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LOCAL ANALYSIS OF A ONE STEP AHEAD ADAPTIVE CONTROLLER*

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Abstract

A local analysis of the one step ahead adaptive controller of [1] is presented. We study the behavior of the closed loop adaptive system, in presence of periodic inputs, around a trajectory defined as the solution of a nominal system. We establish that if this solution is unique, it is uniformly asymptotically stable and we can tolerate a wide class of small perturbations from this nominal system.

1. Introduction

Consider the following adaptive controller [1].

$$\phi(k)' \equiv (u(k),\dots,u(k-m-d+1),y(k),\dots,y(k-n+1))$$

$$\hat{\theta}(k) = \hat{\theta}(k-1) + \frac{\gamma(k)\phi(k-d)(y(k)-\hat{\theta}(k-1)'\phi(k-d))}{1+\gamma(k)\phi(k-d)'\phi(k-d)} \quad (1.1)$$

$$\phi(k)'\hat{\theta}(k) = y_m(k+d) \quad (1.2)$$

where $\gamma(k)$ is a scalar gain, $y_m(k+d)$ is a reference output known d steps in advance and u(k), y(k) are the input and output of a plant which can be described by:

$$A(q^{-1})y(k) = q^{-1}B(q^{-1})u(k) + w(k) \quad (1.3)$$

$A(q^{-1})$ and $B(q^{-1})$ are polynomials of unknown degree. $\{w(k)\}$ is a sequence which may depend upon $\{u(k)\}$ and $\{y(k)\}$. The nominal plant is given when w(k) = 0.

Goodwin, Ramadge and Caines have shown in [2] if n, m are not less than the degree of $A(q^{-1}), B(q^{-1})$, respectively, d is the plant delay, the plant is minimum phase then the adaptive controller ensures global stability and the tracking error tends to zero for the nominal plant. Egardt in [3] extended this result for the plant with: $|w(k)| < w$. This was done by projecting the parameters into a compact set to guarantee their boundedness.

We have shown in [4] that Egardt's result can be extended to the case where:

$$|w(k)| < \delta s(k) + w$$

$$s(k) = \mu s(k-1) + \max(\|\phi(k-d)\|, s)$$

$$s > 0 \quad , \quad 0 \le \mu < 1$$

when δ is sufficiently small. This result holds using the projection technique of Egardt and taking:

$$\gamma(k) = \frac{1}{s(k)^2} \quad .$$

In [5] global stability is shown if there exists a constant parameter vector θ^* which gives in closed loop with the nominal plant a tracking transfer function, (which may be noncausal), having the following properties:

i) The Nyquist plot of its causal part lies entirely within a circle of radius 1 and center 1. This is a conicity condition.

ii) The noncausal part is sufficiently small.

It is important to note that the preceding results made no assumptions concerning the reference output. But, as an extension of the work of Anderson and Johnson [6], Kosut and Anderson [7], Astrom [8], Krause [9], Anderson, Bitmead, Johnson and Kosut [10], Riedle and Kokotovic [11], Kokotovic, Riedle and Praly [12], ... on the stability of error models, it has recently been established in [13] that if this signal is taken into account, then the classical conicity condition can be relaxed to a signal dependent conicity condition. This implies that in our case we can expect the following sufficient condition: The transfer function evaluated only at frequencies contained in $y_m(k)$ lies in a circle of radius 1 and center 1. However, this type of statement is only known for $\gamma(k)$ small. This corresponds to the case of slow adaptation.

In this paper we will study a problem similar to [4]. However, instead of assuming that there exists a constant parameter vector θ^* which gives for the nominal plant a tracking transfer function equal to 1 for any frequency, we now assume that this holds only at the frequencies contained in $y_m(k)$. This is not possible in general, but is possible for some particular sequences $\{y_m^*(k)\}$ called test reference outputs (see definition in Section 2).

