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STOCHASTIC ADAPTIVE CONTROLLERS WITH AND WITHOUT A POSITIVITY CONDITION*

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ABSTRACT

The study of robust adaptive controllers has led us to introduce a new modified least squares algorithm. It incorporates a normalization signal, a covariance matrix regularization, and a parameter projection. In this paper we investigate properties of minimum variance controllers using this parameter adaptation. First, we show that for any mean square bounded driving noise, the input output signals are mean square bounded. Secondly, if the noise is a moving average and its noise model parameters satisfy a very strict passivity condition, then the controller is asymptotically optimal. The price paid to remove the passivity condition, in the first part, is the a priori knowledge of a compact set containing a stabilizing regulator and the sign and a lower bound on its leading coefficient.

1. INTRODUCTION

Research about the behavior of adaptive systems in the presence of output disturbance has led to two types of results.

In [6], Egardt studies the case of uniformly bounded disturbance with no assumption about its stationarity and autocorrelation. First, he shows that instability can occur due to escape of the adapted parameters. Second, he establishes that if the adapted parameters are bounded then the input-output signals are also bounded. This justifies introduction, in the adaptation law, of mechanisms to keep bounded parameters: dead zone when an upper bound of the disturbance is known, projection of the parameters into a compact set when upper bounds of stabilizing parameters are known.

In [1], the case of mean square bounded disturbance is treated. This type of disturbance leads to use vanishing adaptation gain. This is obtained by normalizing the input-output signals and the error signal used in the adaptation law by the ℓ_2-norm of the input-output signals. An assumption about the autocorrelation and the stationarity of the disturbance is introduced in a stochastic framework: the disturbance is considered as a moving average process, the autocorrelation condition is expressed in terms of a very strictly passive operator. With this assumption two types of results are obtained: the escape of adapted parameters does not occur, the minimum variance objective is achieved. Nevertheless, the very strict passivity condition is usually considered as restrictive. Several propositions have been made to remove this condition (such as filtering the error signal [9]). They generally lead to loss of the optimality property in the ideal case (i.e., the VSP condition is satisfied).

In this paper we will show that the analysis of [6] can be extended to the mean square bounded disturbance case. We use an adaptation law incorporating both normalization and projection. Following [6], this projection allows us to remove the autocorrelation-stationarity assumption in the proof of mean square boundedness of the signals. Moreover, in the ideal case, we prove that our algorithm achieves the minimum variance objective. To be complete, let us mention that this adaptive controller can be proved robust to a wide class of unmodeled effects (see [3]).

To summarize, this paper has three objectives:

1. To extend the result of Egardt to the mean square bounded disturbance case.
2. To prove the optimality of a robust adaptive controller.
3. To provide a theorem, to be used in place of the stochastic key technical lemma of [1]. In particular, it can be used in the proof of the robustness of stochastic adaptive controllers [5].

In Section 2, we state the problem. In Section 3, we present our algorithm and establish one of its key properties. In Section 4, we prove the mean square boundedness using the theorem mentioned above. In Section 5, we study the optimality. In Section 6, we give our conclusion.

Notation: In the following the superscript $\sim$ means random variable. This notation will be used only when confusion is possible.

2. PROBLEM STATEMENT

Consider the adaptive control of a linear time invariant finite dimensional single-input, single-output plant having autoregressive representation of the form

$$A(q^{-1})y(t) = q^{-d}B(q^{-1})u(t) + x(t) \tag{2.1}$$

where $y(t)$, $u(t)$, $x(t)$ denote the output, input, and disturbance, respectively. $A(q^{-1})$, $B(q^{-1})$ are scalar polynomials in the unit delay operator q^{-1}

$$A(q^{-1}) = 1 + a_1q^{-1} + \cdots + a_{n_A}q^{-n_A} \tag{2.2}$$

$$B(q^{-1}) = b_0 + b_1q^{-1} + \cdots + b_{n_B}q^{-n_B}, \quad b_0 \neq 0 \tag{2.3}$$

q^{-d} represents a pure time delay. The sequence $x(t)$ will be taken to be a stochastic process defined on a probability space (Ω,F,P) such that

$$\sup_T \frac{1}{T}\sum_{t=1}^{T} x(t)^2 < \infty \quad \text{a.s.} \tag{2.4}$$

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Classical analysis of stochastic recursion schemes (such as those of [1]), consider $x(t)$ as a moving average of a martingale difference sequence $w(t)$. This case will be considered in Section 5. For the time being, no assumption about the stationarity and autocorrelation of $x(t)$ is needed.

