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TOWARDS A DIRECT ADAPTIVE SCHEME FOR A DISCRETE-TIME CONTROL OF A MINIMUM PHASE CONTINUOUS-TIME SYSTEM*

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Abstract

A direct scheme is proposed for discrete-time control of a very rapidly sampled continuous-time system. Knowledge of the relative degree and of an upper bound of the order of the continuous-time system proves to be sufficient *a priori* knowledge for the derivation of the scheme. Convergence to zero of the tracking error is established for the sampled system.

1. Introduction

The subject we want to address in this paper is that of the existence of a discrete-time, model-reference adaptive controller for a very rapidly sampled continuous-time system. We assume that the given system is minimum phase and that an upper bound of its order and its relative degree are known. With very fast sampling, the corresponding discrete time system is nonminimum phase if the relative degree is larger than 1 (see Astrom, et al. [1]). A very rich literature has been devoted to the problem of adaptive control of nonminimum phase systems (see the survey given by M'Saad, Ortega, and Landau [2]). The difficulty which arises in this setting is that the tracking transfer function must retain unstable zeros. As a consequence, the controller cannot be obtained by a linear estimation scheme. One way to circumvent this problem and to linearize the estimation scheme is to incorporate some *a priori* knowledge about the system (see [3] for another type of solution). As very often conjectured, Johansson in [4] has established that the minimum *a priori* knowledge required for model matching is that of the positions of the unstable zeros. Explicit use of *a priori* knowledge in an indirect scheme has been proposed by Clary and Franklin in [5]. They have not solved, however, the problem created by singularity of the Diophantine equation.

In this paper, we shall derive a direct scheme which uses the *a priori* information given for very fast sampling of continuous time systems by Astrom, et al. [1]. They have established that if n is the number of poles and m is the number of zeros of the continuous time system, then:

i) the sampled-date representation of the system will have n-1 zeros and n poles; and, furthermore,

ii) as the sampling period tends to zero, m zeros tend to 1 and are stable. The remaining n-m-1 zeros tend to known values which depend upon only the relative degree n-m, and if they are unstable, they are real and lie on the interval $(-\infty,-1]$.

In Section 2 we present our algorithm. Section 3 is devoted to a convergence analysis in an ideal case. Finally our conclusion is given in Section 4.

2. A Direct Adaptive Scheme

Under the aforementioned circumstances, we shall make the following assumptions.

A1: The sampled-data system may be represented by

$$A(q^{-1})y(t) = B_u(q^{-1})B(q^{-1})u(t) \tag{1}$$

where $A(q^{-1})$, $B(q^{-1})$, and $B_u(q^{-1})$ are polynomials in the backward shift operator q^{-1}; $A(q^{-1})$ is monic and $A(q^{-1})$ and $B_u(q^{-1})$ are relatively prime; the zeros of $B(q^{-1})$ are stable and the sign of $B(0)$ is known (say, positive); $\{y(t)\}$ and $\{u(t)\}$ are the output and input sequences, respectively, of the sampled system.

A2: Integers n and m are known such that

$$n \geq \deg A(q^{-1}) \quad \text{and} \quad m \geq \deg B(q^{-1}).$$

A3: $B_u(q^{-1})$ is known, and has degree d. Its zeros are denoted by z_{ui} where i=1 to d. If the system has delays, some of them are infinite. For the purposes of this paper, we shall assume that all z_{ui} are real.

Now let $C(q^{-1})$ and $T(q^{-1})$ of degree N-m and N, respectively, be chosen polynomials with

$$N = \max\{n, m+d\}.$$

The relative primeness of $A(q^{-1})$ and $B_u(q^{-1})$ implies that there exist polynomials $\bar{S}(q^{-1})$ and $R(q^{-1})$ of degree N-m and N-1, respectively, such that

$$A(q^{-1})\bar{S}(q^{-1}) + B_u(q^{-1})q^{-1}R(q^{-1}) = C(q^{-1})T(q^{-1}). \tag{2}$$

Let us denote $S(q^{-1})$ as the following polynomial of degree N and with leading coefficient b_o strictly positive:

$$S(q^{-1}) = \bar{S}(q^{-1})B(q^{-1}).$$

Examination of (2) will show that, if $C(q^{-1})T(q^{-1})$ and $B_u(q^{-1})$ are relatively prime, then $\bar{S}(q^{-1})$ and $B_u(q^{-1})$ are relatively prime. This implies that $S(q^{-1})$ and $B_u(q^{-1})$ are prime. This fact motivates our next assumption.