In our problem formulation, the nominal case is the data of $A(q^{-1}), B(q^{-1})$ and a test reference output $\{y_m^*(k)\}$. The constant parameter vector θ^* and the corresponding signals $u^*(k), y^*(k)$ are called the tuned solution of the adaptive system. We establish that under the assumption of uniqueness of the tuned solution, this solution is exponentially stable. It follows that the stability is robust to a wide class of small perturbations of the nominal plant or the nominal test reference output.

Since we will analyze only the local behavior of the system around the tuned solution, the projection and the normalization ($\gamma(k) = s(k)^{-2}$) introduced in [3,4], respectively, are not active. We will concentrate our attention on slow adaptation, i.e., with ε small:

$$\gamma(k) = \varepsilon$$

2. Problem Statement

In the following a more useful representation of the plant (1.3) will be the nominimal state space representation defined as (cancellation of poles and zeros at 0):

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$$X(k+1) = FX(k) + G_1u(k) + G_2w(k)$$

$$y(k) = h'X(k) \qquad (2.1)$$

with

$$X(k) = [y(k),\ldots,y(k-n^*+1),u(k-1),\ldots,u(k-m^*)]'$$

and n^* and m^* are large enough for $X(k)$ to contain all the elements of $\phi(k-d)$. It follows that a matrix H exists satisfying

$$\phi(k-d) = HX(k) \quad .$$

We also introduce the following notation:

$$\phi(k)' \equiv (u(k) \mid \phi_r(k)')$$

$$\hat{\theta}(k)' = (\hat{\beta}(k) \mid \hat{\theta}_r(k)') \quad .$$

With this notation there exists a matrix J such that the control u(k) can be written explicitly as:

$$u(k) = \frac{1}{\hat{\beta}(k)}[-\hat{\theta}_r(k)'JX(k) + y_m(k+d)] \quad .$$

It follows that the complete closed loop adaptive system can be written as a nonlinear nonautonomous system:

$$X(k+1) = [F - \frac{G_1\hat{\theta}_r(k)'J}{\hat{\beta}(k)}]X(k) + \frac{G_1}{\hat{\beta}(k)}y_m(k+d) + G_2\,w(k)$$

$$\hat{\theta}(k) = \hat{\theta}(k-1) - \frac{\varepsilon HX(k)X(k)'[h - H'\hat{\theta}(k-1)]}{1+\varepsilon X(k)'H'HX(k)} \quad . \qquad (2.2)$$

We are interested in the existence of a bounded solution and in its stability. For the existence problem, let us make the following

<u>Definition</u>: A test reference output is an N-periodic sequence $\{y_m^*(k)\}$ for which there exists an N-periodic sequence $\{X^*(k)\}$ and constant θ^* with $\beta^* \neq 0$ satisfying:

i) $$X^*(k+1) = (F - \frac{G_1\theta_r^{*\prime}J}{\beta^*})X^*(k) + \frac{G_1}{\beta^*}y_m^*(k+d) \qquad (2.3)$$

$$X^*(k)'(h - H'\theta^*) = 0 \quad . \qquad (2.4)$$

ii) All eigenvalues of
$$(F - \frac{G_1\theta_r^{*\prime}J}{\beta^*})$$
are strictly inside the unit circle.

Equation 2.4 is called the <u>tuning condition</u>. In the appendix we show that this condition is generically satisfied if $y_m^*(k)$ does not contain more than $(m+d+n/2)$ frequencies. $(\theta^*,X^*(k))$ is called the <u>tuned solution</u> of the adaptive system. If it exists, it is a solution of (2.2) (with $y_m(k+d)$ replaced with $y_m^*(k+d)$ and $w(k)$ set to 0). We also introduce the following notation:

$$\phi^*(k-d) = HX^*(k) \quad .$$

Notice that, with condition ii, $X^*(k)$ is uniquely defined. The following assumption will be used.

<u>Assumption A1</u>: For the nominal plant there exists a test reference output $\{y_m^*(k)\}$.

We will now concentrate our attention upon the behavior of system (2.2) around the tuned solution. For this we consider the actual output reference $y_m(k)$ as being obtained from a test output reference $y_m^*(k)$ by:

$$y_m(k) = y_m^*(k) + v(k)$$

where $\{v(k)\}$ is a bounded sequence.