Our objective is to find a control law which stabilizes the system and aims to asymptotically minimize the variance of $y(t)-y^m(t)$, where $y^m(t)$ is a bounded deterministic desired output sequence.

If the polynomials $A(q^{-1})$, $B(q^{-1})$ were known and if $x(t)$ were a finite moving average process, it is well-known [2] that our objective would be achieved using a linear time invariant control law of the type

$$S(q^{-1})u(t) + R(q^{-1})y(t) = C(q^{-1})y^m(t+d) \tag{2.5}$$

where $S(q^{-1})$, $R(q^{-1})$, $C(q^{-1})$ are polynomials in q^{-1} with degree n_S, n_R, n_C, respectively, and $C(q^{-1})$ is monic. In the following we will denote θ the vector in $\mathbb{R}^n$ $(n = n_S+n_R+n_C+2)$ of their coefficients

$$\theta = (s_0,\ldots,s_{n_S}\ r_0\ \ldots\ r_{n_R}\ -c_1\ \ldots\ -c_{n_C})^T. \tag{2.6}$$

The following assumptions will be made about the system (2.1):

A1. The delay d is known.

A2. $B(q^{-1})$ has all its zeros inside the open unit disk.

A3. Integers n_S, n_R are known such that there exist (unknown) polynomials $S_*(q^{-1})$, $R_*(q^{-1})$ satisfying

$$B(q^{-1}) = S_*(q^{-1})A(q^{-1}) + q^{-d}R_*(q^{-1})B(q^{-1}). \tag{2.7}$$

In the following, θ_* will be the vector obtained from $S_*(q^{-1})$, $R_*(q^{-1})$, $C_*(q^{-1}) = 1$. We also define $Q_*(q^{-1})$ as the following polynomial (see (2.7))

$$Q_*(q^{-1}) = S_*(q^{-1})/B(q^{-1}). \tag{2.8}$$

A4. A vector θ_C in $\mathbb{R}^n$ and a scalar K are known such that

$$(\theta_C-\theta_*)^T(\theta_C-\theta_*) \le K. \tag{2.9}$$

A5. A scalar σ_0 is known such that

$$s_{0_*} \ge \sigma_0 \tag{2.10}$$

where s_{0_*} is the first component of θ_*, i.e., the leading coefficient of $S_*(q^{-1})$.

Comment: Assumption A2 is necessary because our control objective considers the minimization of a cost function only involving $y(t)$, not $u(t)$.

Assumption A3 is satisfied if an upper bound of the order of the plant is known.

Assumption A4 means that the stabilizing controller defined by A3 has finite gains and an upper bound (may be large) is known.

Assumption A5 is more restrictive. It refers to the usual assumption about the sign of the leading coefficient of the plant. It allows bypassing the problem of singularity in the transformation: adapted parameters → controller parameters.

3. AN ADAPTIVE CONTROLLER

In order to achieve our control objective we will consider an adaptive minimum variance controller based on a least squares estimation incorporating parameter projection, covariance matrix regularization, and signal normalization.

As usual, let $\theta(t)$, $\varphi(t)$ be the following vectors in $\mathbb{R}^n$

$$\theta(t) = (s_0(t)\ \ldots\ s_{n_S}(t)\ r_0(t)\ \ldots\ r_{n_R}(t)\ -c_1(t)\ldots-c_{n_C}(t))^T \tag{3.1}$$

$$\varphi(t) = (u(t)\ \ldots\ u(t-n_S)y(t)\ \ldots\ y(t-n_R)\hat{y}(t+d-1)\ \ldots\ \hat{y}(t+d-n_C))^T, \tag{3.2}$$

with $\hat{y}(t)$ as (a posteriori predicted output)

