0^1(t) :=$ first element of the vector $\theta^1(t)$,
$F_1(t) :=$ first column of the matrix $F(t)$, and
$F_{11} := (1,1)$–th element of the matrix $F(t)$.
The parameter estimate $\theta(t)$ is then defined recursively by

$$\theta(t) := \theta^c + (\theta^2(t)-\theta^c)\min(1,\frac{K\lambda_1}{\lambda_0\|\theta^2(t)-\theta^c\|}\ .$$

Having obtained a parameter estimate $\theta(t)$ at time t, the control input $u(t)$ is chosen so that,

$$\theta^T(t)\phi(t) = y^m(t+d)\ . \tag{1}$$

This completes the description of the adaptive controller.

III. DISCUSSION OF THE ADAPTIVE CONTROLLER

The adaptive contoller described above incorporates three modifications from the standard "least squares parameter estimator plus certainty equivalent control" adaptive controllers.

First, the signals ϕ and e are normalized by the normalizing sequence $\rho^{1/2}$. The resulting normalized signals $\bar{\phi}$, $\bar{e}$, etc. are then utilized by the adaptive controller. The reason for doing this lies in the fact that when unmodeled dynamics are present, their contribution to the output is roughly comparable to $\rho^{1/2}$ in an l_2 sense; see also Egardt [11].

Second, the eigenvalues of the matrix $F(t)$ are kept in the interval $[\lambda_0,\lambda_1]$. This is also well motivated since it is known that the unbounded condition number of $F(t)$, as $t\to\infty$, can cause problems even in recursive parameter identification; see Lai and Wei [12] and Kumar and Varaiya [6].

Last, the parameter estimator $\theta(t)$ is forced to stay in a (large) sphere with center θ^c and radius $\frac{K\lambda_1}{\lambda_0}$, and the first component of $\theta(t)$, which corresponds to an estimate of the "high frequency gain" is kept bounded below by σ_0. This is motivated by the problem of keeping the parameter estimates in a bounded set, which is known to be important to control laws of the form (1); see Egardt [11].

Though all our modifications are well motivated and appear reasonable, it is an open question whether adaptive controllers without these modifications possess good performance and robustness properties in theory.

IV. MAIN RESULTS

Our first result shows that the inputs and outputs are mean-square bounded when the disturbance in a nominal plant is mean square bounded. Thus no statistical properties of the disturbance, or a positive real condition, are assumed here.

Theorem 1. Suppose the true system satisfies the following properties:

(i) $A(q^{-1})y(t) = q^{-d}B(q^{-1}) + d(t)$ where A and B are polynomials in the delay operator q^{-1}.

(ii) $\sup_T \frac{1}{T}\sum_{t=1}^{T} d^2(t) < \infty$,

(iii) All the zeroes of the polynomials B and C are strictly outside the unit circle.

(iv) There exist polynomials S^* and R^* of degrees n_s and n_R, respectively, such that
$$S^*(q^{-1})A(q^{-1}) + q^{-d}R^*(q^{-1})B(q^{-1}) = B(q^{-1})\ .$$

(v) The first component s_0^c of the vector θ^c and $S^*(0)$ are each $\geqslant \sigma_0$.

(vi) $\|\theta^c-\theta^*\| \leqslant K$ where $\theta^* := (s^*_0,s^*_1,\ldots,s^*_{n_R},r^*_0,\ldots,r^*_{n_R},0,0)^T$ where s^*_i and r^*_i are the coefficients of q^{-i} in the polynomials $S^*(q^{-1})$ and $R^*(q^{-1})$, respectively.

Then, the adaptive controller gives rise to inputs and outputs which satisfy,

$$\sup_T \frac{1}{T}\sum_{t=1}^{T}(y^2(t)+u^2(t)) < +\infty\ . \qquad \square$$

Our second result shows that the adaptive controller yields optimal performance, vis-a-vis the time average of the square of the tracking error, when the disturbance satisfies a positive real condition on its spectrum.

Theorem 2. Suppose that in addition to the properties in Theorem 1, the true system satisfies the following additional assumptions:

(i) The disturbance $d(t)$ is given by,
$$d(t) = C(q^{-1})w(t)\ .$$

(ii) $C(q^{-1})$ is a polynomial of degree n_c in the delay operator q^{-1}, with $C(0)=1$.

(iii) $\sup_\omega |C(e^{i\omega})-1| < \frac{1}{\sqrt{1+\lambda_1}}$.

(iv) $E(w(t)\,|\,y^{t-1},u^{t-1}) = 0$, $E(w^2(t)\,|\,y^{t-1},u^{t-1}) = \sigma^2$, $\sup_t E(\,|w(t)|^{2+\delta}|y^{t-1},u^{t-1}) < +\infty$ for some $\delta > 0$.

(v) There exist polynomials S, R and Q of degrees n_s, n_R and d-1, respectively, such that

$$S(q^{-1})A(q^{-1}) + q^{-d}R(q^{-1}) = C(q^{-1})B(q^{-1})$$

and

$$S(q^{-1}) = Q(q^{-1})B(q^{-1}) .$$

(vi) The vector

$$\theta^0 := (s_0,..,s_{n_R}, r_0, \ldots, r_{n_R}, -c_1, \ldots, -c_{n_c}) ,$$

where s_i, r_i and c_i are the coefficients of q^{-i} in the polynomials $S(q^{-1})$, $R(q^{-1})$ and $C(q^{-1})$, respectively, satisfies,

$$\|\theta^0 - \theta^c\| \leqslant K ,$$

and

$$s_0 \geqslant \sigma_0 .$$

Then, the time average of the square of the tracking error is,

$$\lim \frac{1}{T}\sum_{t=1}^{T}(y(t)-y^m(t))^2 = \sigma^2 \sum_{i=0}^{d-1} q_i^2$$

where q_i is the coefficient of q^{-i} in $Q(q^{-1})$. □

Our next result shows that the adaptive controller continues to provide mean-square stability whenever the true system is merely in a certain graph-topological neighborhood of the class of all "ideal" systems.