Let us define the incremental variables

$$x(k) = X(k) - X^*(k)$$

$$\theta(k) = \hat{\theta}(k) - \theta^* \quad .$$

After some manipulations we obtain the following equivalent incremental form of (2.2):

$$\begin{bmatrix} x(k+1) \\ \theta(k) \end{bmatrix} = M(k,\varepsilon)\begin{bmatrix} x(k) \\ \theta(k-1) \end{bmatrix} + G\begin{bmatrix} v(k+d) \\ w(k) \end{bmatrix} + \begin{bmatrix} R_x(\theta(k),x(k),v(k+d),k) \\ \varepsilon R_\theta(\theta(k-1),x(k),k) \end{bmatrix} \qquad (2.5)$$

where:

$$M(k,\varepsilon) = \begin{bmatrix} I & 0 \\ 0 & I+\varepsilon\phi^*(k-d)\phi^*(k-d)' \end{bmatrix}^{-1}$$

$$\begin{bmatrix} F - \frac{G_1\theta_r^{*\prime}J}{\beta^*} & -\frac{G_1}{\beta^*}\phi^*(k)' \\ \varepsilon\phi^*(k-d)(h-\theta^* H) & I \end{bmatrix}, \quad G = \begin{bmatrix} \frac{G_1}{\beta^*} & G_2 \\ 0 & 0 \end{bmatrix}$$

$$R_x(\theta,x,v,k) = \frac{G_1}{\beta^*(\beta^*+\beta)}[(\beta^*\theta_r - \beta\theta_r^*)'(\frac{\beta}{\beta^*}\phi_r^*(k) - Jx) - \beta(v - \frac{\beta}{\beta^*}y_m^*(k+d))]$$

$$R_\theta(\theta,x,k) = \frac{-H}{1+\varepsilon(\phi^*(k)+Hx)'(\phi^*(k)+Hx)}$$

$$[[(I - \frac{\varepsilon x^*(k)(2x^*(k)+x)'H'H}{1+\varepsilon(\phi^*(k)'\phi^*(k)})x$$

$$((\theta^{*\prime}H - h')x + \phi^*(k-d)'\theta)] + (x^*(k)+x)x'H'\theta] \quad .$$

<u>Problem Statement</u>: In this paper we study the stability of system (2.5) around the origin. Our objective is to show that local stability is preserved in the presence of small perturbations of the nominal plant and of the test reference output. The reference output perturbations are characterized by the bounded sequence v(k). The plant perturbations are characterized by the sequence w(k). To still encompass a wide class of unmodelled effects, we can assume that the sequence w(k) satisfies the following noise to signal ratio inequality:

$$|w(k)| \leq \delta\Big(\sum_{i=0}^{k}\Big(\frac{\mu^{k-i}}{\sum_{i=0}^{k}\mu^{k-i}}\Big)\|x(i)\| + w\Big)$$

with

$$\delta > 0 \;,\; w \geq 0 \;,\; 0 \leq \mu < 1$$

It is not difficult to see that w(k) may include the effects of neglected: poles or zeros close to zero, nearly cancellable stable poles/zeros, some nonlinearities, some time variations...

3. Exponential Stability of the Tuned Solution

The stability properties of the system (2.5) around the origin are dictated by the following linear time varying system:

$$\begin{bmatrix} x(k+1) \\ \theta(k) \end{bmatrix} = M(k,\varepsilon)\begin{bmatrix} \theta(k) \\ \theta(k-1) \end{bmatrix} \quad . \qquad (3.1)$$

The matrix $M(k,\varepsilon)$ being N-periodic, we can study this system applying the Floquet theory. We have:

<u>Lemma</u>: Under Assumption A1, if $\sum_{k=0}^{N-1}\phi^*(k)\phi^*(k)'$ is invertible then the characteristic values of $M(k,\varepsilon)$ are:

$$\lambda\{F - \frac{G_1\theta_r^{*\prime}J}{\beta^*}\} + O(1) \quad ,$$

$$1 - \varepsilon\lambda\{\frac{1}{N}\sum_{k=0}^{N-1}\phi^*(k-d)\phi^*(k-d)'\} + \varepsilon\,O(1) \quad ,$$

where O(1) are continuous functions of ε which leads to 0 as ε tends to 0.