$$\hat{y}(t) = \theta'(t)^T\varphi(t-d). \tag{3.3}$$

The algorithm is: a priori prediction error

$$e(t) = y(t) - \theta(t-d)^T\varphi(t-d) \tag{3.4}$$

usual update:

$$g(t) = 1/1 + \bar{\varphi}(t-d)^T F(t-d)\bar{\varphi}(t-d) \tag{3.5}$$

$$\theta'(t) = \theta(t-d) + g(t)F(t-d)\bar{\varphi}(t-d)\bar{e}(t) \tag{3.6}$$

$$F'(t) = F(t-d) - g(t)F(t-d)\bar{\varphi}(t-d)\bar{\varphi}(t-d)^T F(t-d) \tag{3.7}$$

matrix regularization:

$$F(t) = (1-\lambda_0/\lambda_1)F'(t) + \lambda_0 I, \qquad 0 < \lambda_0 \le \lambda_1 \tag{3.8}$$

leading coefficient regularization:

$$\theta''(t) = \theta'(t) + \mathrm{Max}(0,\ \sigma_0 - s_0'(t))F_{.1}(t)/F_{11}(t). \tag{3.9}$$

projection into the sphere θ_C, $K\sqrt{\lambda_1/\lambda_0}$

$$\theta(t) = \theta_C + (\theta''(t)-\theta_C)\mathrm{Min}(1, K\sqrt{\lambda_1/\lambda_0}/\|\theta''(t)-\theta_C\|) \tag{3.10}$$

control law

$$y^M(t+d) = \theta(t)^T\varphi(t) \tag{3.11}$$

where $F_{.1}(t)$ is the first column of $F(t)$, $F_{11}(t)$ is the first component of $F_{.1}(t)$; $\bar{\varphi}(t-d)$, $\bar{e}(t)$ are normalized signals as defined below. In the following, we call adaptation law equations (3.5) to (3.10).

The properties of this adaptive controller are discussed in [3]. Let us mention here the presence of d distinct interleaved algorithms since $\theta(t)$ is updated in terms of $\theta(t-d)$. Though mean square boundedness can be proven without this multiple recursion, this will highly simplify the notations. Moreover, we will need it in the study of optimality in Section 5.

Normalization Procedure: Before entering the adaptation law, the signals are normalized as follows: let $\rho(t)$ be the output of a first order filter with $\varphi(t-d)^T\varphi(t-d)$ as input or more precisely,

$$\rho(t) = \mu\rho(t-1) + \max(\|\varphi(t-d)\|^2, \rho), \qquad \rho > 0. \tag{3.12}$$

A sequence $v(t)$ is normalized as

$$\bar{v}(t) = \rho(t)^{-\frac{1}{2}}v(t). \tag{3.13}$$

In the following, we denote $(\bar{.})$ the normalized signals.

Usually μ is chosen smaller than 1 (see [3]). However, here, the presence of only mean square bounded disturbances leads to take $\mu = 1$. As a consequence the adaptation gain decreases to zero. This also allows us to solve the minimization problem of the control objective.

The interest of the modifications we have introduced in our algorithm is demonstrated by the following property.

Property of the Adaptation Law: Let Θ be the intersection of the closed sphere with center θ_c and radius K and the closed half space $s_0 \geq \sigma_0$. Let θ be any vector in Θ. Depending on θ, we define $\omega_\theta(t)$ as the "fixed prediction" error given by this parameter vector (note that $\phi(t)$ is built from $\theta(t)$ and not θ. Therefore $\omega_\theta(t)$ is not exactly a prediction error);

$$\omega_\theta(t) = y(t) - \theta^T\phi(t-d). \tag{3.14}$$

Note that (with (2.1),(2.7),(2.8)):

$$\theta = \theta_* \Rightarrow \omega_{\theta_*}(t) = Q_*(q^{-1})x(t). \tag{3.15}$$

We have

Property 3: Whatever $\bar{\phi}(t)$ may be, the adaptation law leads to the following inequality:

$$V_\theta(t) \leq V_\theta(t-d) + \bar{\omega}_\theta(t)^2 - g(t)\bar{e}(t)^2 \tag{3.15}$$

with

$$V_\theta(t) = (\theta(t)-\theta)^T F(t)^{-1}(\theta(t)-\theta). \tag{3.16}$$

Proof: See [4], for example.