Theorem 3. Assume that the true system satisfies the following conditions:

(i) $A(q)y(t) = B(q)u(t-1)+d(t)$.

(ii) A and B are proper rational functions whose poles are in the open unit disk, and such that their norms, defined by $\|A\| = \sup_{|q| \geqslant 1} |A(q)|$, are bounded.

(iii) $\frac{B}{A}$ is a proper rational function.

(iv) $A(\infty) = 1$.

(v) The disturbance d(t) satisfies,

$$\sup_T \frac{1}{T}\sum_{t=1}^{T} d^2(t) < V$$

where $V < +\infty$.

(vi) Let $B(q) = \sum_{i=0}^{\infty} h_i q^{-i}$ and assume that both D(q) and $\frac{1}{D(q)}$ are proper rational functions with bounded norm, where D(q) is defined as,

$$D(q) := \sum_{i=0}^{\infty} h_{i+d-1} q^{-i} .$$

Then, there exists an open neighborhood 0 (which can be explicity specified, but which we omit here), of the set of ideal systems satisfying Theorem 2, such that if the true system $\frac{B}{A}$ lies within 0, then the inputs and outputs satisfy,

$$\sup_T \frac{1}{T}\sum_{t=1}^{T}(y^2(t)+u^2(t)) < +\infty . \quad \square$$

Our last result shows that if the adaptation gain is prevented from going to zero, so that the parameter estimates are ever-changing and open to "adaptation" always, then the adaptive controller still provides robust stability, appropriately defined.

Theorem 4. Choose $0<\mu<1$ and modify the adaptive controller defined in Section II by redefining the normalizing sequence $\rho(t)$ as,

$$\rho(t) = \mu^2\rho(t-1) + \max(\rho, \|\phi(t-d)\|^2) .$$

Also suppose that the disturbance d(t) in (i) of Theorem 3 satisfies,

$$\sup_t |d(t)| \leqslant V ,$$

where $V < +\infty$. Finally, replace the phrase, "proper rational function" by the phrase "proper rational function with poles in the open disk of radius μ," and the definition of norm by $\|A\| = \sup_{|q| \geqslant \mu} A(q)$. Then the result of Theorem 3 is sharpened to,

$$\sup_t (|y(t)| + |u(t)|) < +\infty . \quad \square$$

V. CONCLUDING REMARKS

It is an open question whether standard unmodified adaptive controllers possess good performance and robustness properties. However, at the present time it is not even known whether an adaptive controller with a least squares parameter estimator followed by a certainty equivalent control (i.e., the original self-tuning regulator of Aström and Wittenmark [13]) is stable, let alone optimal. An answer to this question still remaines elusive.

ACKNOWLEDGEMENTS

The research of the second and third authors has been supported by the National Science Foundation under Grant No. ECS-84-14676 and by the Joint Services Electronics Program under Contract No. N00014-85-C-0149.

REFERENCES

[1] Goodwin, G. C., P. J. Ramadge and P. E. Caines, "*Discrete time stochastic adaptive control*", SIAM J. Control and Optimization 19-6, 1981, pp 829-853.

[2] Becker, A. H., P. R. Kumar and C. Z. Wei, "*Adaptive control with the stochastic approximation algorithm:* geometry and convergence", IEEE Trans. AC-30 April 1985, pp 330-338.

[3] Kumar, P. R. and L. Praly, "*Self-tuning trackers*", SIAM J. Control and Optimization, vol. 25, no. 4, pp.1053-1071, July 1987.

[4] Solo, V., "*The convergence of AML*", IEEE Trans. AC-24, 1979, pp. 958-962.

[5] Ljung, L. and T. Söderström, "*Theory and practice of recursive identification*", MIT Press, Cambridge, 1983.

[6] Kumar, P. R. and P. Varaiya, "*Stochastic systems*", Prentice Hall, 1986.

[7] Praly, L., S.-F.Lin and P. R. Kumar, "*A Robust Adaptive Minimum Variance Controller*," submitted for publication, July 1987.

[8] Ioannou, P. and P. Kokotovic, "*An asymptotic error analysis of identifiers* and adaptive observers and in the presence of parasitics", IEEE Trans. AC-27, 1982, pp.921-927.

[9] Rohrs,C., L. Valavani, M. Athans and G. Stein, "*Analytical verification of undesirable properties of direct model*", Proc. of 20th IEEE Conf. on Decision and Control, 1981 pp 1272-1284.

[10] Vidyasagar, M., "*The graph metric for unstable plants and robustness estimates for feedback stability*", IEEE Trans. AC-29, May 1984, pp 403-418.

[11] Egardt, B., "*Stability of adaptive controllers*", Lecture Notes in Control and Information Sciences, Springer-Verlag, 1979.

[12] Lai, T. L. and C. Z. Wei, "*Least squares estimate in stochastic regression with applications to identification and control of dynamic systems*", Ann. Stat. 10, 1982, pp 154-166.

[13] Aström, K. J. and B. Wittenmark, "*On self-tuning regulators*", Automatica 9, 1973, pp 195-199.