<u>Proof</u>: A more general result has been established in [13]. We give here a proof more adapted to our problem.

Since $M(k,\varepsilon)$ is N-periodic, there exists an N-periodic Lyapunov transformation which maps $M(k,\varepsilon)$ into a time invariant matrix whose eigenvalues are the characteristic values of the system. In particular for $\varepsilon = 0$, this transformation is given by

$$\begin{bmatrix} I & -L(k) \\ 0 & I \end{bmatrix}$$

where $L(k)$ is the unique N-periodic solution of:

$$L(k+1) = \left(F - \frac{G_1 \theta_r^{*\prime} J}{\beta^*}\right) L(k) - \frac{G_1}{\beta^*} \phi^*(k)' \quad .$$

Using the same transformation for $\varepsilon \neq 0$ gives:

$$\bar{M}(k,\varepsilon) = \begin{pmatrix} I & -L(k+1) \\ 0 & I \end{pmatrix} M(k,\varepsilon) \begin{pmatrix} I & L(k) \\ 0 & I \end{pmatrix}$$

$$= \begin{pmatrix} F - \frac{G_1 \theta_r^{*\prime} J}{\beta^*} + \varepsilon\Delta_1(k,\varepsilon) & \varepsilon\Delta_2(k,\varepsilon) \\ \varepsilon\Delta_3(k,\varepsilon) & I - \varepsilon\phi^*(k-d)\psi^*(k) + \varepsilon^2\Delta_4(k,\varepsilon) \end{pmatrix}$$

where the $\Delta_i(k,\varepsilon)$'s are bounded N-periodic matrices and $\psi^*(k)$ is defined by:

$$\psi^*(k) = -(h' - \theta^{*\prime}H)L(k) + \phi^*(k-d)' \quad .$$

In appendix, we prove:

Property 1: Under Assumption A1, we have:

$$\psi^*(k) = \phi^*(k-d)' \quad .$$

To complete our proof, let us take the product from $k = 0$ to $N-1$ of $\bar{M}(k,\varepsilon)$. One can see by induction that:

$$\prod_{k=0}^{N-1} \bar{M}(k,\varepsilon) = \begin{pmatrix} I + A_{22} & A_{21} \\ \varepsilon A_{12} & I + \varepsilon A_{11} \end{pmatrix}$$

with

$$A_{22} = \left(F - \frac{G_1 \theta_r^{*\prime} J}{\beta^*}\right)^N - I + \varepsilon\Delta_1(\varepsilon)$$

$$A_{21} = \varepsilon\Delta_2(\varepsilon) \ , \ A_{12} = \Delta_3(\varepsilon)$$

$$A_{11} = \sum_{k=0}^{N-1} \phi^*(k-d)\phi^*(k-d)' + \varepsilon\Delta_4(\varepsilon)$$

where the $\Delta_i(\varepsilon)$'s are bounded.

The result follows by noticing that, as a consequence of Lemma 1 of [15], this product is similar to:

$$\begin{pmatrix} I + A_{22} + \varepsilon^2 L A_{12} & 0 \\ \varepsilon A_{12} & I + \varepsilon A_{11} + \varepsilon^2 A_{12} L \end{pmatrix}$$

where L is a matrix whose existence and boundedness are guaranteed for ε small enough satisfying (see (11) of [15]):

$$\|A_{22}^{-1}\| \le \frac{1}{3\varepsilon} \left(\|(A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1}\| + \|A_{12}\| \cdot \|A_{22}^{-1}A_{21}\|\right)^{-1}$$

where invertibility of A_{22}, A_{11} is given by assumption.

This lemma leads us to introduce a new assumption.