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A4: A convex set $\mathfrak{C} \in \mathbb{R}^N$ is known such that

i) the vector of coefficients of $S(q^{-1})$ belong to $\mathfrak{C}$, and

ii) for any polynomial $S'(q^{-1})$ corresponding to any vector in $\mathfrak{C}$,

$$|S'(z_{ui}^{-1})| > \varepsilon, \qquad 1 \le i \le d$$

and

$$\lim_{z\to\infty} |S'(z^{-1})| = |b_o'| > \varepsilon.$$

Later we will see that $S(q^{-1})$ is the denominator of our controller. Since this controller is a sampled-data version of a continuous time controller, we know that, as the sampling period goes to zero, the poles of this controller tend to 1; moreover, in general the zeros of the system z_{ui}, are in $[-\infty,-1]$. The convex set $\mathfrak{C}$ of assumption (A4) therefore, can be described by the following inequalities (using $b_o > \varepsilon$):

$$S'(z_{ui}^{-1}) \ge \varepsilon, \qquad 1 \le i \le d$$

$$\lim_{z\to\infty} S'(z^{-1}) = b_o' \ge \varepsilon.$$

System Reparameterization: With these assumptions, we can now rederive the parametric model given by Johansson in Chapter 5 of [4] (see also Astrom [6]). Applying (2) to y(t) yields

$$C(q^{-1})T(q^{-1})y(t) = S(q^{-1})B_u(q^{-1})u(t) + q^{-1}R(q^{-1})B_u(q^{-1})y(t). \qquad (3)$$

Define a regression vector $\phi(t)$ by

$$T(q^{-1})\phi(t) = [u(t) \ldots u(t-N)y(t-1) \ldots y(t-N)]. \qquad (4)$$

Define a parameter vector θ by (with $s_o = b_o$):

$$\theta = [s_o \ldots s_N \; r_1 \ldots r_N] \qquad (5)$$

where s_i (i=0 to N) are the coefficients of $S(q^{-1})$ and r_j (j=1 to N) are the coefficients of $R(q^{-1})$.

Then, if the roots of $T(q^{-1})$ are stable and all initial conditions are at zero, we have the equation

$$C(q^{-1})y(t) = \theta^T B_u(q^{-1})\phi(t). \qquad (6)$$

or equivalently,

$$C(q^{-1})y(t) = B_u(q^{-1})\theta^T\phi(t). \qquad (7)$$

Equation (6) can be used to estimate the vector θ.

Control Law: Let $C(q^{-1})$ and $T(q^{-1})$ have stable roots, our control objective is to match, with r(t), a reference input

$$C(q^{-1})y(t) = B_u(q^{-1})r(t). \qquad (8)$$

In view of (7) this can be achieved by choosing the following (implicit) control law

$$r(t) = \theta^T\phi(t). \qquad (9)$$

Adaptive Controller: From (6) and (9) we can propose the following adaptive controller.

Let $\hat{\theta}(t)$ be an estimate of θ and define $\varphi(t)$ as

$$\varphi(t) = B_u(q^{-1})\phi(t). \qquad (10)$$

The algorithm is

$$\hat{\theta}(t) = \mathcal{P}_{\mathfrak{C}}\Big(\hat{\theta}(t-1) + \frac{P(t-1)\;\;(t)}{1 + \;\;(t)^T P(t-1)\;\;(t)}\,(C(q^{-1})y(t) - \hat{\theta}(t-1)^T\;\;(t))\Big) \qquad (11)$$

where $\mathcal{P}_{\mathfrak{C}}$ is the projection on $\mathfrak{C}$ proposed by Goodwin and Sin, p. 92 of [7];