This property allows us to relate the mean square (adaptive) a priori error to the minimum mean square (fixed) "prediction" error.

Theorem 3: There exists a positive constant V such that for any $(q,k) \in \mathbb{N}^2$

$$\sum_{t=q+1}^{q+k} \bar{e}(t)^2 \leq V + (1+\lambda_1)\operatorname{Min}_{\theta\in\Theta}\left(\sum_{t=q+1}^{q+k} \bar{\omega}_\theta(t)^2\right). \tag{3.17}$$

Proof: Straightforward from Property 3. Noting that $V_\theta(t)$ is uniformly bounded independently of $\theta \in \Theta$ and that $g(t) \leq 1/(1+\lambda_1)$.

Comment: From a technical point of view this theorem shows that the adaptation law may be considered as an operator: $\rho(t) \to e(t)^2$ for which (3.17) provides a bound on the average value of its instantaneous gain $\bar{e}(t)^2$. This property will be used in the next section to establish mean square boundedness by a small gain theorem.

From a practical point of view this theorem seems to indicate that the minimization involved in our control objective could be realized. However, $\omega_\theta(t)$ is not truly a prediction error. In Section 5 we will need to introduce assumptions about the autocorrelation and stationarity of $x(t)$ in order to complete the proof of optimality.

4. MEAN SQUARES BOUNDEDNESS ANALYSIS

Using the adaptive controller (3.2)-(3.11) for the plant described by (2.1), we can write the signals included in $\phi(t)$ in terms of $e(t)$, $x(t)$, $y^m(t)$ only

$$y(t) = y^m(t) + e(t) \tag{4.1}$$

$$\hat{y}(t) = y^m(t) + (\theta'(t)-\theta(t-d))^T\phi(t-d) \tag{4.2}$$

$$= y^m(t) + g(t)\bar{\phi}(t-d)F(t-d)\bar{\phi}(t-d)e(t) \tag{4.3}$$

$$B(q^{-1})u(t-d) = A(q^{-1})y^m(t) + A(q^{-1})e(t) - x(t). \tag{4.4}$$

Let $\phi^r(t)$ in $\mathbb{R}^{n-1}$ be equal to $\phi(t)$ without its component $u(t)$:

$$\phi^r(t) = (u(t-1) \ldots u(t-n_S)y(t) \ldots y(t-n_R)$$
$$\hat{y}(t+d-1) \ldots \hat{y}(t+d-n_C))^T. \tag{4.5}$$

We have the following property.

Lemma 4.1: Provided assumption A2 holds, there exists a finite positive constant γ_1 such that

$$\sum_{t=d}^{T} \phi^{rT}_{t-d}\phi^r_{t-d} \leq \gamma_1 \sum_{t=1}^{T-1}(y^m(t)^2 + e(t)^2 + x(t)^2). \tag{4.6}$$

Proof: It follows the line of the proof of Lemma 11.3.1 of [1], noticing that

$$g(t)\bar{\phi}(t-d)F(t-d)\bar{\phi}(t-d) \leq \frac{\lambda_1}{1+\lambda_1}. \tag{4.7}$$

Now if we write the control law (3.11) explicitly in terms of $u(t)$ with $\theta^r(t)$ defined as $\phi^r(t)$, we have

$$u(t) = (-\theta^r(t)^T\phi^r(t) + y^m(t+d))/s_0(t). \tag{4.8}$$

But since $s_0(t) \geq \sigma_0$, we have established the following theorem (review definition (3.12) of $\rho(t)$).