Assumption A2: The test reference output $y_m^*(k)$ is such that the corresponding tuned solution satisfies:

$$\sum_{k=0}^{N-1} \phi^*(k)\phi^*(k)' > \alpha \ I \ , \alpha > 0 \quad .$$

This is the classical persistent spanning assumption. In the appendix we show that this condition is equivalent to the uniqueness of the tuned solution and generically to the fact that the test reference output is persistently exciting of order at least equal to the number of parameters.

Theorem 1: Under Assumptions A1, A2, there exists ε_* such that for all ε, $0 < \varepsilon < \varepsilon_*$, the tuned solution is an exponentially stable solution of the nominal system ($v(k) = w(k) = 0$). In particular, there exists a constant C such that:

$$\left\| \prod_{j=i+1}^{k} M(j,\varepsilon) \right\| \le C(1 - \varepsilon\alpha)^{k-i} \quad .$$

Proof: This is a direct consequence of Lemma 1. In particular, for ε sufficiently small, the spectral radius of

$$\prod_{k=0}^{N-1} M(k,\varepsilon) \text{ is } 1 - \varepsilon \ \lambda\min \sum_{k=0}^{N-1} \phi^*(k)\phi^*(k)' + \varepsilon \ O(1) \ .$$

This implies the existence of C_o such that

$$\left\| \prod_{k=0}^{N-1} M(k,\varepsilon) \right\| \le C_o\left(1 - \varepsilon \ \lambda\min \sum_{k=0}^{N-1} \phi^*(k)\phi^*(k)' + \varepsilon \ O(1)\right)$$

$$\le C_o(1 - \varepsilon\alpha N) \le C_o(1 - \varepsilon\alpha)^N$$

where the second inequality is obtained from Assumption A3 and the property of $O(1)$, for ε small enough. Then using the N-periodicity we obtain:

$$\left\| \prod_{j=i+1}^{k} M(j,\varepsilon) \right\| \le \left(C_o \prod_{j=0}^{N-2} \|M(j,\varepsilon)\|\right)\left((1 - \varepsilon\alpha)^N\right)^{\frac{k-i}{N}} \quad .$$

This theorem extends the result of Anderson and Johnson [6] which was established for the ideal case: any reference output is a test reference output.

It is remarkable that for this one step ahead adaptive controller, the number of frequencies and not their location is important. Both the tuning condition (Assumption A1) and the persistent spanning condition (Assumption A2) do not generically constrain the location of the frequencies.

However, Assumption A1 imposes also that the eigenvalues of $\left(F - \frac{G_1 \theta_r^{*\prime} J}{\beta^*}\right)$ lie in the unit circle, which is related to the frequency location via θ^*. It has been demonstrated by Astrom in ([8],[14]) that violation of this stability condition can by itself create instability.

4. Total Stability of the Tuned Solution

Since the tuned solution is an exponentially stable solution of the nominal system, it has also stability properties with respect to all kinds of permanent perturbations (see §1.8 of [16]). In particular, we have:

Theorem 2: Under Assumptions A1, A2 and for all ε, $0 < \varepsilon < \varepsilon_*$ there exist $N_o(\varepsilon), v_o(\varepsilon), \delta_o(\varepsilon)$ such that if the initial conditions satisfy:

$$\|x(0) - x^*(0)\| + \|\hat{\theta}(0) - \theta^*\| \le N_o(\varepsilon)$$

and if we have the following noise to signal ratio bounds

$$|v(k)| \le v_o(\varepsilon)\alpha$$

$$|w(k)| \le \delta_o(\varepsilon)\alpha\left(\sum_{i=0}^{k} \left(\frac{\mu^{k-i}}{\sum_{i=0}^{k} \mu^{k-i}}\right)\|x(i)\| + w\right)$$

then the solution $x(k)$, $\theta(k)$ to the actual system (2.2) satisfies:

$$\|x(k) - x^*(k)\| + \|\hat{\theta}(k) - \theta^*\| \le N(\varepsilon)$$

for some value $N(\varepsilon)$.