$$P(t) = P(t-1) - \frac{P(t-1)\;\;(t)\;\;(t)^T P(t-1)}{1 + \;\;(t)^T P(t-1)\;\;(t)}, \qquad (12)$$

$$P(0) > 0;$$

$$r(t) = \hat{\theta}(t)^T\phi(t). \qquad (13)$$

The use of the projection $\mathcal{P}_{\mathfrak{C}}$ guarantees that the polynomial $\hat{S}(q^{-1},t)$ obtained from $\hat{\theta}(t)$ satisfies the condition that

$$|\hat{S}(z_{ui}^{-1},t)| > \varepsilon, \qquad 1 \le i \le d$$

and

$$|\hat{s}_o(t)| > \varepsilon$$

where $\hat{s}_o(t)$ is the leading coefficient of $\hat{S}(z^{-1},t)$.

Remark: From our simulations it seems that in most cases it is not necessary to implement the projection $\mathcal{P}_{\mathfrak{C}}$.

3. Convergence of the Algorithm

The following theorem gives the properties of our algorithm.

Theorem: Under the assumptions A1 through A4, the algorithm described by (11), (12), and (13) has the properties that:

i) $\hat{\theta}(t)$ is bounded,

ii) u(t) and y(t) are bounded for bounded r(t), and

iii) $\lim_{t\to+\infty}[C(q^{-1})y(t) - B_u(q^{-1})r(t)] = 0.$

Proof: We shall use the techniques of proof introduced by Goodwin, Ramadge, and Caines [8]. The presence of $B_u(q^{-1})$, however, will impose some additional difficulties. We shall first state some technical lemmas.

LEMMA 1 (see Lemma 3.3.6 of [7]):

Under our aforementioned assumptions,

i) $\|\hat{\theta}(t)-\theta\| \le K_1$, for all t;

ii) $\lim_{t\to\infty} |C(q^{-1})y(t) - \hat{\theta}(t)^T\varphi(t)| = 0;$

iii) $\lim_{t\to\infty} \|\hat{\theta}(t) - \hat{\theta}(t-1)\| = 0.$

LEMMA 2 (see Lemma 4.3.2 of [7]):

Let $P(q^{-1},t)$ and $Q(q^{-1},t)$ be polynomials with degree n_p, n_q, respectively, and with time-varying coefficients $p_i(t)$ (i=0 to n_p) and $q_j(t)$ (j=0 to n_q) such that $p_i(t)$ and $q_j(t)$ are bounded and satisfy

$$\lim_{t\to\infty} |q_j(t) - q_j(t-1)| = 0.$$

For any sequence $v(t)$, let $F(P;Q,v,t)$ be the sequence defined by

$$F(P,Q,v,t,) = P(q^{-1},t)[Q(q^{-1},t)v(t)] - [P(q^{-1},t)Q(q^{-1},t)]v(t), \tag{14}$$

where the first term on the right-hand side implies $P(q^{-1},t)$ operates on $Q(q^{-1},t)u(t)$, while the second term implies that the product $[P(q^{-1},t)Q(q^{-1},t)]$ operates on $u(t)$. Then for any $\varepsilon > 0$, there exists $T(\varepsilon)$ such that, for all $t > T$

$$|F(P,Q,v,t,)| \le \varepsilon\pi n_p(n_p-1)n_1 \sup_{0<\ell\le n_p+n_q} |v(t-\ell)| \tag{15}$$

where

$$\pi = \sup_{i,t}|p_i(t)|.$$

LEMMA 3: If

$$|\hat{S}(z_{ui}^{-1},t)| > \varepsilon, \qquad 1 \le i \le d,$$

then there exist time-varying polynomials $\alpha(q^{-1},t)$ and $\beta(q^{-1},t)$ such that

$$[\alpha(q^{-1},t)\hat{S}(q^{-1},t)] + [\beta(q^{-1},t)B_u(q^{-1})] = 1$$

and the coefficients of $\alpha(q^{-1},t)$ and $\beta(q^{-1},t)$ are bounded.