Theorem 4: There exists a constant γ such that

$$T\rho \leq \rho(T) \leq \gamma\sum_{t=1}^{T-1}(y^m(t)^2 + e(t)^2 + x(t)^2). \tag{4.9}$$

With Theorems 3 and 4, we are now in position to establish the mean square boundedness of the signals. Following the discussion of the previous section, we consider the boundedness problem as shown in Fig. 1. The property of the operator H: $\varepsilon(t) \to \rho(t)$ is given by inequality (4.9). It is strictly causal with ℓ_∞-gain γT at time T. In order to use a small gain theorem we need the instantaneous gain $\bar{e}(t)$ to tend to zero. In fact we have:

Lemma 4.2: Provided $x(t)$ is almost surely mean square bounded (assumption (2.4)), there exists positive random variables almost surely finite $\tilde{L},\tilde{\omega}$ such that: For any $\varepsilon > 0$, if

$$t/\rho(t) < \varepsilon \qquad \forall t \in [q+1, q+k], \quad q \geq 1 \tag{4.10}$$

then

$$\sum_{t=q+1}^{q+k} \bar{e}(t)^2 \leq \tilde{L} + \varepsilon\tilde{\omega}\operatorname{Log}(q+k/q). \tag{4.11}$$

Proof: Let us take $\theta = \theta_*$ in Theorem 3. We have

$$\sum_{t=q+1}^{q+k} \bar{e}^2(t) \leq V + (1+\lambda_1)\sum_{t=q+1}^{q+k} \bar{\omega}_{\theta_*}(t)^2. \tag{4.12}$$

But since $x(t)$ is a.s. mean square bounded and $Q_*(q^{-1})$ is a polynomial, there exists an a.s. finite random variable $\tilde{\omega}$ such that

$$\operatorname*{Sup}_T \frac{1}{T} \sum_{t=1}^{T} \omega_{\theta_*}^2(t) \le \tilde{\omega}. \tag{4.13}$$

The conclusion follows from Lemma A.1 in the Appendix, noticing with (4.10) that

$$\bar{\omega}_{\theta_*}(t)^2 \le \varepsilon\, \omega_{\theta_*}(t)^2/t, \tag{4.14}$$

This lemma shows that the mean square value of the instantaneous gain $\bar{e}(t)$ on any time interval tends to zero as the length of this interval tends to infinity. This property is sufficient to prove our main theorem.

Main Theorem: Subject to assumptions A1 to A5, the adaptive controller (3.2)-(3.11) in closed loop with the system (2.1) with assumption (2.4) leads to mean square bounded signals in the following sense:

$$\operatorname*{Sup}_T \frac{1}{T} \sum_{t=1}^{T} y(t)^2 < +\infty \qquad \text{a.s.} \tag{4.15}$$

$$\operatorname*{Sup}_T \frac{1}{T} \sum_{t=1}^{T} u(t)^2 < +\infty \qquad \text{a.s.} \tag{4.16}$$

Proof: Let $\tilde{\varepsilon}$, $\tilde{\alpha}$ be random variables such that

$$\tilde{\alpha} = 1 - \gamma\tilde{\varepsilon}\tilde{\omega} \tag{4.17}$$

$$0 < \tilde{\alpha} < 1 \qquad \text{a.s.} \tag{4.18}$$

Let us consider a time interval $(T_0,T_1]$ such that

$$\forall t \in (T_0,T_1]: t/\rho(t) \le \tilde{\varepsilon} \quad , \quad T_0/\rho(t_0) > \tilde{\varepsilon} \tag{4.19}$$

T_1 may be infinite. Outside such types of intervals (4.15),(4.16) are satisfied.

Since $x(t)$ is a.s. mean square bounded and $y^m(t)$ is bounded, there exists an a.s. finite random variable $\tilde{M}_1$ such that (from Theorem 4)

$$\rho(T) \le \tilde{M}_1 T + \gamma \sum_{t=1}^{T-1} e(t)^2. \tag{4.20}$$

From the definition of θ_*, it follows that

$$e(t) = (\theta_* - \theta(t-d))^T \varphi(t-d) + \omega_{\theta_*}(t). \tag{4.21}$$

Since $\theta(t)$ is uniformly bounded and $\omega_{\theta_*}(t)$ is a.s. mean square bounded, there exist a constant M_2 and an a.s. random variable $\tilde{M}_3$ such that

$$\sum_{t=1}^{T_0} e(t)^2 \le M_2\rho(T_0) + \tilde{M}_3 T_0. \tag{4.22}$$

This yields with $\tilde{M}_4 = \tilde{M}_1 + \tilde{M}_3$

$$\rho(T) \le \tilde{M}_4 T + M_2\rho(T_0) + \gamma \sum_{t=T_0+1}^{T-1} \bar{e}(t)^2 \rho(t). \tag{4.23}$$