This theorem states that if the nominal plant and its adaptive controller are such that there exists a test reference output which is persistently exciting of order equal to the number of adapted parameters then generically the actual system has a bounded solution provided that the permanent perturbations $v(k)$ and $w(k)$ satisfy noise to signal ratio limitations. In particular, the disturbance on the reference output is limited by the degree of persistent spanning of $\phi^*(k)$.

In [7] Kosut and Anderson were aiming at such a result for a continuous time scheme, but their analysis was incomplete. Here we have obtained more complete results but in a more restricted case, because we studied the local behavior around a particular solution of the nominal adaptive scheme.

Proof: Let $Y(k+1)$ be the complete state of system (2.5). By the variation of constants formula, we have

$$Y(k+1) = \prod_{i=0}^{k} M(i)Y(0) + \sum_{i=0}^{k} \prod_{j=i+1}^{k} M(i)$$

$$\left[G \begin{pmatrix} v(i+d) \\ x(i) \end{pmatrix} + \begin{pmatrix} R_x(\theta(i),x(i),v(i+d),i) \\ \varepsilon R_\theta(\theta(i-1),x(i),i) \end{pmatrix} \right] \quad . \qquad (4.1)$$

We define

$$N(k) = \|Y(k)\| \quad .$$

With Theorem 1, we obtain using abbreviated notations

$$N(k+1) \le C[\rho^k N(0) + \sum_{i=0}^{k} \rho^{k-i} [\|G\|(|v(i+d)| + |w(i)|)$$

$$+ \|R_x(i)\| + \varepsilon\|R_\theta(i)\|] \quad ,$$

with $\rho = 1 - \varepsilon\alpha$. Now, the proof proceeds by induction. We want to show

$$N(i) \le N \quad \forall i \quad 0 \le i \le k \Rightarrow N(k+1) \le N$$

with an appropriate choice of N. Bound on $w(i)$: Since

$$\|x(i)\| \le \sup_k \|x^*(k)\| + \|x(j)\|$$

we have with the induction assumption:

$$|w(k)| \le \delta_o \alpha (N + w^*) \quad .$$

with

$$w^* = \sup_k \|x^*(k)\| + w \quad .$$

Bound on R_x, R_θ: Assuming that $|\beta(k)| < \frac{|\beta^*|}{2}$ and using the Schwarz and triangle inequalities, it is easy to derive from the expressions for R_x and R_θ the following inequalities:

$$\|R_x(i)\| \le \alpha v_o N(i) P_2(N(i)) + N^2(i) P_4(N(i))$$

$$\|R_\theta(i)\| \le N^2(i) P_1(N(i)) \quad .$$

Where P_1, P_2 and P_4 are polynomials with positive coefficients of degree 1, 2 and 4, respectively. These polynomials are increasing functions of $N(k)$ $(N(K) > 0)$. With the induction assumption; it follows that, given a bound $\bar{N}$, there exists a constant K, such that if $|\beta(k)| < |\beta^*/2|$ then:

$$\|R_x(i)\| + \varepsilon\|R_\theta(i)\| \le K\alpha(v_o N + N^2) \ , \ 0 \le i \le k \ , \ N \le \bar{N} \quad .$$

Substituting these bounds into Equation (4.1) gives:

$$N(k+1) \le \frac{C}{\varepsilon}(\rho\varepsilon N_o + \|G\|(v_o + \delta_o w^*) + (\|G\| \delta_o + Kv_o)N + KN^2) \quad .$$

Therefore, $N(k+1)$ will be smaller than N if:

$$CKN^2 + (C\|G\|\delta_o + CKv_o - \varepsilon)N + C(\rho\varepsilon N_o + \|G\|(v_o + \delta_o w^*)) \le 0 \quad .$$

This inequality has a solution in N satisfying $0 < N \le \bar{N}$ iff:

$$C\|G\|\delta_o + CKv_o \le \varepsilon$$

$$(C\|G\|\delta_o + CKv_o - \varepsilon)^2 \ge 4KC^2(\rho\varepsilon N_o + \|G\|(v_o + \delta_o w^*)) \quad .$$

This gives the bounds $\delta_o(\varepsilon)$, $v_o(\varepsilon)$, $N_o(\varepsilon)$. We see that these bounds tend to zero as ε tends to zero.