Proof: The hypothesis on $\hat{S}(q^{-1},t)$ implies that $\hat{S}(q^{-1},t)$ and $B_u(q^{-1})$ are relatively prime for any t. Existence of $\alpha(q^{-1},t)$ and $\beta(q^{-1},t)$ follows from the Bezout identity; moreover, the Bezoutian of $\hat{S}(q^{-1},t)$ and $B_u(q^{-1})$ is (see Kailath, Ex. 1.4-17, p. 159, [9])

$$\prod_{i=1}^{d} \hat{S}(z_{ui}^{-1},t),$$

which is uniformly bounded from below by ε^d. This implies the boundedness of the coefficients of $\alpha(q^{-1},t)$ and $\beta(q^{-1},t)$.

Before proceeding further, let us note that there is no finite escape time, since the mapping from t to t+1 has no singularity. In particular, the projection operation in the control algorithm implies that no division by zero will occur. The proof of the theorem will now be done using the small gain theorem:

Forward Path: $u(t) \to y(t)$:

Let $\eta(t)$ be defined by

$$\begin{aligned}\eta(t) &= C(q^{-1})y(t) - \hat{\theta}(t)^T\varphi(t)\\ &= C(q^{-1})y(t) - \hat{\theta}(t)^T B_u(q^{-1})\phi(t).\end{aligned} \tag{16}$$

Adding and subtracting $B_u(q^{-1})r(t)$ to (16) and using the definition of the control law (13) yields

$$C(q^{-1})y(t) = \eta(t) + B_u(q^{-1})r(t) + \{\hat{\theta}(t)^T[B_u(q^{-1})\phi(t)] - B_u(q^{-1})[\hat{\theta}(t)^T\phi(t)]\}. \tag{17}$$

We remark that $\eta(t)$ is given by Lemma 1 and that, with Lemma 1, we can apply Lemma 2 to the last term of (17). Hence, since there is no finite escape time for any ε, there exists $T_o(\varepsilon)$ such that

$$|\eta(t)| < \varepsilon \quad \text{for all} \quad t \ge T_o \tag{18}$$

and

$$\sup_{T\ge t\ge T_o} |C(q^{-1})y(t)| \le \varepsilon + \sup_{T\ge t\ge T_o} |B_u(q^{-1})r(t)| + \varepsilon K_2 \sup_{T>t\ge T_o} \|\phi(t)\| + K_3(\varepsilon) \tag{19}$$

for some constants K_2 and $K_3(\varepsilon)$.

The exponential stability of $C(q^{-1})^{-1}$ and $T(q^{-1})^{-1}$ then implies that

$$\sup_{T\ge t\ge T_o} |y(t)| \le \varepsilon K_4 \sup_{T>t\ge T_o} |u(t)| + \varepsilon K_5 \sup_{T>t\ge T_o} |y(t-1)| + K_6(\varepsilon), \tag{20}$$

for some constants K_4, K_5, and $K_6(\varepsilon)$. Hence we have obtained

$$\sup_{T\ge t\ge T_o} |y(t)| \le \frac{\varepsilon K_4}{1-\varepsilon K_5} \sup_{T>t\ge T_o} |u(t)| + K_7(\varepsilon) \tag{21}$$

for some $K_7(\varepsilon)$.

Feedback path: $y(t) \to u(t)$:

Let $\alpha(q^{-1},t)$ and $\beta(q^{-1},t)$ be time varying polynomials. Applying $\beta(q^{-1},t)$ to (1) and applying $\alpha(q^{-1},t)B(q^{-1})T(q^{-1})$ to (13) result in the following two equations:

$$[\beta(q^{-1},t)A(q^{-1})]y(t) = [\beta(q^{-1},t)B(q^{-1})B_u(q^{-1})]u(t) \tag{22}$$

$$[\alpha(q^{-1},t)B(q^{-1})T(q^{-1})]r(t) = [\alpha(q^{-1},t)B(q^{-1})T(q^{-1})]\cdot \hat{\theta}(t)^T\phi(t). \tag{23}$$