Let us now use the Bellman-Gronwell lemma

$$\begin{aligned}\rho(t) \le{}& \tilde{M}_4 T + M_2\rho(T_0) \prod_{t=T_0+1}^{T-1} (1+\gamma e(t)^2) \\ &+ \gamma\tilde{M}_4 \sum_{t=T_0+1}^{T-1} t\bar{e}(t)^2 \prod_{i=t+1}^{T-1} (1+\gamma\bar{e}(i)^2).\end{aligned} \tag{4.24}$$

Since $1+x \le \exp x$, we have, using Lemma 4.2:

$$\prod_{t=q+1}^{q+k} (1+\gamma\bar{e}(t)^2) \le \left(\frac{q+k}{q}\right)^{1-\tilde{\alpha}} \exp\gamma\tilde{L}. \tag{4.25}$$

It follows that, with $\tilde{M}_5$ an a.s. finite random variable

$$\rho(T) \le \tilde{M}_5\Big[T + \rho(T_0)\left(\frac{T-1}{T_0}\right)^{1-\tilde{\alpha}} + (T-1)^{1-\tilde{\alpha}} \sum_{t=T_0+1}^{T-1} t^{\tilde{\alpha}}\bar{e}(t)^2\Big]. \tag{4.26}$$

Now using property 3 with $\theta = \theta_*$, we have

$$\sum_{t=T_0+1}^{T-1} t^{\tilde{\alpha}}\, \bar{e}(t)^2 \le (1+\lambda_1) \sum_{t=T_0+1}^{T-1} t^{\tilde{\alpha}} (V(t-d) - V(t) + \bar{\omega}_{\theta_*}(t)^2) \tag{4.27}$$

with $V(t)$ a uniformly bounded sequence. Then from (4.14), Lemmas A.1, A.2 of the Appendix, we know that there exists an a.s. finite random variable $\tilde{M}_6$ such that

$$\sum_{t=T_0+1}^{T-1} t^{\tilde{\alpha}}\, \bar{e}(t)^2 \le \tilde{M}_6 (T-1)^{\tilde{\alpha}}. \tag{4.28}$$

Hence

$$\frac{\rho(T)}{T} \le \tilde{M}_5\left(1 + \frac{\rho(T_0)}{T_0}\left(\frac{T_0}{T-1}\right)^{\tilde{\alpha}} + \tilde{M}_6\right). \tag{4.29}$$

The definitions of T_0, $\tilde{\alpha}$, $\tilde{\varepsilon}$ lead to the conclusion

$$\rho(T)/T \le \tilde{M}_5(1 + \tilde{\varepsilon} + \tilde{M}_6). \tag{4.30}$$

5. ASYMPTOTIC OPTIMALITY

In order to study optimality of our adaptive scheme, let us be more specific about the type of allowed disturbances.

A6. We assume that there exists an exponentially stable monic polynomial $C_+(q^{-1})$ with degree $\le n_c$ such that

$$x(t) = C_+(q^{-1})w(t) \tag{5.1}$$

where $w(t)$ is a stochastic process adapted to the sequence of increasing sub-sigma algebras of F (F_t, $t \in \mathbb{N}$) generated by the observations up to and including time t. Moreover, we assume that

$$E(w(t)|F_{t-1}) = 0 \qquad \text{a.s.} \tag{5.2}$$

$$E(w(t)^2|F_{t-1}) = w^2 \qquad \text{a.s.} \tag{5.3}$$

$$\operatorname*{Sup}_T \frac{1}{T} \sum_{t=1}^{T} w(t)^2 < \infty \qquad \text{a.s.} \tag{5.4}$$

From (5.1),(5.4) it follows that (2.4) is satisfied. Hence, from our main theorem, the signals are mean square bounded.