Appendix: Properties of the Tuned Solution

Here we assume that the sequence $y_m^*(k)$ can be written as:

$$y_m^*(k) \quad \sum_{i=-N_1}^{N_2} y_{m_i}^* \, z_i^k \quad , \quad \forall k$$

with

$$N_1 \le \frac{1}{2} N \ , \ N_2 \le \frac{1}{2}(N-1) \ , \ z_i = \exp(j\frac{2\Pi i}{N}) \ , \ y_{m,-i}^* = \bar{y}_{mi}^*$$

where $\bar{y}_{mi}^*$ is the complex conjugate of y_{mi}^*.

In the following we denote by E the set

$$E = \{i : y_{mi}^* \neq 0\} \quad .$$

Its cardinal is n_E.

The vector $V(z^{-1})$ is defined as:

$$V(z^{-1}) = (A(z^{-1}), \ldots, z^{-m-d+1}A(z^{-1}), z^{-1}B(z^{-1}), \ldots, z^{-n} \cdot B(z^{-1}))' \quad .$$

Property A1: If

$$B(z_i^{-1}) \neq 0 \ , \ \forall i \in E \quad .$$

Then the tuning condition is equivalent to the following linear system:

$$z_i^{d-1} \, B(z_i^{-1}) = V(z_i^{-1})'\theta^* \quad , \quad \forall i \in E \quad .$$

Corollary A1: Under the same assumption the tuning condition is satisfied iff:

$$(B(z_i^{-1}))'_{i \in E} \in \mathbb{R}\text{-Range } (V(z_i^{-1})')_{i \in E} \quad .$$

This condition is only expressed in terms of the nominal plant. It shows that the tuning condition is satisfied generically if:

$$n_E \le m + d + n \quad .$$

This means that $y_m^*(k)$ contains no more than $\frac{m+d+n}{2}$ frequencies.

Proof: The tuning condition can be written in:

$$y^*(k) = \theta^{*\prime}\phi^*(k-d) \quad , \quad \forall k$$

and the control is obtained from

$$y_m^*(k+d) = \theta^{*\prime}\phi^*(k) \quad , \quad \forall k \quad .$$

It follows readily that:

$$y^*(k) = y_m^*(k) \quad , \quad \forall k \quad .$$

Now we notice that the characteristic polynomial of the tuned closed loop system is:

$$P(z^{-1}) = V(z^{-1})'\theta^* \quad .$$

Let $T_y(z^{-1})$ be the transfer function between $y_m^*(k+d)$ and $y^*(k)$. From the state space representation, we have:

$$T_y(z^{-1}) = h'(zI - (F - G_1 \frac{\theta_r^{*\prime}}{\beta^*} J))^{-1} \frac{G_1}{\beta^*} \quad .$$

And from the polynomial representation, we have:

$$T_y(z^{-1}) = \frac{z^{-1}B(z^{-1})}{P(z^{-1})} \quad .$$

The identity of the sequences $\{y^*(k)\}$ and $\{y_m^*(k)\}$ yields:

$$\sum_{i\in E} y_{mi}^*(z_i^d T_y(z_i^{-1}) - 1)z_i^k = 0 \quad , \quad \forall k \quad .$$

But since the sequences $\{z_i^k\}$, $i\in E$ are linearly independent, this relation shows that, with the expression of $T_y(z^{-1})$,

$$\frac{z_i^{d-1}B(z_i^{-1})}{V(z_i^{-1})'\theta^*} - 1 = 0 \quad , \quad \forall i\in E \quad .$$

And since $B(z_i^{-1})$ is not zero. We have obtained the following necessary condition:

$$z_i^{d-1}B(z_i^{-1}) = V(z_i^{-1})'\theta^* \quad , \quad \forall i\in E \quad .$$

Clearly, this is also a sufficient condition.