Let $\hat{R}(q^{-1},t)$ and $\hat{S}(q^{-1},t)$ be defined as those polynomials whose respective coefficients are those coefficients in $\hat{\theta}(t)$. Equation (23) may be rewritten as

$$\begin{aligned}[\alpha(q^{-1},t)B(q^{-1})T(q^{-1})]r(t) &= [\alpha(q^{-1},t)B(q^{-1})][\hat{S}(q^{-1},t)\cdot u(t) + \hat{R}(q^{-1},t)y(t-1)] + [\alpha(q^{-1},t)B(q^{-1})]\cdot\\ &\quad \{T(q^{-1})[\hat{\theta}(t)^T\phi(t)] - \hat{\theta}(t)^T[T(q^{-1})\phi(t)]\}.\end{aligned} \tag{24}$$

Now, if $\alpha(q^{-1},t)$, $B(q^{-1},t)$ are as given by Lemma 3, summing respective sides of (22) and (24) yields the equality

$$B(q^{-1})u(t) = [\alpha(q^{-1},t)B(q^{-1})T(q^{-1})]r(t) + [B(q^{-1},t)A(q^{-1})]y(t) - [\alpha(q^{-1},t)B(q^{-1})]\cdot [\hat{R}(q^{-1},t)y(t-1)]$$

$$+ [\alpha(q^{-1},t)B(q^{-1})]\{\hat{\theta}(t)^T[T(q^{-1})\phi(t)]$$

$$- T(q^{-1})[\hat{\theta}(t)^T\phi(t)]\}$$

$$+ [\alpha(q^{-1},t)B(q^{-1})\hat{S}(q^{-1},t)]u(t)$$

$$- [\alpha(q^{-1},t)B(q^{-1})][\hat{S}(q^{-1},t)u(t)]. \qquad (25)$$

From Lemmas 1, 2, 3 and since there is no finite escape time, there exists $T_1(\varepsilon)$ such that

$$\sup_{T\geq t\geq T_1} |B(q^{-1})u(t)| \leq K_8(\varepsilon) + K_9 \sup_{T\geq t\geq T_1} |y(t)| + \varepsilon K_{10} \sup_{T>t\geq T_1} \|T(q^{-1})\phi(t)\| + K_{11} \sup_{T>t\geq T_1} |u(t)| \qquad (26)$$

for some constants $K_8(\varepsilon)$, K_9, K_{10}, and K_{11}.

Exponential stability of $B(q^{-1})^{-1}$, $T(q^{-1})^{-1}$ yields

$$\sup_{T\geq t\geq T_1} |u(t)| \leq K_{12}(\varepsilon)+K_{13} \sup_{T\geq t\geq T_1} |y(t)| + \varepsilon K_{14} \sup_{T>t\geq T_1} |u(t)| \qquad (27)$$

for some constants $K_{12}(\varepsilon)$, K_{13}, and K_{14}. Equation (27) implies that

$$\sup_{T\geq t\geq T_1} |u(t)| \leq \frac{K_{13}}{1-\varepsilon K_{14}} \sup_{T\geq t\geq T_1} |y(t)| + K_{15} \qquad (28)$$

for some constant $K_{15}(\varepsilon)$.

Closing the loop:

The boundedness of $u(t)$ and $y(t)$ follows from application of the Small Gain Theorem to (21) and (28), with a choice of ε such that

$$\frac{\varepsilon K_4}{1-\varepsilon K_5} \frac{K_{13}}{1-\varepsilon K_{14}} < 1. \qquad (29)$$

We may now conclude from (17) that

$$\lim_{t\to\infty} [c(q^{-1})y(t) - B_u(q^{-1})r(t)] = 0. \qquad (30)$$

Conclusion

In this paper, we have presented a discrete-time model reference adaptive controller for a discrete time system with known unstable zeros. Convergence to zero of the tracking error has been established under the assumption of known order. This research was motivated by the problem of controlling a minimum phase continuous-time plant with very fast sampling. In this case, the sampled system has unstable zeros. The knowledge of only the relative degree of the continuous-time plant is sufficient for obtaining approximations of these zeros. The subjects of future work will be to study the robustness of our scheme with respect to small perturbations of the unstable zeros, and to relax the assumption of known order of the plant.

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