Following [1], for example, we know that the minimum variance (which could be achieved if $A(q^{-1})$, $B(q^{-1})$, $C_+(q^{-1})$ were known) can be computed as follows: let $S_+(q^{-1})$, $R_+(q^{-1})$, $Q_+(q^{-1})$ (with degree d-1) be the unique solution of

$$C_+(q^{-1})B(q^{-1}) = S_+(q^{-1})A(q^{-1}) + q^{-d}R_+(q^{-1})B(q^{-1}) \tag{5.5}$$

$$Q_+(q^{-1}) = S_+(q^{-1})/B(q^{-1}). \tag{5.6}$$

We denote θ_+ the vector obtained from $S_+(q^{-1})$, $R_+(q^{-1})$, $C_+(q^{-1})$ as in (2.6) and $\omega_+(t)$ the following signal

$$\omega_+(t) = Q_+(q^{-1})w(t). \tag{5.7}$$

The minimum variance is then the variance of $\omega_+(t)$

$$v^2 = E(\omega_+(t+d)^2/F_t) = w^2 \sum_{i=0}^{d-1} q_{+i}^2. \quad (5.8)$$

We have the following theorem

Theorem 5: Subject to assumptions A1 to A6, if $(1/C_+(q^{-1}) - \frac{1}{2})$ is very strictly passive and if θ_+ satisfies

$$\theta_+ \in \Theta. \quad (5.9)$$

then the controller (3.2)-(3.11) in closed loop with the system (2.1) leads to mean square bounded signals and

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} E[(y(t)-y^m(t))^2 | F_{t-d}] = v^2 \quad \text{a.s.} \quad (5.10)$$

i.e., the conditional tracking error variance is asymptotically optimal.

Proof: Following [1], and with the help of Theorem 4, it is sufficient to prove that

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} (\bar{e}(t) - \bar{\omega}_+(t))^2 < \infty \quad \text{a.s.} \quad (5.11)$$

Since then the stochastic key technical Lemma 8.5.3 of [1] can be used.

In order to establish (5.11) let us introduce the following notations

$$b(t) = (\theta_+ - \theta'(t))^T \phi(t-d) \quad (5.12)$$

$$\eta(t) = y(t) - \hat{y}(t). \quad (5.13)$$

From (3.3), (3.5), (3.6), (3.14) we have

$$\eta(t) = g(t)e(t) \quad (5.14)$$

$$\omega_{\theta_*}(t) = \eta(t) - b(t). \quad (5.15)$$

Now since $\theta_+ \in \Theta$, we can use property 3 (with the help of (3.5)) to write

$$\begin{aligned} V_{\theta_+}(t) &\le V_{\theta_+}(t-d) + \bar{b}(t)^2 - 2\bar{\eta}(t)\bar{b}(t) \\ &\quad - \bar{\phi}(t-d)^T F(t-d)\bar{\phi}(t-d)\bar{\eta}(t)^2. \end{aligned} \quad (5.16)$$

Then the proof follows exactly the lines of [7] and [8]. Moreover, the difficulty in the proof of [8] (see Ex. 11.13 of [1]) does not exist in our case since we know that $V_{\theta_+}(t)$ is uniformly bounded and that the signals are a.s. mean square bounded.

6. CONCLUSION

We have investigated properties of a minimum variance controller for which the adaptation law incorporates a signal normalization, a covariance matrix regularization, and a parameter projection. Our assumptions are: the plant is minimum phase, an upper bound of its order and its delay are known, and we know a compact set containing a stabilizing regulator and the sign and a lower bound of its leading coefficient. We have established that even in the presence of mean square bounded disturbance, the input-output signals are mean square bounded. Moreover, if the noise is a finite moving average process such that its coloring filter satisfies a very strict passivity condition, then our controller is asymptotically optimal. In the derivation of our results, we have proved the following lemma (compare with Lemma 8.5.3 of [1]).