Corollary A2: Under Assumption A1, we have:

$$\{u^*(k-d)\} = T_y(q^{-1})\{u^*(k)\} \ , \ \{y^*(k-d)\} = T_y(q^{-1})\{y^*(k)\} \ .$$

Proof: The second relation has already been established. For the first one, we notice that $u^*(k)$ is also N-periodic and, therefore:

$$u^*(k) = \sum_{i\in E} u_i z_i^k \quad , \quad \forall k \quad .$$

The conclusion follows from the identity

$$T_y(z_i^{-1}) = z_i^{-d} \quad , \quad \forall i\in E \quad .$$

Property A2: If the tuning condition is satisfied and

$$B(z_i^{-1}) \neq 0 \quad , \quad \forall i\in E \quad .$$

The uniqueness of θ^* and persistent spanning (Assumption A2) are equivalent properties.

Proof: Let $T_u(z^{-1})$ be the transfer function between $y_m^*(k+d)$ and $u^*(k)$, we have:

$$T_u(z^{-1}) = \frac{A(z^{-1})}{P(z^{-1})} \quad .$$

Let $T_\phi(z^{-1})$ be the transfer function between $y_m^*(k+d)$ and $\phi^*(k)$. From the state representation, we have:

$$T_\phi(z^{-1}) = z^d H(zI - (F - G_1 \frac{\theta_r^{*\prime}}{\beta^*} J))^{-1} \frac{G_1}{\beta^*} \quad .$$

But also comparing the definitions of $\phi^*(k)$ and $V(z^{-1})$ and using the expression of $T_u(z^{-1}), T_y(z^{-1})$, we have:

$$T_\phi(z^{-1}) = \frac{V(z^{-1})}{P(z^{-1})} \quad .$$

This expression allows us to rewrite the persistent spanning condition in the frequency domain as:

$$\sum_{k=0}^{N-1} \phi^*(k)\phi^*(k)' = \sum_{i\in E} |y_{mi}^*|^2 \frac{\overline{V(z^{-1})}V(z_i^{-1})'}{|P(z_i^{-1})|^2} > \alpha I \quad .$$

And we know that both y_{mi}^* and $P(z_i^{-1})$ are not equal to zero for $i\in E$. Therefore, the fact that this matrix is positive definite is equivalent to the fact that the matrix $(V(z_i^{-1})'_{i\in E})$ is full rank. But with Property A1 we know that this is equivalent to the uniqueness of θ^*.

Let us now prove Property 1.

Lemma: For any θ^*, we have the following identity:

$$\theta^{*\prime}H(zI - (F - G_1 \frac{\theta_r^{*\prime}}{\beta^*} J))^{-1} \frac{G_1}{\beta^*} = z^{-d} \quad .$$

Proof: This transfer function is just $z^{-d}\theta^{*\prime}T_\phi(z^{-1})$ where $T_\phi(z^{-1})$ is defined in the proof of Property A2. Since we have

$$T_\phi(z^{-1}) = \frac{V(z^{-1})}{V(z^{-1})'\theta^*} \quad .$$

The conclusion follows readily.

Now to prove Property 1, we have to show that the transfer function:

$$H(z^{-1}) = z^{-d} + (h' - \theta_*'H)(zI - (F - G_1 \frac{\theta_r^{*\prime}}{\beta^*} J))^{-1} \frac{G_1}{\beta^*}$$

with $\{\phi^*(k)\}$ as input, has $\{\phi^*(k-d)\}$ as an ouput. From this lemma, we can rewrite $H(z^{-1})$ simply as:

$$H(z^{-1}) = h'(zI - (F - G_1 \frac{\theta_r^*}{\beta^*} J))^{-1} \frac{G_1}{\beta^*} \quad .$$

But this is just $T_y(z^{-1})$. Therefore, the conclusion follows from the definition of $\phi^*(k)$ and Corollary A2.

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