Lemma 6: Let $\rho(t)$, $e(t)$, $\omega(t)$ be sequences such that

$$\rho(T) \le K_1 T + \sum_{t=1}^{T-1} e(t)^2$$

$$e(t)^2 - \omega(t)^2 \le (V(t-1)) - V(t))\rho(t)$$

$$\frac{1}{T} \sum_{t=1}^{T} \omega(t)^2 \le K_2.$$

If $V(t)$ a uniformly bounded sequence, then $\rho(T)/T$ is uniformly bounded. This lemma seems to be a key technical lemma in the study of stochastic adaptive controllers in nonideal cases (see [5] for example).

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APPENDIX

Lemma A.1: Let $v(t)$ be a sequence of positive real numbers such that

$$\frac{1}{T} \sum_{t=1}^{T} v(t) \le V. \quad (A.1)$$

Then for any $q \ge 1$, $k \ge 0$, we have:

$$\text{i)} \quad \sum_{t=q+1}^{q+k} \frac{v(t)}{t} \le V(1 + \log \frac{q+k}{q}) \quad (A.2)$$

$$\text{ii)} \quad \sum_{t=q+1}^{q+k} \frac{v(t)}{t^\alpha} \le \frac{V}{1-\alpha}(q+k)^{1-\alpha}, \qquad 0 \le \alpha < 1. \quad (A.3)$$

Proof: Let $s(t)$ be defined as the sum

$$s(T) = \frac{1}{T}\sum_{t=1}^{T} v(t) \tag{A.4}$$

we have

$$v(t) = ts(t) - (t-1)s(t-1). \tag{A.5}$$

Hence

i) $$\sum_{t=q+1}^{q+k}\frac{v(t)}{t} = s(q+k) - s(q) + \sum_{t=q+1}^{q+k}\frac{s(t-1)}{t} \tag{A.6}$$

$$\leq V\Big(1+\sum_{t=q+1}^{q+k}\frac{1}{t}\Big) \leq V\Big(1+\int_q^{q+k}\frac{dt}{t}\Big) \tag{A.7}$$

ii) $$\sum_{t=q+1}^{q+k}\frac{v(t)}{t^\alpha} \leq (q+k)^{1-\alpha}s(q+k) - q^{1-\alpha}s(q)$$

$$+\sum_{t=q+1}^{q+k}(t-1)\Big(\frac{1}{(t-1)^\alpha}-\frac{1}{t^\alpha}\Big)s(t-1). \tag{A.8}$$

But from the mean value theorem and the monotonicity of $t^{\alpha-1}$ we have

$$t^\alpha - (t-h)^\alpha \leq \alpha h(t-h)^{\alpha-1}, \quad h > 0 \tag{A.9}$$

This yields

$$(t-h)\big((t-h)^{-\alpha} - t^{-\alpha}\big) \leq \alpha h\, t^{-\alpha}. \tag{A.10}$$

The conclusion follows taking the continuous summation.

Lemma A.2: Let $v(t)$ be a sequence of positive real numbers bounded by V. For any α, $0 \leq \alpha < 1$, there exists a constant C such that: $\forall(q,k)$

$$\sum_{t=q+1}^{q+k} t^\alpha\big(v(t-d)-v(t)\big) \leq C(q+k)^\alpha. \tag{A.11}$$

Proof: (In the case $d=1$ to simplify notations.) We have

$$\sum_{t=q+1}^{q+k} t^\alpha\big(v(t-1)-v(t)\big) = -(q+k)^\alpha v(q+k)$$

$$+\, q^\alpha v(q) + \sum_{t=q+1}^{q+k}\big((t+1)^\alpha - t^\alpha\big)v(t). \tag{A.12}$$

Then the conclusion follows with the same arguments as those of the proof of Lemma A.1.

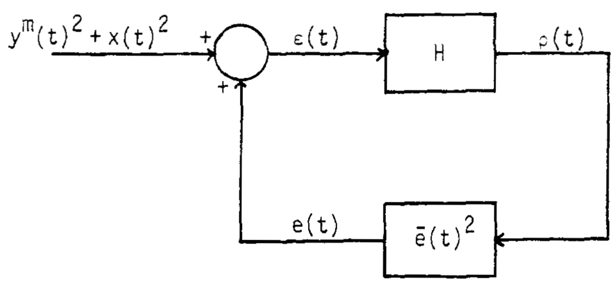

Figure 1